UNIT 9 ANNUITIES

Structure

9.0 OBJECTIVES

After reading this unit, you will be able to:

- define the term 'annuity';
- distinguish between the terms 'annuity-certain' and 'contingent annuities';
- outline the meaning of the term 'annuity-immediate' expressing the relationship between their 'present and accumulated values';
- explain the meaning of the term 'annuity-due' bringing out the relationship between 'annuity-immediate' and 'annuity-due';
- state the meaning of the term 'deferred annuity';
- differentiate between 'increasing annuity' and 'decreasing annuity' deriving expressions for their present values;
- discuss the concepts of 'varying interest annuity' and 'varying annuity-due';
- write a note on 'perpetuity';

- indicate how the computation of present value is made under a 'continuous annuity'; and
- describe the significance of 'amortization' with a distinction on the methods adopted for balancing the accumulated loan payments.

9.1 INTRODUCTION

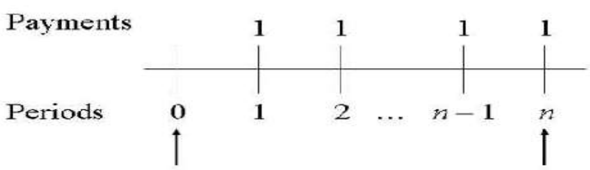
An annuity refers to a series of periodic payments made at equal intervals of time. Examples of annuities are house rent, mortgage payments on homes, installments payments on automobiles, etc. In principle, annuity payments can occur at various times. However, while dealing with standard annuity cases we usually consider equal payments made in equal increments of time. In order to evaluate these payments, we need a set 'points of time' to calculate payments. We use the term 'present time' to refer to the time in which the payment schedule goes into effect. This is not necessarily the time at which the payments start. Most of the standard annuities are *level*, meaning that the payments are equal in amounts. Annuities can be categorised into: (i) regular annuities (or annuity-certain) where payments are guaranteed to occur for a fixed period of time and (ii) contingent annuities, where the beneficiary does not begin receiving payments until a specified event occurs. In case of annuity-certain, the term 'certain' refers to the legal obligation rather than a force of will. If you cease making payments, the lender will recoup the outstanding balance in some other way and/or write it off as a loss i.e. marking the debt as 'paid'.

9.2 TYPES OF ANNUITIES

Broadly, there are two main types of annuities viz. 'annuity – immediate' and 'annuity – due'. There is a third type called 'annuity deferred'. In this section, we will discuss these three types of annuities.

9.2.1 Annuity-Immediate

An annuity-immediate is defined as one under which payments of 1 are made at the end of each period with i as the 'effective rate of interest' for that period. The word 'immediate' conveys that the payments are to be made at the end of the period (which is contrary to a direct meaning of the word immediate). It is also common to assume that the payments cease at some point like n periods (years). The cash stream represented by such an annuity can be visualized on a 'time diagram' as below.



In the figure above, the first arrow shows the beginning of the first period, at the end of which the first payment is due under the annuity. The second arrow indicates the last payment date i.e. just after the last payment has been made. The present value of the annuity-immediate at time $\mathbf{0}$ is denoted by $a_{\overline{n|}\,i}$ or simply $a_{\overline{n|}}$. Using the 'equation of value' with the comparison date kept at time $t = 0$, we can write:

$$a_{\overline{n|}} = v + v^2 + \cdots + v^n \qquad (9.1)$$

i.e. the present value of the annuity is the sum of the present values of each of the n payments. Note that the expression on the right-hand side of (9.1) is a geometric progression. Multiplying both sides of the equation by v we get:

$$va_{\overline{n|}} = v^2 + v^3 \ldots + v^n + v^{n+1} \qquad (9.2)$$

Subtracting (9.2) from (9.1) and simplifying we get:

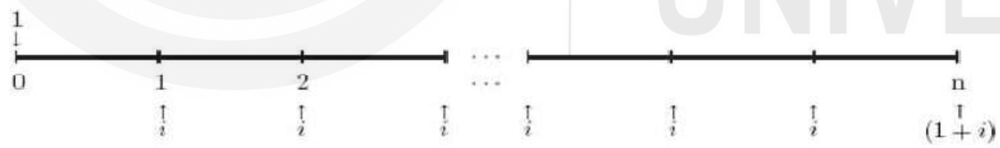
$(1 - v)a_{\overline{n|}} = v(1 - v^n)$. Hence:

$$a_{\overline{n|}} = v.\frac{1-v^n}{1-v} = v.\frac{1-v^n}{iv} = \frac{1-(1+i)^{-n}}{i} \quad (\text{since } v = \frac{1}{1+i}) \qquad (9.3)$$

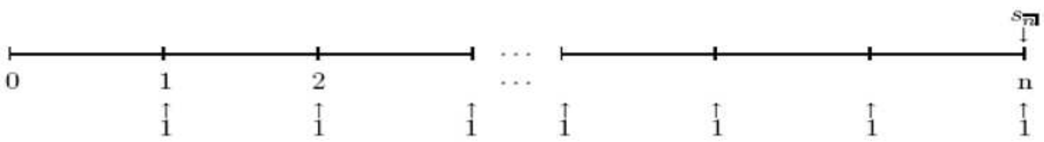
If payments are P instead of 1, then the present value of the annuity (i.e. the present value of all payments) is $Pa_{\overline{n|}}$. It means the certainty of payments in amounts of P made according to the schedule is worth $Pa_{\overline{n|}}$ presently. Note that equation (9.3) is same as:

$$1 = v^n + ia_{\overline{n|}} \qquad (9.4)$$

Equation (9.4) is the equation of value at time $t = 0$ of an investment of 1 for n periods during which an interest of i is received at the end of each period and is reinvested at the same rate i and at the end of the n periods the original investment of 1 is returned. The time diagram of such a transaction is as below.



Let us now determine the accumulated value of an 'annuity-immediate' right after the n^{th} payment is made. It is denoted by $s_{\overline{n|}}$ as in the time diagram below.



Writing the equation of value at the comparison date $t = n$ we find $s_{\overline{n|}} = 1 + (1 + i) + (1 + i)^2 + \cdots + (1 + i)^{n-1}$. That is, $s_{\overline{n|}}$ is the sum of the accumulated value of each of the n payments. We can write $s_{\overline{n|}} = 1 + (1 + i) + (1 + i)^2 + \cdots + (1 + i)^{n-1} = \frac{(1+i)^n-1}{(1+i)-1} = \frac{(1+i)^n-1}{i}$.

The relationships between the Present and the Accumulated values of 'annuity-immediate', for given $a_{\overline{n}|}$ and $s_{\overline{n}|}$, can be expressed as follows:

i) $\boldsymbol{s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}}$

[since $\boldsymbol{s_{\overline{n}|}} = \frac{(1+i)^n-1}{i} = (1+i)^n . \frac{1-(1+i)^{-n}}{i} = (1+i)^n a_{\overline{n}|}$]

(**i. e.** the accumulated value of a principal of $\mathbf{a_{\overline{n}|}}$ after '**n**' periods is $\mathbf{s_{\overline{n}|}}$.)

ii) $\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$ [since $\frac{1}{s_{\overline{n}|}} + i = \frac{i}{(1+i)^n-1} + i = \frac{+i1+i^n-i}{(1+i)^n-1} = \frac{i}{1-v^n} = \frac{1}{a_{\overline{n}|}}$].

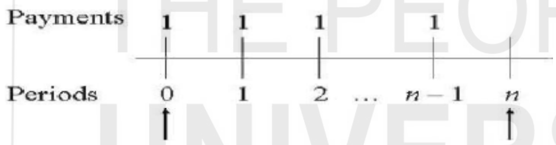
Therefore, we have:

$$s_{\overline{n}|} = (1+i)^n a_{\overline{n}|} \qquad (9.5a)$$

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i \qquad (9.5b)$$

9.2.2 Annuity-Due

The word 'due' conveys that payments are made at the beginning of the year. Hence, in case of an annuity-due, payments are made at the beginning of the payment periods. Assume that a series of payments are made in unit amount (Re.1) at the beginning of every period (or year) and that i is the effective rate of interest for that period (or year). Let us also assume that payments cease after some fixed number of years n. The cash stream represented by the annuity can be visualised as in the time diagram below.



In the time diagram above, the first arrow shows the beginning of the first period at which the first payment is made under the annuity. The second arrow shows n^{th} period i.e. beginning of the last payment period. Let i denote the interest rate per period. The present value of the 'annuity-due' at time '0' will be denoted by $\ddot{a}_{\overline{n}|}$ · To determine $\ddot{a}_{\overline{n}|}$, we consider the equation of value at time $t = 0$ as: $\ddot{a}_{\overline{n}|} = 1 + v + v^2 \dots + v^{n-1}$. That is, $\ddot{a}_{\overline{n}|}$ is equal to the sum of the present values of each of the n payments. Hence, if payments are P instead of Re.1, then the present value of the annuity is $P\ddot{a}_{\overline{n}|}$ · The right-hand side in $\ddot{a}_{\overline{n}|}$. is a geometric progression. Multiplying both sides by v we obtain, $v\ddot{a}_{\overline{n}|} = 1 + v + v^2 + v^3 \dots + v^{n-1} + v^n$. Subtracting $v\ddot{a}_{\overline{n}|}$ from the equation for $\ddot{a}_{\overline{n}|}$ we get $(1-v)\ddot{a}_{\overline{n}|} = (1 - v^n)$. Hence:

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{1-v} \qquad (9.6)$$

Since $1 - v = d$, we have $\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d} = \frac{1-(1+i)^{-n}}{d} = \frac{1-(1-d)^n}{d}$ (9.6a)

We, therefore, have the relationship between the present and the accumulated values of annuity-due as:

a) $\ddot{s}_{\overline{n}|} = \frac{(1+i)^n-1}{d} = (1+i)^n + \frac{1-(1+i)^{-n}}{d} = (1+i)^n \ddot{a}_{\overline{n}|}$ (9.7a)

b) $\frac{1}{\ddot{s}_{\overline{n}|}} + d = \frac{d}{(1+i)^{n-}} + d\frac{(1+i)^n-1}{(1+i)^n-1} = \frac{d+d[(1+i)^n-1]}{(1+i)^n-1} = \frac{d^{(1+i)^n}}{(1+i)^n-1} = \frac{d}{1-(1+i)^{-n}} = \frac{1}{\ddot{a}_{\overline{n}|}}$

(9.7b)

Equation (9.7), in particular, states that if the present value at time 0, $\ddot{a}_{\overline{n}|}$, is accumulated forward to time *n,* then we will have its future value, $\ddot{s}_{\overline{n}|}$. Now, four relationships between 'annuity-immediate' and 'annuity-due' can be easily arrived at as follows.

a) Since $d = \frac{i}{i+1}$, we have $\ddot{a}_{\overline{n}|} = \frac{1-(1+i)^{-n}}{d} = (1+i).\frac{1-(1+i)^{-n}}{i} = (1+i)a_{\overline{n}|}$.

b) Since $d = \frac{i}{i+1}$, we have: $\ddot{s}_{\overline{n}|} = \frac{(1+i)^n-1}{d} = (1+i).\frac{(1+i)^n-1}{i} = (1+i)s_{\overline{n}|}$

c) Since $\ddot{a}_{\overline{n}|} = \frac{1-(1+i)^{-n}}{d} = \frac{i+}{i}[1-(1+i)^{-n}] = \frac{1-(1+i)^{-n+1}+i}{i} = 1 + a_{\overline{n-1}|}$.

d) Since $\ddot{s}_{\overline{n-1}|} = \frac{(1+i)^{-n}-1}{d} = \frac{i+1}{i}[(1+i)^{n-1}-1] = \frac{(1+i)^{-n}-1-i}{i} = s_{\overline{n}|}$ -1

Thus, the four relationships are:

a) $\ddot{a}_{\overline{n}|} = (1+i)\, a_{\overline{n}|}$ (9.8a)

b) $\ddot{s}_{\overline{n}|} = (1+i)\, s_{\overline{n}|}$ (9.8b)

c) $\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$. (9.8c)

d) $s_{\overline{n}|} = \ddot{s}_{\overline{n-1}|} + 1$. (9.8d)

9.2.3 Deferred Annuity

A 'deferred annuity' is an annuity in which the payments start at some future time. A standard deferred annuity immediate is one in which payments are deferred for k periods i.e. its first payment of 1 made at time $k+1$. More precisely, since all payments are made at the end of the time period, the first payment of 1 is also made at the end of year $k+1$. The fact that an annuity has been deferred m years is conveyed by having the symbol '$m|$' appear in the lower left of an annuity symbol. For instance, the n-year term annuity, of which the first *k*-year are deferred for payment, is denoted by $_{k|}\ddot{a}_{\overline{n}|}$. Note that the present value of a *k*-year deferred annuity is just v^k times the present value of the annuity where the payments begin at present. Hence, $_{k|}\ddot{a}_{\overline{n}|} = v^k \ddot{a}_{\overline{n}|}$. From the perspective of a person standing at time k, this deferred annuity-immediate looks like a standard n period annuity-immediate.

9.2.4 Illustrations

Let us now consider some illustrations which will help you to understand the application of above equations in practice.

a) *For a given interest rate i, $a_{\overline{n|}}$= 8.3064 and $s_{\overline{n|}}$ = 14.2068. Calculate (i) i and (ii) n.*

i) $i = \frac{1}{8.3064} - \frac{1}{14.2068} = \mathbf{5\%}$

ii) $\boldsymbol{n} = \frac{1}{\ln(1+i)} \ln\left(\frac{s_{\overline{n|}}}{a_{\overline{n|}}}\right) = \mathbf{11}$

b) *Calculate the future value of an annuity-immediate of amount Rs. 100 paid annually for 5 years at the rate of interest of 9%.*

$$100s_{\overline{5|}} = 100 \times \frac{(1.09)^5 - 1}{0.09} \approx 598.47$$

c) *Show that* $\boldsymbol{a_{\overline{m+n|}} = a_{\overline{m|}} + v^m a_{\overline{n|}} = a_{\overline{n|}} + v^n a_{\overline{m|}}}$. *Interpret the result.*

$$\boldsymbol{a_{\overline{m|}} + v^m a_{\overline{n|}} = \frac{1 - v^m}{i} + v^m . \frac{1 - v^n}{i} = \frac{1 - v^m + v^m - v^{m+n}}{i}}$$
$$\boldsymbol{= \frac{1 - v^{m+n}}{i} = a_{\overline{m+.n|}}}$$

The present value of the first $\boldsymbol{m}$ payments of an $(\boldsymbol{m + n})$-year annuity-immediate of 1 is $\boldsymbol{a_{\overline{m|}}}$. The remaining $\boldsymbol{n}$ payments have value $\boldsymbol{a_{\overline{n|}}}$ at time $\boldsymbol{t = m}$. Discounted to the present, it is $\boldsymbol{v^m a_{\overline{n|}}}$ at time $\boldsymbol{t = 0}$.

d) *Calculate the present value of an annuity-immediate of amount Rs. 100 paid annually for 5 years at the rate of interest of 9%.*

$\boldsymbol{100a_{\overline{5|}} = 100\frac{1-(1.09)^{-5}}{0.09} \approx 388.97}$.

e) *Calculate the present value of an annuity-due paying annual payments of 1200 for 12 years with the first payment two years from now. The annual effective interest rate is 6%.*

$1200(\mathbf{1.06})^{-\mathbf{2}}\ddot{a}_{\overline{12|}} = \mathbf{1200}\left(\ddot{a}_{\overline{14|}} - \ddot{a}_{\overline{2|}}\right) = \mathbf{1200(9.8527 - 1.9434) \approx \ 9,491.16}$

f) *For four years, an annuity pays Rs. 200 at the end of each year with an effective 8% rate of interest. Find the accumulated value of the annuity 3 years after the last payment.*

$$200(1 + 0.08)^3 s_{\overline{4|}} = 200\left(s_{\overline{7|}} - s_{\overline{3|}}\right) = 200(8.9228 - 3.2464)$$
$$= Rs.1135.28$$

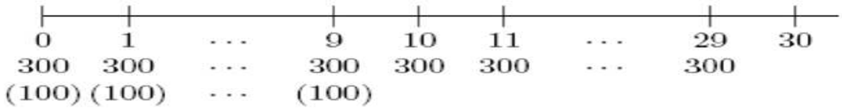
It is also possible to work with annuities-due instead of annuities-immediate. You may verify that $\boldsymbol{(1 + i)^m \ddot{s}_{\overline{n|}} = \ddot{s}_{\overline{m+n|}} - \ddot{s}_{\overline{m|}}}$.

g) *A monthly annuity-due pays 100 per month for 12 months. Calculate the accumulated value 24 months after the first payment using a nominal rate of 4% compounded monthly.*

$$\mathbf{100\left(1 + \frac{0.04}{12}\right)^{12} \ddot{s}_{\overline{12|}\frac{0.04}{12}} = 1,276.28}$$

h) *Over the next 30 years, you deposit money into a retirement account at the beginning of each year. The first 10 payments are 200 each. The remaining 20 payments are 300 each. The effective annual rate of interest is 9%. (i) Find the present value of these payments. (ii) Find* $\ddot{a}_{\overline{n}|}$ *if the effective rate of discount is 10%.*

i) The time diagram of this situation is as follows.



$\therefore 300\ddot{a}_{\overline{30}|} - 100\ddot{a}_{\overline{10}|} = \mathbf{1.09\left(300a_{\overline{30}|} - 100a_{\overline{10}|}\right) = Rs. 2659.96}$

ii) Here, d = 0.10. Therefore, $\mathbf{1 = v - d = 0.9}$ **Hence,** $\ddot{a}_{\overline{8}|} = \frac{1-(0.9)^8}{0.1} = \mathbf{5.6953279.}$

j) *Estimate the amount you must invest today at 6% interest rate compounded annually so that you can withdraw Rs.5,000 at the beginning of each year for the next 5 years?*

$5000\ddot{a}_{\overline{n}|} = 5000.\frac{1-(1.06)^{-5}}{0.06(1.06)^{-1}} = \mathbf{22,325.53}.$

Check Your Progress 1 [answer within the space given in about 50-100 words]

1) What is an ‘annuity’? Give illustrations.

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

2) Distinguish between ‘annuity-certain’ and ‘contingent annuities’.

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

ignou THE PEOPLE'S UNIVERSITY

3) What does $a_{\overline{n|}}$ and $s_{\overline{n|}}$ denote? How are they related?

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

4) What is the expression for the calculation of $a_{\overline{n|}}$?

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

5) How is an 'annuity-due' different from that of an 'annuity-immediate'?

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

6) What do the expressions $\ddot{a}_{\overline{n|}}$ and $\ddot{s}_{\overline{n|}}$ represent?

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

7) Write the expression for calculating $\ddot{a}_{\overline{n|}}$ in terms of 'i'.

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

8) Indicate the expressions for the relationship between the present and accumulated values of 'annuity-due'.

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

ignou THE PEOPLE'S UNIVERSITY

9) State the relationships between 'annuity-immediate' and 'annuity-due'.

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

10) What is meant by a 'deferred annuity'?

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

9.3 INCREASING AND DECREASING ANNUITY

An 'increasing annuity immediate' with a term of n periods pays 1 at the end of the first period, 2 at the end of the second period, 3 at the end of the third period, & so on, so that it pays 'n' at the end of the nth period. The present value of such an annuity is $(Ia)_{\overline{n|}} = \sum_{j=1}^{n} jv^j$. Although this is not a geometric series, using the same technique as used before, we get:

$(Ia)_{\overline{n|}} - v(Ia)_{\overline{n|}} = v + v^2 + \ldots + v^n - nv^{n+1}$.

$\therefore (Ia)_{\overline{n|}} = \frac{\left(a_{\overline{n|}} - nv^{n+1}\right)}{1-v} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i}$.

Following the method of increasing annuity above, a decreasing annuity immediate can also be similarly worked out. For instance, a decreasing annuity immediate with a term of 'n' periods pays n at the end of the first period, $n-1$ at the end of the second period, $n-2$ at the end of the third period, & so on, so that, it pays 1 at the end of the n^{th} period. To find out the present value of such an annuity, $(Da)_{\overline{n|}}$, we take $(Ia)_{\overline{n|}} + (Da)_{\overline{n|}} = (n+1)a_{\overline{n|}}$. This gives, $(Da)_{\overline{n|}} = \frac{(n - a_{\overline{n|}})}{i}$.

Likewise, an annuity immediate with $2n-1$ payments pays 1 at the end of the first period, 2 at the end of the second period, & so on. It pays 'n' at the end of the n^{th} period, $n-1$ at the end of the $(n+1)^{st}$ period, . . . , 1 at the end of the $(2n-1)^{st}$ period. Its present value is therefore:

$$(Ia)_{\overline{n|}} + v^n(Da)_{\overline{n-1|}} = \frac{\left(1 + a_{\overline{n-1|}} - v^n - v^n a_{\overline{n-1|}}\right)}{i} = (1 - v^n)\frac{\left(1 + a_{\overline{n-1|}}\right)}{i} = \ddot{a}_{\overline{n|}} a_{\overline{n|}}.$$

9.3.1 Varying Interest Annuity

Let us consider situations in which interest can vary each period with compound interest in effect. Let us denote $\boldsymbol{i_k}$ as the rate of interest applicable from time $\boldsymbol{k-1}$ to time $\boldsymbol{k}$. We first consider the present value of an annuity-immediate for '$\boldsymbol{n}$ −period' with two variations: the first is when $\boldsymbol{i_k}$ is applicable only for period $\boldsymbol{k}$ regardless of when the payment is made i.e. the rate $\boldsymbol{i_k}$ is used only in period $\boldsymbol{k}$ for discounting all payments. In this case, the present value, $\boldsymbol{a_{\overline{n|}}}$, is given by:

$$(1+i_1)^{-1} + (1+i_1)^{-1}(1+i_2)^{-1} + \cdots + (1+i_1)^{-1}(1+i_2)^{-1} \ldots (1+i_n)^{-1}.$$

The second variation we consider is when a payment is made at time $\boldsymbol{k}$ with the rate $\boldsymbol{i_k}$ used as the effective rate of interest for each period $\boldsymbol{i \le k}$. In this case, the present value is:

$$a_{\overline{n|}} = (1+i_1)^{-1}(1+i_2)^{-1} + \cdots + (1+i_n)^{-n}.$$

The present value of annuity-due can be obtained from the present value of annuity immediate by using $\boldsymbol{\ddot{a}_{\overline{n|}} = 1 + a_{\overline{n-1|}}}$. Let us now turn to accumulated values where we will consider an annuity-due. Again, we consider two different situations. If $\boldsymbol{i_k}$ is applicable only for period $\boldsymbol{k}$ regardless of when the payment is made, then the 'accumulated value' is given by:

$$\ddot{s}_{\overline{n|}} = (1+i_1)(1+i_2) \ldots (1+i_n) + \cdots + (1+i_{n-1})(1+i_n) + (1+i_n).$$

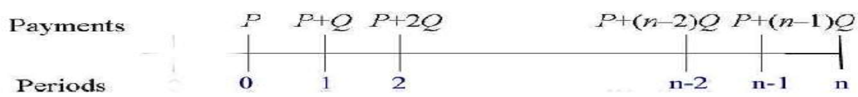
For $\boldsymbol{i_k}$ (applicable for all periods $\boldsymbol{i \le k}$), the 'accumulated value' is:

$$\boldsymbol{\ddot{s}_{\overline{n|}} = (1+i_1)^n + (1+i_2)^{n-1} + \cdots + (1+i_n)}.$$

Now, accumulated values of annuity-immediate can be obtained from the accumulated values of annuity-due by using the relationship: $\boldsymbol{s_{\overline{n+1|}} = \ddot{s}_{\overline{n|}} + 1}$

9.3.2 Varying Annuity-Due

Let us consider the case of an increasing annuity-due where an annuity with the first payment is $\boldsymbol{P}$ at the beginning of year 1 and the payments increase by $\boldsymbol{Q}$ thereafter, continuing for $\boldsymbol{n}$ years. A time diagram of this situation is as below.



The present value for this annuity-due is:

$$\boldsymbol{PV = P + (P+Q)v + (P+2Q)v^2 + \cdots + [P+(n-1)Q]v^{n-1}} \quad \textbf{(9.9a)}$$

Multiplying by $\boldsymbol{v}$, we get

$$\boldsymbol{vPV = Pv + (P+Q)v^2 + (P+2Q)v^3 + \cdots + [P+(n-1)Q]v^n} \quad \textbf{(9.9b)}$$

Subtracting (9.9b) from (9.9a) we get:

$$(1-v)PV = P(1-v^n) + (v + v^2 + \cdots + v^n)Q - nv^nQ$$

$$\text{i.e. } PV = P\ddot{a}_{\overline{n}|} + Q\frac{[a_{\overline{n}|} - nv^n]}{d} \quad (9.9c)$$

The 'accumulated value' of these payments at time n is:

$$AV = (1+i)^n PV = P\ddot{s}_{\overline{n}|} + Q\frac{[s_{\overline{n}|} - n]}{d} \quad (9.10a)$$

In the special case when $P = Q = 1$ we have:

$$(I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d} \quad (9.10b)$$

$$\text{and } (I\ddot{s})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d} = \frac{s_{\overline{n+1}|} - (n+1)}{d} \quad (9.10c)$$

9.3.3 Illustrations

Let us now consider some empirical illustrations as before.

a) *Find the accumulated value of a 12-year annuity-immediate of Rs.500 per year, if the effective rate of interest (for all money) is 8% for the first 3 years, 6% for the following 5 years, and 4% for the last 4 years.*

The accumulated value of the first 3 payments to the end of year 3 is:

$$500s_{\overline{3}|0.08} = 500(3.2464) = Rs.1623.20$$

The accumulated value of the first 3 payments to the end of year 8 at 6% and then to the end of year 12 at 4% is:

$$\mathbf{1623.20(1.06)^5(1.04)^4 = Rs.2541.18.}$$

The accumulated value of payments 4, 5, 6, 7, and 8 at 6% to the end of year 8 is:

$$500s_{\overline{5}|0.06} = 500(5.6371) = Rs.2818.55$$

The accumulated value of payments 4, 5, 6, 7, and 8 to the end of year 12 at 4% is:

$$\mathbf{2818.55(1.04)^4(1.04)^4 = Rs.3297.30.}$$

The accumulated value of payments 9, 10, 11, and 12 to the end of year 12 at 4% is:

$$500s_{\overline{4}|0.04} = 500(4.2464) = Rs.2123.23$$

The accumulated value of the 12-year annuity immediate is:

$$\mathbf{2541.18 + 3297.30 + 2123.23 = Rs.7961.71.}$$

b) *How much must a person deposit now into a special account in order to withdraw Rs.1,000 at the end of each year for the next fifteen years, if the effective rate of interest is equal to 7% for the first five years, and equal to 9% for the last ten years?*

$$PV = 1000\left(a_{\overline{5}|0.07}\right) + a_{\overline{10}|0.09}(1.07)^{-5} = 1000(4.1002 + 4.5757)$$
$$= 8675.90.$$

c) *Determine the present value and the future value of payments of Rs.75 at time 0, Rs.80 at time 1, Rs.85 at time 2, and so on up to Rs.175 at time 20 years. The annual effective rate is 4%.*

The present value is $\boldsymbol{70\ddot{a}_{\overline{21}|} + 5(I\ddot{a})_{\overline{21}|} = Rs.1,720.05}$. The future value is

$$\mathbf{(1.04)^{21}(1,720.05) = \boldsymbol{Rs.}\,3919.60.}$$

Note that in the case of a decreasing annuity-due (where $\boldsymbol{P = n}$ **and** $\boldsymbol{Q = -1}$), the present value at time 0 is $\boldsymbol{(D\ddot{a})_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{d}}$ and the accumulated value at time $\boldsymbol{n}$ is $\boldsymbol{(D\ddot{s})_{\overline{n}|} = (1+i)^n (D\ddot{a})_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{d}}$.

d) *Determine the present value at time 0 of payments of Rs.10 paid at time 0, Rs.20 paid at time 1 year, Rs.30 paid at time 2 years, and so on, assuming an annual effective rate of 5%.*

$$10(I\ddot{a})_{\overline{\infty}|} = \frac{10}{d^2} = 10\left(\frac{1.05}{0.05}\right)^2 = Rs.4{,}410.00.$$

Check Your Progress 2 [answer within the space given in about 50-100 words]

1) Write the expression for obtaining the 'present value' of an 'increasing annuity immediate' and 'decreasing annuity immediate'.

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

2) What is the expression for calculating the 'present value' of an 'varying interest annuity' when a rate i_k is applied in period k for discounting all payments? What is the corresponding expression for 'present value' if the rate i_k used as the effective rate of interest for each period $i \leq k$?

………………………………………………………………………………

………………………………………………………………………………

……………………………………………………………………………

3) Write the expression for the 'present value' and the 'accumulated value' of a 'varying annuity due' where the first payment is P and the successive payments increase by Q for the next n years.

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

9.4 PERPETUITY

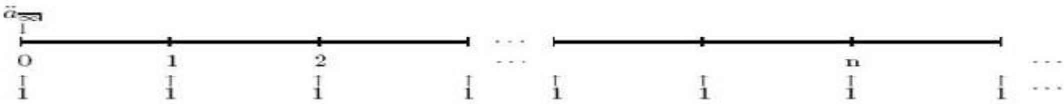
A perpetuity is an annuity whose term is infinite. In other words, it is an annuity whose payments continue forever. The first payment can occur either immediately (perpetuity-due) or one period from now (perpetuity-immediate). The accumulated values of perpetuities do not exist. Let us determine the present value of a perpetuity-immediate at the time of one period before the first payment. Let us assume a payment of 1 is made at the end of each period. The present value, denoted by $\boldsymbol{a_{\overline{\infty}|}}$, bears a time diagram as below.

Using the equation of value at time $t = 0$ we find that:

$$\boldsymbol{a_{\overline{\infty}|} = v + v^2 + \ldots} \quad (9.11)$$

(9.11) is an infinite geometric progression (with $\boldsymbol{v} < 1$). It is equal to $\frac{v}{1-v} = \frac{v}{iv} = \frac{1}{i}$). The verbal interpretation of Equation (9.11) is therefore as follows. If the periodic effective rate of interest is $\boldsymbol{i}$, then one can invest a principal of $\frac{1}{i}$ for one period and obtain a balance of $\mathbf{1} + \frac{1}{i}$ at the end of the first period. A payment of Rs.1 is made and the remaining balance of $\frac{1}{i}$ is reinvested for the next period. This process continues forever. Now, since $\boldsymbol{a_{\overline{n}|}} = \frac{1-v^n}{i}$ and $\lim_{n\to\infty} \boldsymbol{v^n = 0}$ for $0 < \boldsymbol{v} < 1$ we have $\boldsymbol{a_{\overline{\infty}|}} = \lim_{n\to\infty} \boldsymbol{a_{\overline{n}|}} = \frac{1}{i}$.

Analogous to perpetuity-immediate, we can define a perpetuity-due to be an infinite sequence of equal payments where each payment is made at the beginning of the period. If $\boldsymbol{\ddot{a}_{\overline{\infty}|}}$ denotes the present value of such a perpetuity-due at the time of first payment, a time diagram describing the case would be as follows.



The equation of value at time t = 0 is:

$$\ddot{a}_{\overline{\infty}|} = 1+v + v^2 + \cdots = \frac{1}{1-v} = \frac{1}{d} = \lim_{n\to\infty} \ddot{a}_{\overline{n}|} \qquad (9.12)$$

9.4.1 Continuing Annuity

Theoretically, an annuity could be paid continuously i.e. the annuitant can receive money at a constant rate of 1 rupee per unit of time for ever. The present value of such an annuity that pays 1 per unit time for n time periods can be denoted by $\overline{a}_{\overline{n}|}$. Its value at time 0 can be computed as follows. The value of dt rupees, in the time interval 't to $t + dt$', is $v^t dt = e^{-\delta t}dt$. Hence:

$$\bar{a}_{\overline{n}|} = \int_0^n e^{-\delta t} dt = \frac{1-v^n}{\delta} \qquad (9.13)$$

Annuity payments can be made either more or less often than interest is compounded. In such cases, the equivalent rate of interest can be used to compute the value of the annuity. The symbol $a_{\overline{n}|i}^{(m)}$ denotes the present value of an 'annuity-immediate' that pays $\frac{1}{m}$ at the end of each m^{th} part of a period for n periods under the assumption that the effective interest rate is i per period. For instance, if $m = 12$ and the period is a year, payments of 1/12 are made at the end of each month. We need an expression (a formula) for $a_{\overline{n}|}^{(m)}$ assuming the effective rate of interest as i per period. Notice that the payments are made more frequently than interest is compounded. Hence, using the equivalent rate $i^{(m)}$ makes the computations easy. Using geometric series, we have:

$$a_{\overline{n}|}^{(m)} = \frac{1}{m}\sum_j^{nm}\left(1+\frac{i^{(m)}}{m}\right)^{-j} = \left(\frac{1-v^n}{i^{(m)}}\right) = \frac{i a_{\overline{n}|}}{i^{(m)}} \qquad (9.14)$$

The symbol $\ddot{a}_{\overline{n}|i}^{(m)}$ can be used to denote the present value of an 'annuity-due' that pays $\frac{1}{m}$ at the beginning of each m^{th} part of a period for n periods when the effective periodic interest rate is i. For this, we have $\ddot{a}_{\overline{n}|}^{(m)} = \frac{(1-v^n)}{d^m} = d\ddot{a}_{\overline{n}|}$ as the expression for $\ddot{a}_{\overline{n}|}^{(m)}$ if the effective periodic rate of interest is i. The symbol $(Ia)_{\overline{n}|}^{(m)}$ is the present value of an annuity that pays $\frac{1}{m}$ at the end of each m^{th} part of the first period, $\frac{2}{m}$ at the end of each m^{th} part of the second period, . . . , $\frac{n}{m}$ at the end of each m^{th} part of the nth period, & so on. For $(Ia)_{\overline{n}|}^{(m)}$ the equation is:

$$(Ia)_{\overline{n}|}^{(m)} = \frac{\ddot{a}_{\overline{n}|}^{(m)} - nv^n}{i^{(m)}} \qquad (9.15)$$

The LHS of (9.15) is same as $(I^{(m)}a)_{\overline{n}|}^{(m)}$ and is the present value of the annuity. The annuity pays $1 - m^2$ at the end of the first m^{th} of the first period, $2 - m^2$ at the end of the second m^{th} of the first period,. . . , $\frac{nm}{m^2}$ at the end of last m^{th} of the first period. A computational equation for this is equation (9.15). So far, the value of an annuity has been computed at time 0. Another common time point at which the value of an annuity, consisting of n payments of 1, is computed is time n. Denoting by $s_{\overline{n}|}$ the value of an annuity-immediate at time *n* (i.e. payable immediately after the n^{th} payment), we have $s_{\overline{n}|} = (1 + i)^n_{a_{\overline{n}|}}$.

9.4.2 Amortization

The term 'amortization' indicates the paying off of a debt with a *fixed repayment schedule* in regular installments over a period of time. For instance, you are going to buy a house for which the purchase price is Rs. 100,000 and the down payment is Rs.20,000. You will get the Rs. 80,000 financed by borrowing this amount from a bank at 10% interest with a 30 year term. What is your monthly payment? Typically, such a loan is amortized i.e. you will make equal monthly payments for the life of the loan with each payment consisting partially of interest and partially of principal. From the banks point of view, this transaction represents the purchase by the bank of an annuity-immediate. The monthly payment, p, is thus the solution of the equation $80000 = pa_{\overline{360}|\frac{0.10}{12}}$. In this setting, the quoted interest rate on the loan is assumed to be compounded at the same frequency as the payment period. We find the monthly payment and the total amount of the payment made by $pa_{\overline{360}|\frac{0.10}{12}}$ = 113.95 from which we get $p = 702.06$. Thus, the total amount of the payments is $360p = 2{,}52{,}740.60$.

An 'amortization table' is a table which lists the principal and interest portions of each payment for a loan which is being amortized. Such a table can be constructed as follows. Let us denote by b_k the loan balance immediately after the k^{th} payment and by b_0 the original loan amount. Now, the interest part of the k^{th} payment is $p - ib_{k-1}$ and the principal amount of the k^{th} payment is $(P - ib_{k-1})$ where P is the periodic payment amount. Notice that $b_{k+1} = (1 + i)b_k - P$. These relations allow the rows of the amortization table to be constructed sequentially.

A single row of the amortization table that is desired can be arrived at without constructing the whole table. It is called as the 'prospective method' of calculating the loan balance. The loan balance at any point in time is the present value of the remaining loan payments. The 'prospective method' gives the loan balance immediately after the k^{th} payment as $b_k = Pa_{\overline{n-k}|}$. In the 'retrospective method' the loan balance at any point in time is the accumulated original loan amount minus the accumulated value of the past loan payments. The retrospective method gives the loan balance immediately after the k^{th} payment as $b_k = b_0(1 + i)^k - Ps_{\overline{k}|}$. Either method can be used to find the loan balance at an arbitrary time point. Note that for the retrospective method, $b_0(1 + i)^k - Ps_{\overline{k}|} = b_0 + (ib_0 - P)s_{\overline{k}|}$ is a simple identity of $v^{-n} = 1 + i\, s_{\overline{n}|}$. We may check that the prospective and retrospective methods give the same value as follows. Direct computation using the formula gives $a_{\overline{n-k|}} = v^{-k}\left(a_{\overline{n|}} - a_{\overline{k|}}\right)$ and $P = \frac{b_0}{a_{\overline{n|}}}$ gives $Pa_{\overline{n-k|}} = b_0 v^{-k}\frac{\left(a_{\overline{n|}} - a_{\overline{k|}}\right)}{a_{\overline{n|}}} = b_0(1+i)^k - Ps_{\overline{k|}}$. A further bit of insight is obtained by examining the case in which the loan amount is $a_{\overline{n|}}$ so that each loan payment is 1. In this case, the interest part of the k^{th} payment is $ia_{\overline{n-k+1|}} = 1 - v^{n-k+1}$ and the principal part of the k^{th} payment is v^{n-k+1}. This shows that the principal payments form a geometric series. We may hence note that the prospective and retrospective method can be applied to any series of loan payments.

Sinking Fund: A second way of paying off a loan is by means of a 'sinking fund'. Again, consider Rs. 80,000 is borrowed at 10% annual interest. But now, only the interest is required to be paid each month. The principal amount is to be repaid in full at the end of 30 years. Here, the borrower would accumulate a separate fund, called a sinking fund, which will accumulate to Rs. 80,000 in 30 years. The borrower may earn less than 10% interest (say 5%) compounded monthly. In this scenario, the monthly interest payment is $80000(\frac{0.10}{12}) = 666.67$. The contribution c each month into the sinking fund must satisfy $cs_{\overline{360|}\frac{0.05}{12}} = 80000$ from which we get $c = 96.12$. As expected, the combined payment is higher, since the interest rate earned on the sinking fund is lower than 10%.

9.4.3 Illustrations

Let us consider some empirical examples as before.

a) *Suppose a company issues a stock that pays a dividend at the end of each year of Rs. 10 indefinitely. The company's cost of capital is 6%. What is the value of the stock at the beginning of the year?*

   $10.\boldsymbol{a}_{\overline{\infty|}} = 10.\frac{\mathbf{1}}{\mathbf{0.06}}$ = Rs.166.67.

b) *What would you be willing to pay for an infinite stream of Rs. 37 annual payments (cash inflows) beginning now if the interest rate is 8% per annum?*

   $37\boldsymbol{\ddot{a}}_{\overline{\infty|}} = \frac{\mathbf{37}}{\mathbf{0.08(1.08)^{-1}}}$ = Rs.499.50.

c) *The present value of a perpetuity paying 1 at the end of every 3 years is* $\frac{\mathbf{125}}{\mathbf{91}}$. *Find i.*

$\frac{\mathbf{125}}{\mathbf{91}} = \frac{\mathbf{1}}{\boldsymbol{is_{\overline{3|}}}} = \frac{\mathbf{1}}{\mathbf{(1+\boldsymbol{i})^3-1}}$. So, $(\mathbf{1}+\boldsymbol{i})^3 = \frac{\mathbf{91}}{\mathbf{125}} + \mathbf{1} = \frac{\mathbf{216}}{\mathbf{125}}$ Hence, $\boldsymbol{i} = \mathbf{0.20}$.

d) *You are receiving an annuity with payments made continuously at a rate of 1000 per year. The annuity is for 10 years. Calculate the present value of this annuity at an annual effective interest rate of 6%.*

This is a level continuous annuity with $\boldsymbol{i = 6\%}$ *and* $\boldsymbol{\delta = ln\,(1.06)}$. The present value is $\mathbf{1000}\left(\frac{\mathbf{1-v^{10}}}{\boldsymbol{\delta}}\right) = \mathbf{7578.75}$.

e) *A loan of 1000 is being repaid by equal annual installments of 100 together with a smaller final payment at the end of 10 years. If the interest rate is 4%, show that the balance immediately after the fifth payment is* $\mathbf{1000 - 60}\boldsymbol{s_{\overline{5|}}}$.

The retrospective method gives the balance as $1000\boldsymbol{v^{-5}} - 100\boldsymbol{s_{\overline{5|}.04}}$, which re-arranges to the stated quantity using the identity $\boldsymbol{v^{-n} = 1 + is_{\overline{n|}}}$.

f) *A loan of 1200 is to be repaid over 20 years. The borrower is to make annual payments of 100 at the end of each year. The lender receives 5% on the loan for the first 10 years and 6% on the loan balance for the remaining years. After accounting for the interest to be paid, the remainder of the payment of 100 is deposited in a sinking fund earning 3%. What is the loan balance still due at the end of 20 years?*

The amount in the sinking fund at the end of 20 years is $\mathbf{40}\boldsymbol{s_{\overline{10|}}}\mathbf{(1.03)^{10}} + \mathbf{28}\boldsymbol{s_{\overline{10|}}} = \mathbf{937.25.}$ Hence, the loan balance is $\mathbf{1200 - 937.25 = 262.75}$.

Check Your Progress 3 [answer within the space given in about 50-100 words]

1) State the expression for the computation of 'present value' in a continuously paid annuity.

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

2) State what do the symbols $(Is)_{\overline{n|}}$ and $(I\ddot{s})_{\overline{n|}}$ represent?

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

……………………………………………………………………………

3) What is an amortization Table?

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

4) What is meant by a 'sinking fund'?

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

5) Differentiate between 'prospective and retrospective methods' of loan balance.

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

9.5 LET US SUM UP

In this unit, we have discussed the concept of an 'annuity'. Annuity is a financial product that pays out a fixed stream of payments to an individual. There are different types of annuities based on the basis of payments. Annuity-immediate is paid at the end of payment periods whereas annuity-due is paid at the beginning of payment periods. These are the two broad types of annuities. Deferred annuities are those whose payments start at some future time. For all these types of annuities, the computation of their present and accumulated values are discussed in the unit. We have also discussed the 'increasing and decreasing annuities' by varying interest rates. Perpetuities, which are a constant stream of identical cash flows with no end, and continuous annuities (where the annuitant receives payment at a constant rate per unit time), are also discussed in the unit. The concept of amortization, which refers to a plan of paying off a debt with a fixed repayment schedule, is explained. The use of a sinking fund, where an amount deposited periodically accumulates to help pay off the principal amount, is outlined.

Two methods of balancing a loan account viz. the prospective and retrospective methods are explained.

9.6 KEY WORDS

Annuity	:	A financial product that pays out a fixed stream of payments to an individual.
Annuity-Certain	:	Payments guaranteed to occur for a fixed period of time.
Annuity-Due	:	Payments made at the beginning of payment periods.
Annuity-Immediate	:	Payments made at the end of payment periods, so that the interest accrues between the issue of the annuity and the first payment.
Contingent Annuities	:	This is an arrangement in which the beneficiary does not begin receiving payments until a specified event occurs.
Continuous Perpetuity	:	This is an annuity where the Annuitant receives payment at a constant rate of 1 per unit time.
Deferred Annuity	:	An annuity in which the payments start at some future time.
Level	:	A plan paying interest and principal in such a way that the total is same for each payment.
Perpetuity	:	Payment of a constant stream of identical cash flows with no end.
Prospective Method	:	It calculates the loan balance as the present value of all future payments to be made.
Retrospective Method	:	Calculates the loan balance as the accumulated value of the loan at the time of evaluation minus the accumulated value of all instalments paid up to the time of evaluation.
Sinking Fund	:	An amount deposited periodically so as to accumulate to the principal over the duration of the loan period.

9.7 SUGGESTED BOOKS FOR FURTHER READING

1) Marcel B Finan (2017). A Basic Course in the Theory of Interest and Derivatives Markets (internet).

2) Jerry Alan Veeh (2006). Lecture Notes on the Mathematics of Finance.

9.8 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) Annuity is a series of periodic payments at equal intervals of time. They can be payments received or made. Examples of annuities are house rent, mortgage payments, instalments.

2) In the former, payments are guaranteed in the sense of a 'legal obligation' to pay. In the latter, it begins after the occurrence of an event (e.g. attainment of 60 years, retirement from service).

3) $a_{\overline{n}|}$ denotes the 'present value' of an 'annuity-immediate' at time t = 0. $s_{\overline{n}|}$ denotes the 'accumulated value' of an 'annuity-immediate' after n-payments are made. They are related in terms of the expressions: $\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$ and $s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}$.

4) $a_{\overline{n}|} = \frac{1-(1+i)^{-n}}{i}$ where i is the r.o.i. paid.

5) In the former, payments are made at the beginning of the year. In the latter, it is made at the end of the year.

6) They represent the 'present value' at time t = 0 and the 'accumulated value' after n-payments are made of an 'annuity-due' respectively. Note that having 'double dot' is only for distinction and has no special significance.

7) $\ddot{a}_{\overline{n}|} = \frac{1-(1-d)^n}{d}$ where $d = \frac{i}{i+1}$.

8) $\ddot{s}_{\overline{n}|} = \frac{(1+i)^n-1}{d} = (1+i)^n \ddot{a}_{\overline{n}|}$.

9) They are given by Equations (9.8a) to (9.8d).

10) A 'deferred annuity' is an annuity in which the payments start at some future time.

Check Your Progress 2

1) $(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$ and $(Da)_{\overline{n}|} = \frac{(n - a_{\overline{n}|})}{i}$.

2) $a_{\overline{n}|} = (1+i_1)^{-1} + (1+i_1)^{-1}(1+i_2)^{-1} + \cdots + (1+i_1)^{-1}(1+i_2)^{-1} \ldots (1+i_n)^{-1}$ and $a_{\overline{n}|} = (1+i_1)^{-1}(1+i_2)^{-1} + \cdots + (1+i_n)^{-n}$.

3) $PV = P\ddot{a}_{\overline{n}|} + Q\frac{[a_{\overline{n}|} - nv^n]}{d}$ and $AV = (1+i)^n PV = P\ddot{s}_{\overline{n}|} + Q\frac{[s_{\overline{n}|} - n]}{d}$.

Check Your Progress 3

1) Paid at a constant rate of 1 rupee per unit time, the present value is $\int_0^n e^{-\delta t} dt = \frac{1-v^n}{\partial}$.

2) The symbol $(Is)_{\overline{n|}}$ is the value of an increasing annuity immediate computed at time n and $(I\ddot{s})_{\overline{n|}}$ is the value of an increasing annuity due at time n.

3) It is a Table which lists the principal and interest portions of each payment for a loan which is being amortized.

4) It is a fund into which periodic payments are made which, with compound interest earned, will ultimately be sufficient to meet a known future capital commitment or discharge a liability.

5) Retrospective Method for loan balancing is based on payments already made. Contrary to such a method, the prospective method is based on 'looking into the future' i.e. evaluating the value of remaining payments.

ignou
THE PEOPLE'S
UNIVERSITY

ignou
THE PEOPLE'S
UNIVERSITY