BLOCK 3
INTEREST THEORY

INTRODUCTION TO BLOCK 3

Block 3 is on Interest Theory. It has three units (units 7, 8 and 9). **Unit 7** is on Basics of Interest Theory. It first outlines the concept of Accumulation function. Under this, it covers two aspects viz. (i) accumulation factor and interest payment and (ii) nominal interest rate and effective interest rate. The unit then proceeds to provide a distinction between 'linear accumulation functions' and 'exponential accumulation functions'. Concepts of simple interest (SI), compound interest (CI) and relationship between SI and CI are explained. Many illustrations to help understand the computational needs are given.

Unit 8 is on Equations of Value and Time. Beginning with a distinction on 'present value' and 'discount factor', the unit explains the concept of 'effective rate of discount'. The concept of 'force of interest', by way of 'continuous compounding', is explained. Methods of solving for interest rate like: (i) direct method, (ii) analytical method, (iii) linear interpolation and (iv) successive iterations method are explained.

Unit 9 is on Annuities. This unit begins by giving an account on the types of annuities viz. (i) annuity-immediate, (ii) annuity-due and (iii) deferred annuity. The difference between 'increasing and decreasing annuities' is explained by the application of methods like (i) varying interest annuities and (ii) varying annuity-due. The concepts of (i) perpetuity, (ii) continuing annuity and (iii) amortisation are explained towards the end of the unit.

UNIT 7 BASICS OF INTEREST THEORY

Structure

7.0 OBJECTIVES

After reading this unit, you will be able to:

- define the term 'interest' with illustrations;
- outline the properties of an 'accumulation function' with illustrations;
- differentiate between 'accumulating factor' and 'interest earned during a period' with expressions specified for both;
- distinguish between 'nominal interest rate' and 'effective interest rate' with expressions stated for both;
- show that the 'linear accumulation function' is valid for all real numbers;
- explain the different types of 'simple interest' with illustrations;
- calculate the interest rate in different contexts; and
- bring out the relationship between simple interest and compound interest.

7.1 INTRODUCTION

Interest is an amount charged to a borrower for the use of the lender's money over a period of time. For instance, if you borrowed Rs.100, promising to pay back Rs.105 after one year, then the lender is making a profit of Rs. 5. Rs. 5

is the fee or interest for borrowing the money. Calculation of interest therefore involves four quantities: (i) principal amount, (ii) investment period, (iii) interest rate and (iv) amount value (principal + interest). The money invested in transactions will be referred to as the principal and is denoted by P. The amount to which it grows up to with the interest is called the 'amount value' and will be denoted by A. The difference I = A – P is the amount of interest earned during the period of investment. Interest expressed as a percent of the principal is referred to as the 'interest rate'. Interest takes into account the risk of default i.e. risk that the borrower can not pay back the loan. Such a risk can be reduced if the borrowers keep asset as collateral to be used in the event of their default. The 'unit' in which the time of investment is measured is called 'measurement period'. In general, if $A(t)$ denotes the amount value of an investment at time t years, the 'amount of interest earned' from time t to time '$t + s$' is given by '$\boldsymbol{A(t+s)-A(t)}$'. The interest for the period t is then calculated as $\frac{A(t+s)-A(t)}{A(t)}$. This value multiplied by 100 is the 'interest rate'. For instance, suppose you deposit an amount Rs. 1000 to a savings account and one year later the account has accumulated to Rs. 1050. Now the principal is Rs. 1000 and the interest earned is 1050 – 1000 = 50. Hence, the annual interest rate is '50/1000*100' = 5%. Interest rates are usually computed on an annual basis. But they can also be determined for non-annual time periods. For instance, a bank that offers you for your deposits an annual interest rate of 10%, 'compounds it' semi-annually. Interest rate for up to 1 year is called the 'nominal interest rate'. When calculated for sub-periods like 'months and days' it is termed as the 'effective interest rate'. There is a relationship between the 'nominal interest rate' and the 'effective interest rate'. Using this, knowing one, the other can be calculated. We will study about this relationship soon.

7.2 ACCUMULATION FUNCTION

'Accumulation value' is the total amount of an investment over a time period. It includes the capital invested and the interest earned. The 'amount function' is written as $A(t)$. It denotes the value of an investment at time t. Note that in situations of this kind, we can refer $A(0)$ as the principal amount P. In order to compare various amount functions, it is convenient to define a function $a(t) = [A(t))/(A(0)]$. It is termed as the 'accumulation function'. It represents the accumulated value of a principal invested at time t. Accumulation function possesses the following properties: (i) $a(0) = 1$, (ii) $a(t)$ is increasing [i.e. if $t_1 < t_2$ then $a(t_1) \leq a(t_2)$] and (iii) if interest accrues for non-integer values of t (i.e. for any fractional part of a year), then $a(t)$ is a continuous function. If interest does not accrue between interest payment dates, then $a(t)$ possesses discontinuities i.e. the function $a(t)$ stays constant for a period of time, but will take a jump whenever the interest is added to the account, usually at the end of the period. Figure 7.1 graphically illustrates some real life situations of accumulation functions.

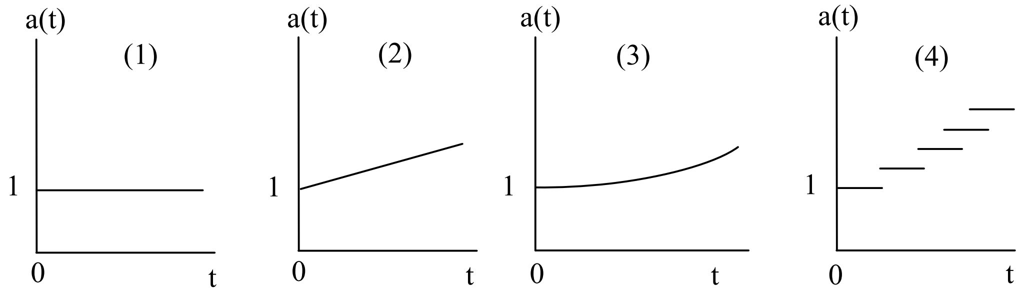

Fig. 7.1: Accumulation Functions

The first figure therein represents an investment that is not earning any interest. The second one shows one which is linear in its growth. We will later see that this fits the case where 'simple interest' is paid. In the third figure, it is an exponential function. This represents the case of 'compound interest growth' where the investment earns 'interest on interest'. The fourth figure is a step function given in horizontal line segments of the length of the period. It will rise whenever interest is paid at fixed periods of time. If the amount of interest paid is constant per time period, the steps will all be of the same height. If the amount of interest increases with time, then the steps will get larger and larger as time increases.

7.2.1 Accumulation Factor and Interest Payment

'Accumulation factor' measures the rate at which the principal grows. Let P = the principal, I = interest, S = the accumulated value of P, r = annual rate of interest, t = time in years. Now, S = P + I and interest I = Prt. Therefore, $\mathrm{S = P + Prt}$ or $\mathrm{S = P(1 + rt)}$. The factor $\mathrm{(1 + rt)}$ is called the 'accumulation factor'. Here, for the moment, note that r is the 'simple interest rate'. The process of calculating s from p is called 'accumulation given the interest rate r'. In general, if an amount $\boldsymbol{k}$ is deposited at time s, then the accumulated value k at time t (>s) is: $k \times \frac{a(t)}{a(s)}$. Here $\frac{a(t)}{a(s)}$ is called the accumulation factor or the growth factor. In other words, the accumulation factor $\frac{a(t)}{a(s)}$ gives the value at time t of Rs. 1 deposited at time s.

Now, let $\boldsymbol{n}$ be a positive integer. The n^{th} period of time is defined to be the period of time between $t = n - 1$ and $t = n$. In other words, this period refers to the time interval '$n - 1 \le t \le n$'. We define the interest earned during the n^{th} period of time as: $I_n = A(n) - A(n - 1)$. This tells us that, the interest earned during a period of time is the 'difference between the amount at the end of the period and at the beginning of the period'. Note that I_n shows the amount of interest 'over an interval of time', whereas $A(n)$ is an amount at a 'specific point in time'. In general, the amount of interest earned on an original investment of $Rs. k$ between time s and t is given by: $I_{[s,t]} = A(t) - A(s) = k[a(t) - a(s)]$. In particular, if the amount function is given by: $A(t) = t^2 + 2t + 1$, then I_n is: $I_n = A(n) - A(n-1) = n^2 + 2n + 1 - (n-1)^2 - 2(n-1) - 1 = 2n + 1$. Let us alternatively write the suffix 'n' in the 'power' i.e. I_n as I^n. Now, note that:

$A(1) - A(0) = I^1$, $A(2) - A(1) = I^2$, ………, $A(n) - A(n-1) = I^n$. Hence:

$$A(n) - A(0) = [A(1) - A(0)] + [A(2) - A(1)] + \cdots \ldots + [A(n) - A(n-1)]$$

$$= I^1 + I^2 + \cdots \ldots \ldots \ldots \ldots \ldots . + I^n.$$

$$\therefore\ A(n) = A(0) + (I^1 + I^2 + \_\_\_ + I^n)$$

Here, $I^1 + I^2 + \_\_\_ + I^n$ is the interest earned on the capital $A(0)$ over the period n. It is the sum of the interest earned in each of the periods separately. Note that for any $0 \leq t < n$, we have $A(n) - A(t) = [A(n) - A(0)] - [A(t) - A(0)] = \sum_{j=0}^{n} I_j - \sum_{j=1}^{t} I_j = \sum_{j=t+1}^{n} I_j$. This means, the interest earned between time t and time n is the 'total interest earned from time 0 to time n' minus the 'total interest earned from time 0 to time t'.

7.2.2 Nominal Interest Rate and Effective Interest Rate

An interest rate is only meaningful in the context of time. Time, in general, is understood as '*per year*' which may be called *the nominal interest rate*. This nomenclature helps us to distinguish the interest earned during other sub-periods of time (like month, week or day), as the '*effective interest rate*'. Nominal interest rate for a period, with effective interest rates in its sub-periods, can be calculated as:

$$i = (1 + i_e)^n - 1 \quad (7.1)$$

where i = nominal interest rate for the period, i_e = effective interest rate for the sub-period and n = number of sub-periods. In particular, the nominal interest rate per year' with a monthly effective rate of 1% (i.e. 1/100 = 0.01) can be calculated as**:** $i_n = (1 + 0.01)^{12} - 1 = 1.01^{12} - 1 = 1.127 - 1 = 0.127 = 12.7\%$. Note that the nominal interest rate for the year is not '12 × 1' (= 12%) but it is 12.7%. Likewise, we can calculate the 'effective interest rate' for sub-periods of a period as:

$$i_e = (i_n + 1)^{1/n} - 1 \quad (7.2)$$

For instance, we can calculate the 'effective interest rate per month' with a nominal interest rate of 10% (i.e. 10/100 = 1/10 = 0.1) as: $i_e = (0.1 + 1)^{1/12} - 1 = (1.1)^{1/12} - 1 = 1.0079 - 1 = 0.0079 = 0.79\%$. Thus, the 'effective rate of interest' i is the amount of money that one unit invested at the beginning of a period will earn during the period assuming that the 'interest is paid at the end of the period'. This definition is equivalent to $i = a(1) - a(0)$. Writing i in terms of the amount function, we get:

$$i = a(1) - a(0) = \frac{a(1) - a(0)}{a(0)} = \frac{A(1) - A(0)}{A(0)} = \frac{I_1}{A(0)}.$$

Thus, the effective rate of interest for a period is the 'amount of interest earned in one period divided by the principal at the beginning of the period'. In its more generalised form the 'effective rate of interest' in the n^{th} period is defined as:

$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{I_n}{A(n-1)}$ where $I_n = A(n) - A(n-1)$. Note that I_n represents the 'amount of growth' of the function $A(t)$ in the n^{th} period whereas i_n is the 'rate of growth' on the amount over the period n. Thus, the effective rate of interest i_n is the ratio of the amount of interest earned during the period to the amount of principal invested at the beginning of the period. Note that $i_1 = i = a(1) - 1$ and for any accumulation function, it holds that $a(1) = 1 + i$. Note also that i_n and $A(n)$ are related as:

$$A\,(n) = A(n-1) + i_n A(n-1) = (1 + i_n)A(n-1).$$

Hence, the fund at the end of the n^{th} period is equal to the fund at the beginning of the period $(n - 1)$ plus the interest earned over the amount at period $(n - 1)$. We can also therefore write this as:

$$A(n) = (1 + i_1)(1 + i_2)\_\_\_(1 + i_n)A(0).$$

Note that i_n can be expressed in terms of $a(t)$ as:

$$\boldsymbol{i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{A(0)a(n) - A(0)a(n-1)}{A(0)a(n-1)} = \frac{a(n) - a(n-1)}{a(n-1)}} \qquad (7.3)$$

7.2.3 Illustrations

Let us consider some empirical illustrations. It will help us to relate the theoretical description given above in practical terms.

(a) *Suppose that $A(t) = t^2\alpha + 10\beta$. If an amount X invested at time 0 accumulates to Rs. 500 at time 4, and to Rs. 1,000 at time 10, what is the amount of the original investment X?*

We have $A(0) = X = 10\beta$; $A(4) = 500 = 16\alpha + 10\beta$; and $A(10) = 1000 = 100\alpha + 10\beta$. Using the first equation in the second and third we obtain the following system of linear equations:

$$16\alpha + X = 500$$

$$100\alpha + X = 1000.$$

Solving for α we get $84\alpha = 500$ or $\alpha = 125/21$. Substituting for β we get: β = 1/10[500 – 16 * 125/21] = 1/210[10500 – 2000] = 8500/210 = 850/21.

$\therefore X = 10 \times \frac{850}{21} = \frac{8500}{21} = 404.76.$

(b) (*i*) *Given that a(t) is like $a(t) = b(1.1)^t + ct^2$, if Rs.100 invested at time t = 0 accumulates to Rs.170 at time t = 3, estimate the accumulated value at time t = 12 of Rs.100 invested at time t = 1. (ii) Show that $a(t)$ is increasing.*

(i) we must have $a(0) = 1$. Thus, $b(1.1)^0 + c(0)^2 = 1$. This implies that $b = 1$. We also have $A(3) = 170 = 100a(3)$. Therefore, $170 = 100[(1.1)^3 + c.3^2]$

Solving for c we get $c = 0.041$. Hence, $a(t) = A(t)/A(0) = (1.1)^t + 0.041t^2$. Hence, $a(1) = 1.141$ and $a(12) = 9.04$.

Now, $100\, a(t)/a(1)$ is the accumulated value of Rs. 100 investment from time $t = 1$ to $t > 1$. Hence,

$$\frac{100a(12)}{a(1)} = 100 \times \frac{9.04}{1.141} = 100(7.925) = 792.50$$

So Rs.100 at time $t = 1$ grows to Rs. 792.50 at time $t = 12$.

(ii) Since $a(t) = (1.1)^t + 0.041t^2$, we have $a'(t) = (1.1)^t \ln 1.1 + 0.082t > 0$ for $t \geq 0$. This shows that $a(t)$ is increasing for $t \geq 0$.

(c) *Assume that* $A(t) = 100(1.1)^t$. *Calculate* i_5.

We have $i_5 = \frac{A(5) - A(4)}{A(4}) = \frac{100(1.1)^5 - 100(1.1)^4}{100(1.1)^4} = 0.1$.

(d) *If* $A(4) = 1000$ and $i_n = 0.01n$, what is $A(7)$?

We have $A(7) = (1 + i_7)A(6) = (1 + i_7)(1 + i_6)A(5)$

$$= (1 + i_7)(1 + i_6)(1 + i_5)\, A(4)$$

$$= (1.07)(1.06)(1.05)(1000) = 1,190.91$$

(e) *Show that if* $a(n) = 1 + in$; $n > 1$, i_n *is a decreasing function of* n.

We have: $i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{[1 + in - (1 + i(n-1))]}{1 + i(n-1)} = \frac{i}{1 + i(n-1)}$.

Since $i_{n+1} - i_n = \frac{i}{1 + in} - \frac{i}{1 + i(n-1)} = \frac{i^2}{(1 + in)(1 + i(n-1))} < 0$, as n increases i_n decreases. Hence, i_n is a decreasing function of *n*.

Check Your Progress 1 [answer within the space given in about 50-100 words]

1) Define the term 'interest' with the usual terms used.

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2) What is an 'accumulation function'? What are its properties?

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3) What is an 'accumulation factor'? What does it represent?

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4) Distinguish between the 'nominal interest rate' and the 'effective interest rate'. How is the relationship between the two expressed? What is its use?

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7.3 LINEAR ACCUMULATION FUNCTIONS

Linear accumulation functions are applicable in case of simple interest. Suppose an investment of 1 earns a 'constant interest' in each period equal to i. Then, at the end of the first period, the accumulated value is: $a(1) = 1 + i$, at the end of the second period it is $a(2) = 1 + 2i$ and at the end of the n^{th} period it is $a(n) = 1 + in, n \geq 0$.

Thus, the accumulation function is a linear function and accrual of interest according to this function $a(n) = 1 + \text{in}, n \geq 0$ is called 'simple interest'.

7.3.1 Simple Interest

In case of a simple interest rate i, the effective interest rate i_n is decreasing. To see how this is, consider:

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{[1 + in - (1 + i(n-1))]}{1 + i(n-1)} = \frac{i}{1 + i(n-1)},$$

$$n \geq 1 \qquad (7.4)$$

Now, $\boldsymbol{i_{n+1}} - \boldsymbol{i_n} = \frac{i}{1+in} - \frac{i}{1+i(n-1)} = -\frac{i^2}{(1+in)(1+i(n-1))} < 0.$ (7.4a)

Thus, even though the rate of simple interest is constant over each period of time, the effective rate of interest is decreasing from one period to the next. In fact, it converges to '0' in the long run. Note that the simple interest is earned in the absolute amount in each time interval i.e. $I_n = a(n) - a(n-1)$. Such an amount remains constant. In contrast, i_n is a decreasing function of n, as 'n' increases.

The accumulation function for simple interest has been defined for integral values of $n \geq 0$. In order for this function to have the graph shown in Figure 7.1 (2), we need to extend $a(n)$ for non integral values of n. It is like crediting interest proportionally over any fraction of a period. When interest is accrued only for completed periods with no credit for fractional periods, the accumulation function becomes a step function as illustrated in Figure 7.1 (4). In order to define $a(t)$ for real numbers $t \geq 0$ we need to redefine the rate of simple interest. This should be done in such a way that the previous definition becomes a particular result of this generalisation. This can be done as follows.

The simple interest earned by an initial investment of Rs. 1 in all time periods of length $t + s$ is equal to the sum of the interest earned for periods of lengths t and s. This means:

$$a(t+s) - a(0) = [a(t) - a(0)] + [a(s) - a(0)].$$

$$\text{Or}, a(t+s) = a(t) + a(s) - a(0) \qquad (7.5)$$

for all non-negative real numbers t and s. Note that the definition assumes the rule to hold for periods of any non-negative length, including that of integer length. To verify this, we need to assume that *a(t)* is a differentiable function. Then:

$$a'(t) = \lim_{s\to 0}\frac{a(t+s) - a(t)}{s} = \lim_{s\to 0}\frac{a(t) + a(s) - a(0) - a(t)}{s} = \lim_{s\to 0}\frac{a(s) - a(0)}{s} = a'(0)$$

……………… ….(7.5a)

is a constant. Hence, for the time derivative of $a(t)$ to be a constant, $a(t)$ must be $a(t) = a'(0)t + C$ where C is a constant. We can determine C, by assigning to t the particular value 0, so that $C = a(0) = 1$. Thus, $a(t) = 1 + a'(0)t$. Letting $t = 1$ and defining $i_1 = i = a(1) - a(0)$, we can write $a(t) = 1 + it, t \geq 0$. As a result, simple interest accumulation function preserves the property in Equation (7.5). Note that the above derivation being a non-negative integer does not depend on t and is valid for all non-negative real numbers t.

Simple interest is useful for approximating compound interest for a short time period such as a fraction of a year. In particular, if the accumulation function for compound interest i is given by the formula $a(t) = (1+i)^t$, using the binomial theorem, we can write the expansion series of $a(t)$ to obtain:

$(1 + i)^t = 1 + it + \frac{t(t-1)}{2!}i^2 + \frac{t(t-1)(t-2)}{3!}i^3 \cdots$. Thus, for $0 < t < 1$ we can write the approximation:

$$(1 + i)^t \approx 1 + it. \tag{7.6}$$

7.3.2 Types of Simple Interest

In simple interest calculations, the length of the investment need not be in integral number of years. It can be even in terms of days or some years and some days. There are three techniques for counting the number of days in a period of investment for simple interest (SI). In all these three methods, time is taken as:

$$time = \frac{\text{No.of days between two dates}}{\text{No.of days in a year}} \tag{7.7}$$

In counting the time or days as above, interest is credited for either of the starting or ending date, but not for both these dates. Usually, it is credited for the terminal date and not the first date. Three types of 'simple interest' that is usually calculated is outlined as follows.

Exact Simple Interest: If the ratio of (7.7) is taken as 'actual/actual', it means the exact number of days for the period of investment divided by 365 days in a non-leap year and 366 for a leap year is taken into account. For this method, it is important to know the number of days in each month. In counting days between two dates, only the last date is included.

Ordinary Simple Interest: In this method, the ratio in (7.7) is taken as: '30/360'. The 30/360 day counting scheme was invented in the days before the computers. Here, the premise is that for the purposes of computation, all months have 30 days and therefore the total number of days over 12 months is $12 \times 30 = 360$ days. When the number of days to be calculated spreads over years and months and days within a month, then the following formula is applied:

$$N = 360(Y_2 - Y_1) + 30(M_2 - M_1) + (D_2 - D_1). \tag{7.8}$$

Banker's Rule: In this method, the ratio of (7.7) is taken as 'actual/360'. This method uses the 'exact number of days for the period of investment' (i.e. the numerator of 7.7) but considers the calendar year as having 360 days for its denominator. The number of days between two dates is found in the same way as for 'exact simple interest'. In this method also we count the last day and not the first day.

7.3.3 Illustrations

Once again, let us consider some empirical illustrations on calculating the interest rate by the different type of simple interest calculation methods.

a) *A fund is earning 5% simple interest. Calculate the effective interest rate in the 6th year.*

The effective interest rate in the 6th year is i_6 which is given by:

$i_6 = \frac{i}{1 + i(n-1)} = \frac{0.05}{1 + 0.05(5)} = 4\%.$

b) *You invest Rs.100 at time 0, at an annual simple interest rate of 10%. Compute the accumulated value after 6 months.*

The accumulated function for simple interest is a continuous function. We take 6 months as: 6/12 = 0.5. Thus:

$A(0.5) = 100[1 + 0.1(0.5)] = \text{Rs.} 105.$

c) *Rs.10,000 is invested for four months at 12.6% compounded annually as $A(t) = 10000\,(1 + 0.126)^t$, where interest is computed using a quadratic equation to approximate an exact calculation. Compute the accumulated value.*

Since the money is invested for 4 months, it is 4/12 = 1/3 years. We need to estimate $A\left(\frac{1}{3}\right) = 10000(1 + 0.126)^{\frac{1}{3}}$. Using the first three terms of the series expansion of $(1 + i)^t$, we have $A\left(\frac{1}{3}\right) = 10000(1.126)^{\frac{1}{3}} \approx 10000\left(1 + \frac{1}{3} \times 0.126 + \frac{\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)}{2!}(0.126)^2\right) = \text{Rs.} 10,402.36.$

d) *Suppose that Rs. 2,500 is deposited on March 8 and withdrawn on October 3 of the same year. The interest rate is 5%. What is the amount of interest earned, if it is computed using 'exact simple interest'. Assume non leap year.*

From March 8 (not included) to October 3 (included) there are 23 + 30 + 31 + 30 + 31 + 31 + 30 + 3 = 209 days. Thus, the amount of interest earned using exact simple interest is $2500(0.05) \cdot \frac{209}{365} = \text{Rs.} 71.58$.

e) *Jack borrows 1,000 from the bank on January 28, 1996 at a rate of 15% simple interest per year. Calculate how much does he owe on March 5, 1996 using 'ordinary simple interest'.*

The amount owed at time t is $A(t) = 1000(1 + 0.15t)$. Using ordinary simple interest with $Y_1 = Y_2 = 1996$, $M_1 = 1$; $M_2 = 3$; and $D_1 = 4$, $D_2 = 28$ and $D_3 = 5$ we get $t = \frac{37}{360}$ and the amount owed on March 5, 1996 is:

$$1,000\left(1 + 0.15 \times \frac{37}{360}\right) = \text{Rs.} 1,015.42$$

f) *Jack borrows 1,000 from the bank on January 1, 1996 at a rate of 15% simple interest per year. Calculate how much does he owe the bank on January 17, 1996 using the 'Banker's rule'?*

Jack owes $1000\left(1 + 0.15 \times \frac{16}{360}\right) = \text{Rs. } 1{,}006.67.$

g) *An investment was made on the date the United States entered World War II, i.e. December 7, 1941, and was terminated at the end of the war on August 8, 1945. For how many days was the money invested under the ordinary simple interest method?*

Applying the formula in (7.8) we get the number of days as:

$360\,(1945 - 1941) + 30(8 - 12) + (8 - 7) = 1321.$

7.4 EXPONENTIAL ACCUMLATION FUNCTIONS

Simple interest has the property that the interest earned is not invested to earn additional interest. In contrast, compound interest has the property that the interest earned at the end of one period is automatically invested in the next period to earn additional interest. For this reason, its accumulation function bears the characteristic of being exponential.

7.4.1 Accumulation Function for Compound Interest

Let us calculate the accumulation function for compound interest i.e. start with an investment of 1 and with a compound interest rate i per period. At the end of the first period, the accumulated value is $1 + i$. At the end of the second period, the accumulated value is $(1 + i) + i(1 + i) = (1 + i)^2$. Continuing this way, we find that the accumulated value after t periods is given by the exponential function:

$$a(t) = (1 + i)^t, \text{ for integral } t \geq 0. \qquad (7.9)$$

Interest accruing according to this function is called 'compound interest'. We call i the 'rate of compound interest'. From the above accumulation function, we can write

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} = 1 + i - 1 = i \qquad (7.10)$$

Thus, the 'effective rate of interest' for compound interest is constant.

7.4.2 Relationship Between Simple Interest and Compound Interest

The accumulation function for compound interest is defined for non-negative integers. In order to extend the domain to fractional periods, we note that the compound interest accumulation function $a(t) = (1 + i)^t$ satisfies the property: $(1 + i)^{t+s} = (1 + i)^t \cdot (1 + i)^s$. Thus, under compound interest we have the accumulation function satisfying the property:

$$a(t + s) = a(t) \cdot a(s), t, s \geq 0. \qquad (7.11)$$

Equation (7.11) gives us a property of 'compound interest' (CI). It says that, for an initial investment of 1, invested over '$t + s$' period, there is no loss of interest if the investment is terminated at the end of 't' period. This is because, Equation (7.11) tells us that the amount of interest earned at time 't' is immediately invested for an additional 's' period. In other words, (7.11) says that, the amount of interest earned by an initial investment of 1 over '$t + s$' periods is equal to 'the amount of interest earned at the end of 't' periods plus interest earned on new principal over the period 's'. Hence, termination of investment at end of 't' does not cause any loss of interest. Because of this property of compound interest, we see that the graphical distinction between simple and compound interest is as in Figure 7.2. That is, the graph of an accumulation function under 'simple interest' is a straight line (a linear function), whereas, the graph of an accumulation function under 'compound interest' is an exponential function. In spite of this difference, which apparently seems to be diverging, there is a close relationship between SI and CI.

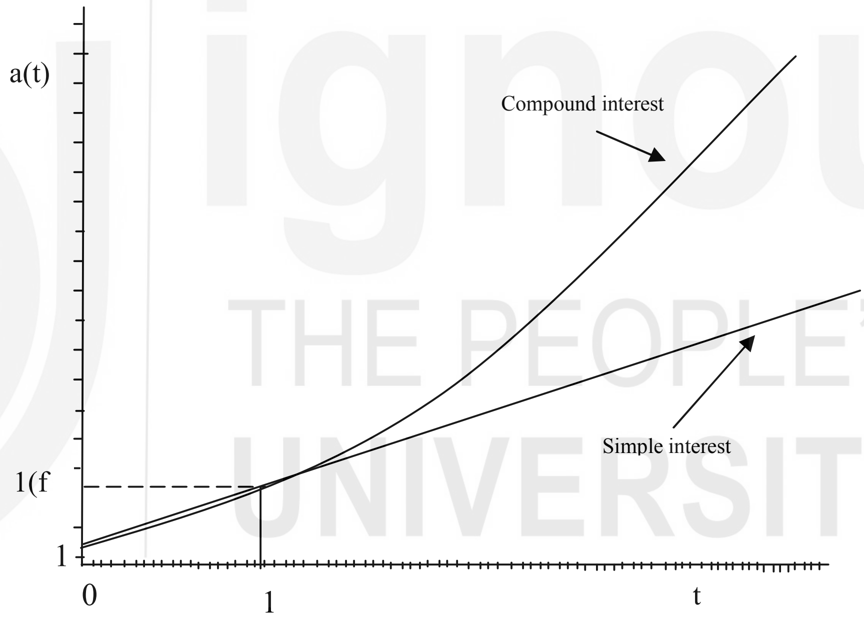

Fig. 7.2: Comparison of Simple and Compound Interest Accumulation

The relationship follows from the following theorem or result. It says that for all 'i' between 0 and 1 i.e. $0 < i < 1$, we have the following three relationships holding true depending on the time period for deposit 't'.

a) $(1 + i)^t < 1 + it$ for $0 < t < 1$;

b) $(1 + i)^t = 1 + it$ for $t = 0$ or $t = 1$;

c) $(1 + i)^t > 1 + it$ for $t > 1$.

Thus, simple and compound interest produce the same result over one measurement period i.e. $t = 0$ or 1. Compound interest produces a larger return than simple interest for periods greater than 1 and smaller return for periods smaller than 1. It can be further seen that:

a) with simple interest, the 'absolute amount of growth' is constant i.e. for a fixed 's' the difference $a(t + s) - a(t) = a(s) - 1$ does not depend on t;

b) with compound interest, the 'relative rate of growth' is constant i.e. for a fixed 's'the ratio $\frac{[a(t+s)-a(t)]}{a(t)} = a(s) - 1$ does not depend on t.

7.4.3 Illustrations

Let us now consider some empirical illustrations as before.

a) *It is known that Rs. 600 invested for two years will earn Rs. 264 in interest. What would be the accumulated value of Rs. 2,000 invested at the same rate of compound interest for three years?*

We have $600(1 + i)^2 = 600 + 264 = 864$. Thus, $(1 + i)^2 = 1.44$ and $i = 0.2$. Thus, the accumulated value of investing Rs. 2,000 for three years (at the rate $i = 20\%$) is $2,000(1 + 0.2)^3 = Rs. 3,456$.

b) *At an annual compound interest rate of 5%, how long will it take you to triple your money? Provide an answer in years, to one decimal place.*

We need to solve the equation $(1 + 0.05)^t = 3$. Thus, $t = \frac{ln\,3}{ln\,1.05} \approx 22.5$.

[Note: t log 1.05 = log 3; t = log 3/log 1.05 = 0.4771/0.0212 = 22.5].

c) *At a certain rate of compound interest, 1 will increase to 2 in a years, 2 will increase to 3 in b years, and 3 will increase to 15 in c years. If 6 will increase to 10 in n years, what will be the expression for n in terms of a, b and c.*

If the common rate of compound interest is i, the hypotheses are:

$$1(1 + i)^a = 2 \rightarrow ln\,2 = a\ln(1 + i)$$

$$2(1 + i)^b = 3 \rightarrow ln\frac{3}{2} = b ln\,(1 + i)$$

$$3(1 + i)^c = 15 \rightarrow ln\,5 = c ln\,(1 + i)$$

$$6(1 + i)^n = 10 \rightarrow ln\frac{5}{3} = n ln\,(1 + i)$$

$ln\frac{5}{3} = ln\,5 - ln\,3 = ln\,5 - (ln\,2 + ln\,1.5)$ (since 2 * 1.5 = 3).

Hence, substituting for the four equations above, we get:

$$n ln\,(1 + i) = c ln\,(1 + i) - a ln\,(1 + i) - b ln\,(1 + i)$$

$$= (c - a - b)\,ln\,(1 + i)$$

Equating the coefficients for *ln (1 + i)*, we get: $n = c - a - b$

d) *You invest 1000 on Feb. 1 in an account earning compound interest at an annual effective rate of 6%. On April 15 of the same year, you withdraw the money. Assuming non-leap year, how much money will you* *withdraw if the bank counts days using: a) actual/actual method, b) 30/360 method and c) actual/360 method. What do you observe from the results of the three?*

a) The number of days is 27 + 31 + 15 = 73. Thus, the amount of money withdrawn is

$$1000(1 + 0.06)^{\frac{73}{365}} = Rs. 1,011.70.$$

b) Using the $30/360$, we obtain:

$1000(1 + 0.06)^{\frac{72}{360}} = Rs. 1011.70.$

c) Using the actual/360 method, we get

$$1000(1 + 0.06)^{\frac{73}{360}} = Rs. 1011.89$$

We observe that, the manner of counting days has an implication for interest. It is however marginal for small period within a year. This is also what we had noted earlier in Equation (7.6).

Check Your Progress 2 [answer within the space given in about 50-100 words]

1) How is 'simple interest' defined? Why is the 'effective interest' in SI decreasing?

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2) How is compound interest defined? For short time period like a 'fraction of a year', how is SI an approximation for the CI?

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3) Indicate the distinction in counting the number of days in the three methods of exact SI, ordinary SI and SI as per the Banker's rule?

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4) In terms of 'effective r.o.i.', what is the difference between SI and CI?

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5) How would you explain the relationship between SI and CI?

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7.5 LET US SUM UP

In this unit, we discussed the 'theory of interest' from the perspective of payment made to lender by a borrower. The difference between initial investment and the accumulated value at a future time point of time is the amount of interest. We learnt that value at time $t \geq 0$ of the principal invested at time 0 indicates the 'amount function' whereas the accumulated value at time t (of initial investment of 1) would give the 'accumulation function'. Two basic concepts of interest rates viz., simple and compound interests were covered with time considered both as an integer as well as a non-integer value. Simple interest is paid only on the original principal and not on the interest accrued. In case of compound interest, the calculation would be not only on the initial principal but also on the accumulated interest of prior periods. Further, the 'nominal interest rate' is defined as a 'stated interest rate' while the 'effective interest rate' is the one that caters to the compounding periods during a payment plan.

7.6 KEY WORDS

Interest	:	Payment from a borrower to a lender of an amount above the amount borrowed.
Nominal Interest Rate	:	The rate quoted in loan and deposit agreements.
Effective Interest Rate	:	Interest rate on a debt that takes into account the effects of compounding.
Principal Amount	:	Money invested (or loaned).
Simple Interest	:	Interest paid only on the original principal, not on the interest accrued.

Accumulation Factor	:	It is a measurement of the rate at which the principal grows. In SI, it is the factor $(1 + nr)$. In CI, it is the factor $(1 + r)^n$, n being the period and r being the interest rate for one period.
Accumulation Function	:	It is the function that describes the accumulated value at time t of initial investment of 1.
Amount Function	:	It is the function that gives the accumulated value of an initial investment of k at time t. It is the value at time $t \geq 0$ of the principal invested at time 0.
Annual Percentage Rate (APR)	:	It is an expression of the 'effective interest rate' that the borrower will pay on a loan, taking into account the one-time fees and standardising the way the rate is expressed.
Compound Interest	:	Interest calculated not only on the initial principal but also on the accumulated interest of prior periods.
Effective Annual Rate of Interest	:	Amount of money that one unit invested at the beginning of the year will earn during the year, when the amount earned is paid at the end of the year.

7.7 SUGGESTED BOOKS FOR FURTHER READING

1) Finan, Marcel B (2017). A Basic Course in the Theory of Interest and Derivatives Markets (see Internet).
2) Slud Eric V (2001). Actuarial Mathematics and Life-Table Statistics, University of Maryland, College Park (see Internet).
3) Veeh, Jerry Alan (2001). Lecture Notes on Actuarial Mathematics (see Internet).

7.8 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

1) If $A(t)$ is the amount invested in time t, and $A(t+s)$ is the amount that $A(t)$ has become over the period s, then $A(t+s) - A(t)$ is defined as the 'interest earned over the period s. $A(t)$ is referred to as the 'principal amount' at time t and $A(t+s)$ is called the accumulated amount over the period s. The amount earned over the period s calculated as a ratio to the principal amount i.e. $\frac{A(t+s) - A(t)}{A(t)}$ is called the 'interest'. The value of interest, multiplied by 100, is the 'interest rate'.

2) $A(t)$ is our 'amount function' at time t. The amount invested in time '0' i.e. $A(0)$ is the principal amount. The ratio or the function $a(t) = A(t)/A(0)$ is called as the 'accumulation function'. Defining such a function i.e. the accumulation function is useful for comparison of different amount values. The accumulation function represents the accumulated value of a principal invested at time t. It possesses the following properties: (i) $a(0) = 1$, (ii) $a(t)$ is increasing [i.e. if $t_1 < t_2$ then $a(t_1) \leq a(t_2)$] and (iii) if interest accrues for non-integer values of t (i.e. for any fractional part of a year), then $a(t)$ is a continuous function.

3) If S is the sum to which the principal P has accumulated over a period t with the 'rate of interest' (r.o.i.) r, then in case of simple interest, we have $S = P(1 + rt)$. The term $(1 + rt)$ is called as the 'accumulation factor'. It represents the rate at which the principal amount grows over time.

4) The former is usually taken as the r.o.i for one full year. The latter is taken as the interest rate over sub-periods like a quarter, month, week, day, etc. It is expressed in terms of Equation (7.1) and (7.2) respectively. The 'effective rate of interest' i is the amount of money that one unit invested at the beginning of a period will earn during the period assuming that the 'interest is paid at the end of the period'. It is also the 'interest earned in one period divided by the principal at the beginning of the period'. It is useful to calculate the sub-period interest rates.

Check Your Progress 2

1) In case of SI, an investment of 1 earns a 'constant interest' in each period equal to i. As a result, the accumulation function is a linear function and accrual of interest is given by $a(n) = 1 + in, n \geq 0$. Now, if we consider the effective r.o.i. $i_{n+1} - i_n$, we find that it is a negative value (as per 7.4a). Hence, the effective interest in SI is decreasing over time.

2) The accumulation function for compound interest i is given by the formula $a(t) = (1 + i)^t$. For short term time period, expanding the right side of this function, we find that: $(1 + i)^t \approx 1 + it$.

3) In exact SI, we take 'actual/actual' in which the denominator is 365 or 366. In ordinary SI, it is taken as 30/360. Banker's rule is a mix of actual and ordinary: it takes numerator as 'actual' but denominator as 360.

4) In case of SI, as noted in the answer to the first CYP above, the effective r.o.i. is decreasing. In case of CI, as noted by Equation (7.10), it is constant.

5) For two time points $t = 0$ and $t = 1$, both are same. For $t > 1$, CI is higher than SI. For $t < 1$, CI is less than SI. Alternatively, in case of SI, the 'absolute amount of growth' is constant, whereas, in case of CI, the 'relative rate of growth' is constant.

UNIT 8 EQUATIONS OF VALUE AND TIME

Structure

8.0 OBJECTIVES

After reading this unit, you will be able to:

- state the meaning of ‘present value’;
- define the term ‘discount factor’;
- express the relationships between the ‘effective rate of discount (d)’, ‘discount rate (v)’ and ‘effective rate of interest (i)’;
- discuss the nature of ‘effective rate of discount (d_n)’ in the context of compound and simple discount rates (d);
- derive the expressions for the ‘discount rates’ applied over m-fractions of a period’ over t years;
- indicate the relationship between effective and nominal rates of interest and discounts;

- outline the concept of 'force of interest' in a 'continuous compounding framework';
- specify the meaning of the term 'equation of value'; and
- discuss the methods of solving for 'interest rates'.

8.1 INTRODUCTION

In this unit we discuss the question of determining the present value of some cash flow in the past or into the future. The underlying idea is that the value of money today (or now) is greater than a reliable promise to receive the same amount at a future date. Such an idea must be clear for making any decision on investment. For this purpose, we need to learn the techniques of deriving 'what the future cash flow is worth today'?

8.2 PRESENT VALUE AND DISCOUNT FACTOR

Present value (PV) refers to the 'current worth of a future stream of cash flows given a specified rate of return'. The concept of a *present value* takes into account the time dimension to money. You may think in terms of the amount of money to be set aside now, which, when compounded with interest, will be sufficient to settle a known future payment. For instance, suppose you want to know what Re.1 a period ago, invested at compound interest rate i per period, is worth today. This means, if X is the accumulated value, then you must have $1(1 + i) = X$ so that Re.1 a period ago is worth $1 + i$ rupees now. Here, as you know from Unit 7, $1 + i$ is the 'accumulation factor'. Similarly, Re.1 a period from now invested at the rate i will be worth $v = \frac{1}{1+i}$ today. We call v the 'discount factor' since it discounts the value of an investment at the end of a period to its present value at the beginning of the period. Let us mark this equation for 'discount factor' since we will be using it often.

$$v = \frac{1}{1+i} \qquad (8.1)$$

Distinction is also often made between 'present value' and 'net present value (NPV)'. The NPV accounts for the initial capital outlay required to fund a project, making it a net figure, whereas the PV calculation only accounts for cash inflows. For instance, assume a given project requires an initial capital investment of Rs. 15,000. The project is anticipated to generate revenues of Rs. 3,500, Rs. 9,400 and Rs. 15,100 in the next three years, respectively, and the company's hurdle rate is 7%. The present value of the anticipated income is $\text{Rs.}\,3{,}500 / (1 + 0.07)^1 + \text{Rs.}\,9{,}400 / (1 + 0.07)^2 + \text{Rs.}\,15{,}100 / (1 + 0.07)^3$, or $\text{Rs.}\,23{,}807$. The NPV of this project can be determined by simply subtracting the initial capital investment from the discounted revenue: Rs. 23,807 – Rs. 15,000 = Rs. 8,807.

Broadly, accumulation and discounting can be seen to be opposite processes. Why? The term $(1 + i)^t$ is the accumulated value of Re.1 at the end of t time periods. The term v^t is the present value (or the discounted value) of Re.1 to be paid at the end of t periods. We can write the discount factor during the n^{th} period in terms of the amount function. This is because, since $i_n = \frac{A(n)-A(n-1)}{A(n-1)}$ and the rate of discount is $(1 + i_n)^{-1}$, we have:

$$\frac{1}{1 + i_n} = \frac{1}{1 + \frac{A(n)-A(n-1)}{A(n-1)}} = \frac{A(n-1)}{A(n)}$$

Further, from $[a(t)]^{-1} = \frac{1}{(1+i)^t} = v^t$, we have $a(t) = (1 + i)^t$. Putting negative values of t, a graph of $a(t)$ can be drawn as in Figure 8.1. The present values of simple interest can be calculated by taking the accumulation function $a(t) = 1 + it$. The present value of 1 for t years in the future is $(a(t))^{-1} = \frac{1}{1 + it}$, $t \geq 0$.

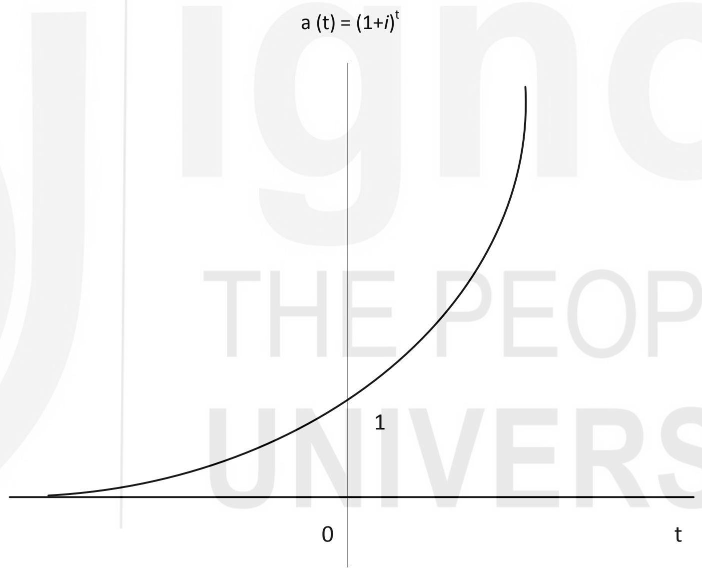

Fig. 8.1: Accumulation Function

8.3 EFFECTIVE RATE OF DISCOUNT

The effective rate of discount is denoted by $d.$ It is a measure of interest paid at the 'beginning of the period'. It is in contrast to the 'effective rate of interest' which is paid at the 'end of the period'. For instance, consider a loan of 'Rs. 1,200 taken for one year at an effective rate of discount of 5%'. It means that the borrower will pay the interest of $1200 \times 0.05 = Rs. 60$ (the amount of discount) at the beginning of the year and repays Rs. 1,200 at the end of the year. So the lender is getting the interest in advance from the borrower.

8.3.1 Compound and Simple Discount Rates

The 'effective rate of interest' is not the same as an 'effective rate of discount' (they are in fact the opposite). But, there is a relationship between the two. Assume that Re. 1 is invested for one year at an effective rate of discount d. Then, the original principal is Rs. $(1 - d)$. The effective rate of interest i for the year is defined to be the ratio of 'the amount of interest divided by the balance at the beginning of the year'. That is:

$$i = \frac{d}{1-d} \qquad (8.2)$$

Solving equation (8.2) for d we get:

$$d = \frac{i}{1+i} \qquad (8.3)$$

Therefore, d is the ratio of the amount of interest that the investment 1 will earn during the year to the balance at the end of the year. Several identities can be derived from Equations (8.1) to (8.3). For instance, since $v = \frac{1}{1+i}$, we have:

$$d = iv \qquad (8.4)$$

That is, we obtain d by discounting i from the end of the period to the beginning of the period by applying the discount factor v. Next, we have:

$$d = \frac{i}{1+i} = \frac{1+i}{1+i} - \frac{1}{1+i} = 1 - v \qquad (8.5)$$

$$\text{or, } v = 1 - d \qquad (8.5a)$$

Both sides of the equation (8.5) represent the present value of 1 to be paid at the end of the period. From $i = \frac{d}{1-d}$, we have $i(1-d) = d$ or $i - id = d$. Therefore:

$$i - d = id \qquad (8.6)$$

Equation (8.6) means that the difference of interest in the two schemes is the same as the interest earned on amount d invested at the rate i for one period. Effective rates of discount can be calculated over any particular measurement period. The effective rate of discount d_n in the n^{th} period is defined to be the 'ratio of the amount of discount and the accumulated value at the end of the period'. That is:

$d_n = \frac{a(n)-a(n-1)}{a(n)}$. Since $A(n) = A(0)a(n)$, we can write $d_n = \frac{A(n)-A(n-1)}{A(n)} = I_n A(n)$. If the compound discount is $d > 0$ per period, then we have $d_n = d$ for all $n \geq 1$. How? Let us see this as follows.

Compound interest and compound discount have equal accumulation functions. Hence:

$a(t) = (1 + i)^t = \left(1 + \frac{d}{1-d}\right)^t = \frac{1}{(1-d)^t}$, for all $t \geq 0$. Therefore:

$$d_n = \frac{a(n) - a(n-1)}{a(n)} = \frac{\frac{1}{(1-d)^n} - \frac{1}{(1-d)^{n-1}}}{\frac{1}{(1-d)^n}} = 1 - (1-d) = d.$$

We can define 'simple discount' in a manner analogous to the definition of 'simple interest'. Let d be a simple discount discounting Re.1 as in Figure 8.2.

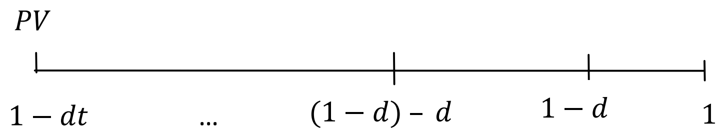

Fig. 8.2: Discount Rate

If $a_s(t)$ is its accumulation function, the original principal which will produce an accumulated value of 1 at the end of t periods is given by: $[a_s(t)]^{-1} = 1 - dt, 0 \leq t < \frac{1}{d}$. Note that it is necessary to keep $[a_s(t)]^{-1} > 0$. Thus, the accumulation function for simple discount d is:

$$a_s(t) = \frac{1}{1-dt}, 0 \leq t < \frac{1}{d} \quad (8.7)$$

Now, assuming a simple rate of discount $d > 0$, d_n is an increasing function of n for $0 < n - 1 < \frac{1}{d}$. We can observe this as follows. Under the simple discount rate d, we have $a_s(n) = \frac{1}{1-dn}$. Thus:

$$d_n = \frac{a_s(n) - a_s(n-1)}{a_s(n)} = \frac{\frac{1}{1-dn} - \frac{1}{1-d(n-1)}}{\frac{1}{1-dn}} = \frac{d}{1 - dn + d}$$

$$= \frac{d}{1 + d(1-n)}$$

d_n is > 0 since $0 < n - 1 < \frac{1}{d}$. As n increases the denominator decreases. Hence, d_n is an increasing function. Now, as done in Unit 7, here too, we may note that for $0 < d < 1$, we have the following three relationships.

a) $(1-d)^t < 1 - dt$ if $0 < t < 1$

b) $(1-d)^t = 1 - dt$ if $t = 0$ or $t = 1$

c) $(1-d)^t > 1 - dt$ if $t > 1$

In order to verify the above, note that we can write the power series expansion of $(1-d)^t$ as:

$$(1-d)^t = 1 - dt + \frac{1}{2}t(t-1)d^2 - \frac{1}{6}t(t-1)(t-2)d^3 + \cdots$$

where if $t < 1$, then all terms after the second are negative. Hence, $(1 - d)^t < 1 - dt$ for $0 < t < 1$. The second relationship follows by straight forward substitution. For the third, suppose $t > 1$. Let $f(d) = 1 - dt - (1-d)^t$. Then $f(0) = 0$ and $f'(d) = -t + t(1-d)^{t-1}$. Since $t > 1$ we have $t - 1 > 0$. Since $0 < d < 1$, we have $1 - d < 1$. Thus, $(1-d)^{t-1} < 1$ and consequently $f'(d) < -t + t = 0$. Figure 8.3 compares the discount function under simple and compound discounts.

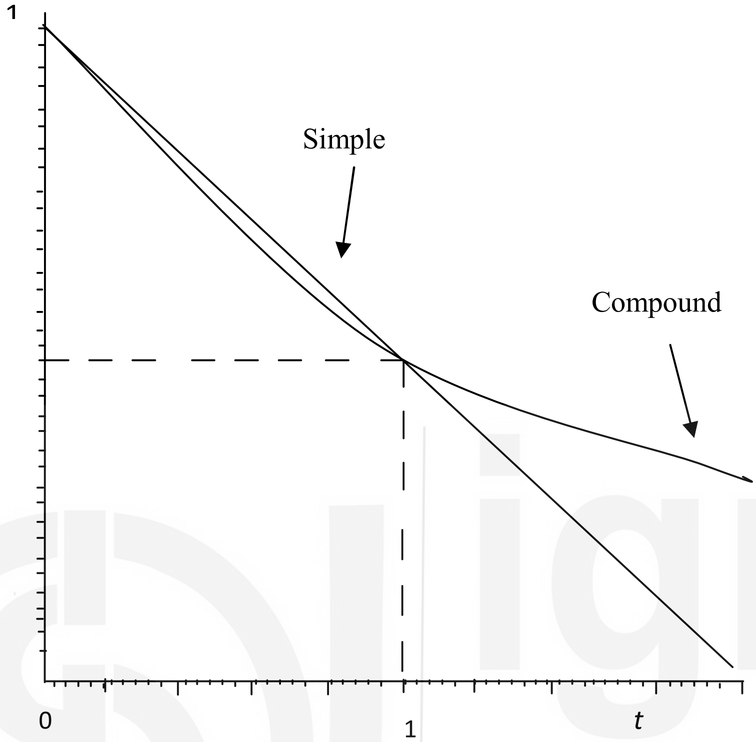

Fig. 8.3: Comparison of Simple and Compound Discount Functions

8.3.2 Frequently Paid Interests

If interest is paid in fractions of a period, then it means that more frequently interest is paid than when it is paid only once per period. A nominal rate of interest $i^{(m)}$ is payable m times per period. This means if a rate of interest is paid monthly, then for each m^{th} of a period, rate of interest paid is $\frac{i^{(m)}}{m}$ and not $i^{(m)}$. For instance, if a nominal rate of 12% compounded monthly is paid, the effective rate of interest per month is 1% since there are twelve months in a year. Now, suppose that Rs. 1 is invested at a nominal rate $i^{(m)}$ compounded m times per period. In such an arrangement, the period is partitioned into m equal fractions. At the end of the first fraction, the accumulated value is $\frac{1+ i^{(m)}}{m}$. At the end of the second fraction, the accumulated value is $\left(\frac{1+ i^{(m)}}{m}\right)^2$. Continuing, we get the accumulated value at the end of the m^{th} fraction of a period, which is also the same as the end of one period, as $\left(\frac{1+ i^{(m)}}{m}\right)^m$. Therefore, at the end of t years the accumulated value becomes $a(t) = \left(\frac{1+ i^{(m)}}{m}\right)^{mt}$. Figure 8.4 illustrates the accumulation at a nominal rate of interest for one measurement period.

Time	0	$1/m$	$2/m$	...	$m-1/m$	1
Balance	1	$1+\frac{i^{(m)}}{m}$	$\left(1+\frac{i^{(m)}}{m}\right)^2$	...	$\left(1+\frac{i^{(m)}}{m}\right)^{m-1}$	$\left(1+\frac{i^{(m)}}{m}\right)^{m}$

Fig. 8.4: Accumulation and Nominal Rate of Interest

Now, if d is the effective discount rate equivalent to $d^{(m)}$ then:

$$1-d = \left(1-\frac{d^{(m)}}{m}\right)^m \qquad (8.8)$$

Since each side of the Equation (8.8) gives the present value of 1 to be paid at the end of the measurement period, rearranging, we have:

$$\boldsymbol{d = 1-\left(1-\frac{d^{(m)}}{m}\right)^m} \qquad (8.8a)$$

Solving equation (8.8a) for $\boldsymbol{d^{(m)}}$ we get:

$$\boldsymbol{d^{(m)} = m[1-(1-d)^{\frac{1}{m}}] = m(1-v^{\frac{1}{m}})} \qquad (8.9)$$

We can now indicate the relationship between effective and nominal rates as follows. If $\boldsymbol{i}$ denotes the effective rate of interest per period equivalent to $\boldsymbol{i^{(m)}}$, then: $\boldsymbol{1+i = \left(\frac{1+i^{(m)}}{m}\right)^m}$. Hence:

for any $\boldsymbol{t \geq 0}$, we have:

$$\boldsymbol{(1+i)^t = \left(1+\frac{i^{(m)}}{m}\right)^{mt}} \qquad (8.10)$$

Now, since $\boldsymbol{1-d = \frac{1}{1+i}}$,

$$\boldsymbol{\left(\frac{1+i^{(m)}}{m}\right)^m = 1+i = (1-d)^{-1} = \left(1-\frac{d^{(n)}}{n}\right)^{-n}} \qquad (8.10a)$$

Equation (8.10a) establishes the relationship between 'nominal rate of interest' and 'nominal rate of discount'. Note that if $\boldsymbol{m = n}$ then (8.10a) reduces to:

$$\boldsymbol{\left(1+\frac{i^{(n)}}{n}\right) = \left(1-\frac{d^{(n)}}{n}\right)^{-1}}.$$

8.3.3 Illustrations

Let us now consider some illustrations which will help you to understand the application of above equations in practice.

a) *What is the present value of Rs.10,000 to be paid at the end of three years if the interest rate is 10% compounded annually?*

   We are required to find out the present value (PV) given the future value (FV). Since $\mathbf{FV = PV(1+i)^3}$, $\mathbf{PV = FV(1+i)^{-3}}$, $\mathbf{PV = 10000(1.10)^{-3} \approx Rs.}$ 7513.15.

b) *Comment on the difference between the following two situations? (i) A loan of Rs. 200 is made for one year at an effective rate of interest of 6%. (ii) A loan of Rs. 200 is made for one year at an effective rate of discount of 6%.*

In the first case the interest is paid at the end of the period. So the borrower was able to use Rs. 200 for the whole year. In the second case, the interest is paid at the beginning of the period. So the borrower had access to only Rs. 188 for the year.

c) *Calculate the accumulated value of 500 after 4 years at a rate of interest of 10% per year convertible monthly.*

The accumulated value is $\mathbf{500\left(1 + \left(\frac{0.10}{12}\right)\right)^{12\times 4} = 744.68}$.

d) *Calculate the accumulated value of Rs. 3000 to be paid at the end of 8 years with a compound rate of interest of 5%: (i) per annum, (ii) convertible quarterly, (iii) convertible monthly.*

(i) $\mathbf{3000\left(1 + \frac{0.05}{1}\right)^{8} \approx \boldsymbol{Rs.}\,4,432.37}$.

(ii) $\mathbf{3000\left(1 + \frac{0.05}{4}\right)^{8\times 4} \approx \boldsymbol{Rs.}\,4,464.39}$.

(iii) $\mathbf{3000\left(1 + \frac{0.05}{12}\right)^{8\times 12} \approx \boldsymbol{Rs.}\,4,471.76.}$

Compute the present value of Rs. 10000 to be paid at the end of 5 years at an annual compound interest rate of 8% convertible semi-annually.

$$\mathbf{\frac{10000}{\left(1 - \frac{0.08}{2}\right)^{5\times 2}} \approx Rs.\,15,041.38}$$

e) *What is the present value of Rs. 2000 to be paid at the end of 5 years at 5% per year payable in advance and convertible semi-annually?*

$\mathbf{2000\left(1 - \frac{0.05}{2}\right)^{10} = \boldsymbol{Rs.}\,1552.66}$.

f) *Show that* $\frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} = \frac{i^{(m)}}{m} \cdot \frac{d^{(m)}}{m}$

Since $\left(1 + \frac{i^{(m)}}{m}\right) = \left(1 - \frac{d^{(m)}}{m}\right)^{-1}$

We can write $\left(1 + \frac{i^{(m)}}{m}\right) \cdot \left(1 - \frac{d^{(m)}}{m}\right) = 1$

Expanding we obtain $1 - \frac{d^{(m)}}{m} + \frac{i^{(m)}}{m} - \frac{i^{(m)}}{m} \cdot \frac{d^{(m)}}{m} = 1$

Hence, $\frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} = \frac{i^{(m)}}{m} \cdot \frac{d^{(m)}}{m}$

g) *Calculate the accumulated value of Rs. 500 invested for five years at 8% annum convertible quarterly.*

$$500\left[1+\frac{.08}{4}\right]^{4\cdot 5} = 500[1.02]^{20}$$

Note that this situation is equivalent to one in which Rs. 500 is invested at a rate of interest of 2% for 20 years.

h) *Compute the present value of Rs.1000 to be paid at the end of six years at 6% per annum payable in advance and convertible semi-annually.*

$$1000\left[1-\frac{.06}{2}\right]^{2\cdot 6} = 1000(0.97)^{12}$$

Note that this situation is equivalent to one in which the present value of Rs. 1000 to be paid at the end of 12 years is calculated at a rate of discount of 3%.

Check Your Progress 1 [answer within the space given in about 50-100 words]

1) State the meaning of the term 'present value'.

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2) Define the term 'discount factor'.

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3) Distinguish between the terms 'net present value (NPV)' and 'present value (PV)'.

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4) What is meant by 'effective rate of discount'?

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5) State the relationships between d, i and v [where d is 'effective rate of discount', v is 'discount rate' and i is 'effective rate of interest'].

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8.4 FORCE OF INTEREST

If the interest is compounded continuously, the accumulated function $A(t)$ is a continuous function of 't'. The nominal rate of interest is then called the 'force of interest' and is denoted by δ_t (sometimes $i^{(\infty)}$). Let us see how this works in theory.

8.4.1 Continuous Compounding

To understand the concept, let us consider the case of a nominal compound interest $\boldsymbol{i^{(m)}}$ paid $\boldsymbol{m}$ times over a period. The force of interest, denoted by $\boldsymbol{\delta}$, is taken to be the limit of $\boldsymbol{i^{(m)}}$ since the number of times we credit the compounded interest theoretically goes to infinity. This means: $\boldsymbol{\delta = lim_{m\to\infty} i^{(m)}}$. Therefore, if $\boldsymbol{i}$ is the effective interest rate equivalent to $\boldsymbol{i^{(m)}}$, we have:

$$\boldsymbol{i^{(m)} = m[(1 + i)^{\frac{1}{m}} - 1] = \frac{(1+i)^{\frac{1}{m}} - 1}{\frac{1}{m}}}.$$

Hence, $\boldsymbol{\delta = lim_{m\to\infty} \frac{(1+i)^{\frac{1}{m}} - 1}{\frac{1}{m}}}$.

The above limit is of the form $\frac{\boldsymbol{0}}{\boldsymbol{0}}$. Hence, we can apply L'Hospital's rule to obtain:

$$\boldsymbol{\delta = lim_{m\to\infty} \frac{\frac{d}{dm}\left[(1 + i)^{\frac{1}{m}} - 1\right]}{\frac{d}{dm}\left(\frac{1}{m}\right)} = lim_{m\to\infty}\left[(1 + i)^{\frac{1}{m}} ln\,(1 + i)\right]}$$

$$\boldsymbol{= ln\,(1 + i)}$$

(since $\boldsymbol{lim_{m\to\infty}(1 + i)^{\frac{1}{m}} = 1}$).

Therefore, the effective interest rate $\boldsymbol{i}$ can be written as a series expansion of $\boldsymbol{\delta}$. Thus:

$$\boldsymbol{i} = \boldsymbol{e}^{\boldsymbol{\delta}} - \mathbf{1} = \boldsymbol{\delta} + \frac{\boldsymbol{\delta}^2}{\mathbf{2}!} + \cdots + \frac{\boldsymbol{\delta}^n}{\boldsymbol{n}!} + \cdots.$$

Similarly, $\delta = ln\,(1 + i) = i - \frac{i^2}{2!} + \cdots + (-1)^n \frac{i^n}{n} + \cdots.$

Now, using the accumulation function of compound interest $\boldsymbol{a(t)} = (\mathbf{1} + \boldsymbol{i})^{\boldsymbol{t}}$, we get $\boldsymbol{\delta} = \boldsymbol{ln}\,(\mathbf{1} + \boldsymbol{i}) = \frac{\frac{d}{dt}\boldsymbol{a(t)}}{\boldsymbol{a(t)}}$. The definition of force of interest in terms of a compound interest accumulation function can be extended to any accumulation function. This means, for an accumulation function $\boldsymbol{a(t)}$ we can define the 'force of interest at time $\boldsymbol{t}$' as $\boldsymbol{\delta_t} = \frac{\boldsymbol{a'(t)}}{\boldsymbol{a(t)}}$.

From the definition of the derivative we have: $\frac{\frac{d}{dt}\boldsymbol{a(t)}}{\boldsymbol{a(t)}} = \boldsymbol{lim}_{\boldsymbol{m}\to\infty} \frac{\frac{a\left(t+\frac{1}{n}\right)-a(t)}{a(t)}}{\frac{1}{n}}$.

Therefore, the expression $\frac{a\left(t+\frac{1}{n}\right)-a(t)}{a(t)}$ is just the 'effective rate of interest' over a very small time period $\frac{\mathbf{1}}{\boldsymbol{n}}$ so that $\frac{\frac{a\left(t+\frac{1}{n}\right)-a(t)}{a(t)}}{\frac{1}{n}}$ is the 'nominal annual interest rate' converted $\boldsymbol{n}$ times a year with each time period of length $\mathbf{1}/\boldsymbol{n}$ corresponding to that effective rate.

8.4.2 Equation of Value

Interest problems generally involve four quantities: principal(s), investment period(s) or lengths of time, interest rate(s), accumulated value(s). If any three of these quantities are known, then the fourth quantity can be determined. In this section, we consider equations of this kind. In calculations involving interest, the value of an amount of money at any given point in time depends upon the length of time. This principle of giving recognition to the 'time value of money' makes the principal amount invested at a certain 'rate of interest' (r.o.i.) grow differently for differing amounts. We assume that this principle does not include the effect of inflation. Inflation reduces the purchasing power of money over time. Hence, investors expect a higher rate of return to compensate for inflation. As a consequence of the above principle, various amounts of money payable at different points in time cannot be compared. In order to compare them, we must accumulate or discount the amounts to a common date, called the 'comparison date'. The equation which accumulates or discounts each payment to the comparison date is called the 'equation of value'.

8.4.3 Illustrations

Let us now consider some empirical illustrations as before.

a) *What is the accumulated value of Rs. 1000 invested for ten years if the force of interest is 5%?*

$$1000e^{(0.05)(10)} = 1000e^{(0.5)}$$

b) *A loan of Rs. 3000 is taken out on June 23, 1997. If the force of interest (i.e. continuous compounding) is 14%, calculate the following:* (i) the value of the loan on June 23, 2002, (ii) the value of $\boldsymbol{i}$ and (iii) the value of $\boldsymbol{i}^{(12)}$.

i) $\mathbf{3\,000(1 + \boldsymbol{i})^5 = 3\,000}\boldsymbol{e}^{5\delta} = \mathbf{3\,000}\boldsymbol{e}^{0.7} \approx \boldsymbol{Rs.}\,\mathbf{6,041.26}$.

ii) $\boldsymbol{i} = \boldsymbol{e}^{\delta} - \mathbf{1} = \boldsymbol{e}^{0.14} - \mathbf{1} \approx \mathbf{0.15027}$. Thus, the effective r.o.i. is 15%.

iii) We have $\left(\mathbf{1} + \frac{\boldsymbol{i}^{(12)}}{\mathbf{12}}\right) = \mathbf{1} + \boldsymbol{i} = \boldsymbol{e}^{0.14}$.

Solving for $\boldsymbol{i}^{(12)}$ we get $\boldsymbol{i}^{(12)} \approx \mathbf{0.14082}$. Thus, for each fraction of the period the interest rate is 14.1%.

c) *(i) Calculate the effective interest rate over the first four-year period from now if the effective rate of interest is 5% for the first three years, 4.5% for the next three years, and 4% for the last three years. (ii) What about over the six-year period from now?*

(i) Since $(\mathbf{1} + \boldsymbol{i})^{4} = (\mathbf{1.05})^{3}(\mathbf{1.045})$, solving for $\boldsymbol{i}$ we get
$\boldsymbol{i} = [(\mathbf{1.05})^{3}(\mathbf{1.045})]^{\frac{1}{4}} - \mathbf{1} = \mathbf{4.87\%}$

(ii) Since $(\mathbf{1} + \boldsymbol{i})^{6} = (\mathbf{1.05})^{3}(\mathbf{1.045})^{3}$, solving for $\boldsymbol{i}$ we get
$\boldsymbol{i} = [(\mathbf{1.05})^{3}(\mathbf{1.045})^{3}]^{\frac{1}{6}} - \mathbf{1} = \mathbf{4.75\%}$

d) *Calculate the accumulated value of 1 at the end of n periods where the effective rate of interest for the k^{th} period,* $\mathbf{1} \le \boldsymbol{k} \le \boldsymbol{n}$, *is defined by* $\boldsymbol{i}^{\boldsymbol{k}} = (\mathbf{1} + \boldsymbol{r})^{\boldsymbol{k}}(\mathbf{1} + \boldsymbol{i}) - \mathbf{1}$.

$$\begin{aligned}\boldsymbol{a}(\boldsymbol{n}) &= (\mathbf{1} + \boldsymbol{i}_1)(\mathbf{1} + \boldsymbol{i}_2)\_\_\_(\mathbf{1} + \boldsymbol{i}_n) \\ &= (\mathbf{1} + \boldsymbol{r})(\mathbf{1} + \boldsymbol{i})(\mathbf{1} + \boldsymbol{r})^2(\mathbf{1} + \boldsymbol{i})\_\_\_(\mathbf{1} + \boldsymbol{r})^{\boldsymbol{n}}(\mathbf{1} + \boldsymbol{i}) \\ &= (\mathbf{1} + \boldsymbol{r})^{1+2+\_+n}(\mathbf{1} + \boldsymbol{i})^{\boldsymbol{n}} = (\mathbf{1} + \boldsymbol{r})^{\frac{n(n+1)}{2}}(\mathbf{1} + \boldsymbol{i})^{\boldsymbol{n}}\end{aligned}$$

e) *In return for a payment of Rs. 1,200 at the end of 10 years, a lender agrees to pay Rs. 200 immediately, Rs.400 at the end of 6 years, and a final amount at the end of 15 years. Estimate the amount of the final payment at the end of 15 years if the nominal rate of interest is 9% converted semi-annually?*

The comparison date is chosen to be $t = 0$. The time diagram is as given in Figure 8.5. The equation of value is:

$$\mathbf{200 + 400(1 + 0.045)^{-12}} + \boldsymbol{X}\mathbf{(1 + 0.045)^{-30} = 1200(1 + 0.045)^{-20}}.$$

Solving this equation for $\boldsymbol{X}$ we get $\boldsymbol{X} \approx \boldsymbol{Rs.}\,\mathbf{231.11}$.

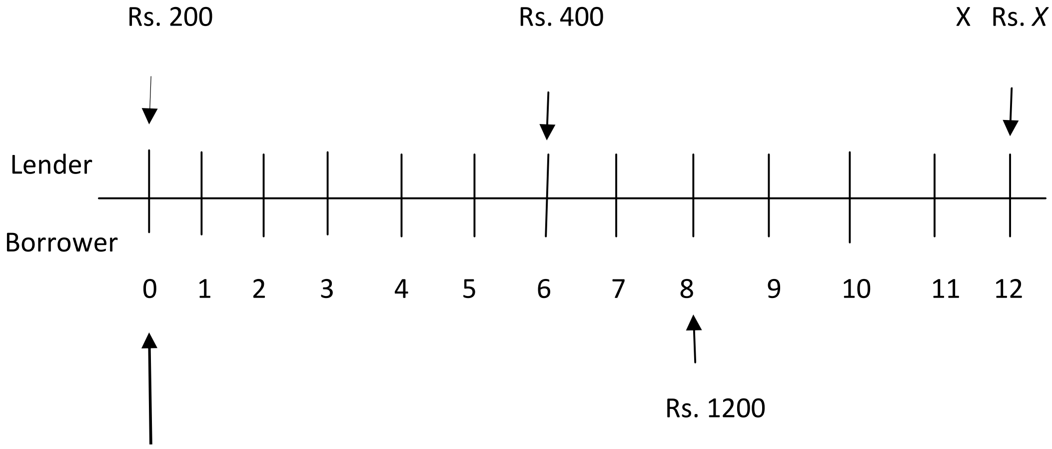

Fig. 8.5: Time Diagram for Payment

f) *Investor **A** deposits 1,000 into an account paying 4% compounded quarterly. At the end of three years, he deposits an additional 1,000. Investor **B** deposits **X** into an account with force of interest* $\boldsymbol{\delta_t} = \frac{\mathbf{1}}{\mathbf{6+t}}$. *After five years, investors **A** and **B** have the same amount of money. Find **X**.*

Consider investor ***A***'s account first. The initial 1,000 accumulates at 4% compounded quarterly for five years. The accumulated amount of this is:

$$\mathbf{1,000}\left(\mathbf{1} + \frac{\mathbf{0.04}}{\mathbf{4}}\right)^{\mathbf{4\times5}} = \mathbf{1000(1.01)^{20}}.$$

The second 1,000 accumulates at 4% compounded quarterly for two years. It accumulates to $1,\mathbf{000}\left(\mathbf{1} + \frac{\mathbf{0.04}}{\mathbf{4}}\right)^{\mathbf{4\times2}} = \mathbf{1000(1.01)^{8}}$.

The value in investor A's account after five years is $\boldsymbol{A} = \mathbf{1000(1.01)^{20} + 1000(1.01)^{8}}$.

The accumulated amount of investor B's account after five years is given by:

$$\boldsymbol{B} = \boldsymbol{X}e^{\int_0^5 \frac{dt}{6+t}} = \boldsymbol{X}e^{\ln\left(\frac{11}{6}\right)} = \frac{\mathbf{11}}{\mathbf{6}}\boldsymbol{X}.$$

The equation of value at time $\boldsymbol{t} = \mathbf{5}$ is $\boldsymbol{X} = \mathbf{1000(1.01)^{20} + 1000(1.01)^{8}}$.

Solving for $\boldsymbol{X}$ we have $\boldsymbol{X} \approx \boldsymbol{Rs.}\,\mathbf{1,256.21}$.

g) *Instead of making payments of 300, 400, and 700 at the end of years 1, 2, and 3, the borrower prefers to make a single payment of 1400. Calculate the time for which this payment should be made if the interest rate is 6% compounded annually?*

Computing all of the present values at time 0 shows that the required time *t* should satisfy the equation of value $300(1.06)^{-1} + 400(1.06)^{-2} + 700(1.06)^{-3} = 1400(1.06)^{-t}$. Hence, the solution is $t = 2.267$.

h) *Calculate the compounded quarterly rate of interest required for a deposit of 5000 today to accumulate to 10,000 after 10 years?*

The equation of value is $5000(1 + i/4)^{40} = 10000$ from which it follows that $i = 0.0699$.

Check Your Progress 2 [answer within the space given in about 50-100 words]

1) Why is the r.o.i. in continuous compounding called as 'force of interest'?

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2) How is the 'accumulation function' for the 'force of interest' defined?

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3) State the meaning of the term 'equation of value'.

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8.5 SOLVING FOR INTEREST RATE

If the rate of interest $\boldsymbol{i}$ is unknown in an equation of value we have the following methods for determining $\boldsymbol{i}$.

8.5.1 Direct Method

When a single payment is involved, the best method is to directly solve for $\boldsymbol{i}$ from the equation of value (using exponential and logarithmic functions). In this situation, the equation of value takes either the form $\boldsymbol{A = P(1 + i)^n}$ or $\boldsymbol{A = Pe^{\delta_n}}$. In the first case, the interest is found by using the exponential function by obtaining $\boldsymbol{i = \left(\frac{A}{P}\right)^{\frac{1}{n}} - 1}$. In the second case, the interest is found using the logarithmic function i.e. by obtaining $\boldsymbol{\delta = \frac{1}{n} ln\left(\frac{A}{P}\right)}$.

8.5.2 Analytical Method

When multiple payments are involved, we consider the equation of value in the form $\boldsymbol{f(i) = 0}$. For instance, let us consider solving for the effective rate of interest that results in the present value of Rs. 1000 at the end of 3 years plus Rs. 2000 at the end of 6 years to be equal to Rs. 2700? This means, the equation of value at time $\boldsymbol{t = 0}$ is $\boldsymbol{2000v^6 + 1000v^3 = 2700}$. If we put $v^3 = i$, it amounts to solving the quadratic equation $2i^2 + i - 2.7 = 0$. Using i we can then go back to calculate v.

8.5.3 Linear Interpolation

Interpolation determines the value of an unknown data point based on the values of known surrounding data points. Linear interpolation is a simple method of interpolation that assumes a straight line (linear) relationship between the known points. It essentially means averaging the two rates over the interpolation period. When the known algebraic methods are not helpful to solve the equation $f(i) = 0$, we use the linear interpolation methods. Theoretically, it means that if the values of a function $f(x)$ at distinct points x_1 and at x_2 are known, and $f(x_1) \neq f(x_2)$ *and* $|x_1 - x_2|$ is small, the graph of f is approximately linear between x_1 and x_2. This means for $x_1 < x < x_2$, we can assume that:

$$f(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

If we now wish to determine an approximate value of $\boldsymbol{x}$ where $f(x_2)$ has a specific value $\boldsymbol{y_0}$, we can use the above equation to obtain:

$\boldsymbol{x = x_1 + (x_2 - x_1)\frac{y_0 - f(x_1)}{f(x_2) - f(x_1)}}$. We can now find a zero of $\boldsymbol{f}$ near the point $\boldsymbol{y_0 = 0}$. A good approximation is: $\boldsymbol{x \approx x_1 - f(x_1)\frac{x_2 - x_1}{f(x_2) - f(x_1)}}$. Applying this for points between $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$, we can perform the linear interpolation.

8.5.4 Successive Iteration Method

Iteration method can be applied when an equation of the form $i = g(i)$ exists and converges to the true value of i. We assume some starting value i_0, then generate a value i_1 and so on. This is like:

$$i_1 = g(i_0) \rightarrow i_2 = g(i_1) \cdots \rightarrow i_n = g(i_{n-1}) \cdots$$

If the iteration converges, then the successive $i_0, i_1, i_2, \ldots$ will converge to the true value i. There are two iteration methods viz. the 'bisection method' and the 'Newton-Raphson method'. The bisection method is based on the result that a differentiable function f satisfying $f(\alpha)f(\beta) < 0$ must satisfy the equation $f(x_0) = 0$ for some x_0 between α and β. The first step in the method therefore consists of finding two starting values $x_0 < x_1$ such that $f(x_0)f(x_1) < 0$. Usually, these values are found by trial and error. In the second step, we bisect the interval using the mid-point $x_2 = \frac{x_0+x_1}{2}$. If $f(x_0)f(x_2) < 0$, then we apply the bisection process to the interval $x_0 \leq x \leq x_2$. We continue the bisection process as many times as necessary to achieve the desired level of accuracy. The problem with the 'bisection method' is that the rate of convergence is slow. An iteration method with a faster rate of convergence is the Newton-Raphson method given by the iteration formula: $j_{n+1} = \frac{j_n - f(j_n)}{f'(j_n)}$.

8.5.5 Illustrations

Let us consider some empirical examples as before.

a) *At what interest rate convertible semi-annually would Rs. 600 accumulate to Rs. 900 in 4 years?*

Let $\boldsymbol{j} = \frac{i^{(2)}}{2}$ be the effective rate for six months. It is given that $\mathbf{600\ (1 + j)^8 = 900. Therefore,} \boldsymbol{j} = \left(\frac{9}{6}\right)^{\frac{1}{8}} - \mathbf{1} \approx \mathbf{0.050199}$.

Thus, $\boldsymbol{i^{(2)}} = \mathbf{2j} = \mathbf{0.104} = \mathbf{10.4}\%$.

b) *Obtain the interest rate convertible semi-annually on an investment of Rs. 1000 now and Rs. 2000 three years from now, so as to accumulate Rs. 5000 ten years from now.*

The equation of value at time $\boldsymbol{t} = \mathbf{10}$ years is

$\mathbf{1000(1 + j)^{20} + 2000(1 + j)^{14} = 5000}$ where $\boldsymbol{j} = \frac{i^{(2)}}{2}$.

To solve for $\boldsymbol{j}$ we equate the above equation, $f(j) = 0$.

Now, we first get $\boldsymbol{f}(\mathbf{0.03}) = \mathbf{-168.71}$ and $\boldsymbol{f}(\mathbf{0.035}) = \mathbf{227.17}$. Since $\boldsymbol{f}$ is continuous, $\boldsymbol{j}$ is between these two values. Using linear interpolation we get $\boldsymbol{j} \approx \mathbf{0.03 + 168.71} \times \frac{0.005}{227.17 + 168.71} \approx \mathbf{0.0321}$ and $\boldsymbol{f}(\mathbf{0.0321}) = \mathbf{-6.11}$. Therefore, $\boldsymbol{i^{(2)}} = \mathbf{2j} = \mathbf{0}.0642 = 6.42\%$.

c) *Find out an interest rate, convertible semi-annually, to yield from an investment of Rs. 1000 now and Rs.2000 three years from now and accumulate to Rs. 5000 ten years from now using the bisection method.*

The equation of value at time $\boldsymbol{t} = \mathbf{10}$ years is $\mathbf{1000(1 + j)^{20} + 2000(1 + j)^{14} = 5000}$, where $\boldsymbol{j} = \frac{i^{(2)}}{2}$. Hence: $\boldsymbol{f}(\boldsymbol{j}) = \mathbf{1000(1 + j)^{20} + 2000(1 + j)^{14} - 5000}$ should be equated to zero and solved. Descartes' rule of signs says that the maximum number of positive roots to a polynomial equation is equal to the number of sign changes in the coefficients of the polynomial. In this case, Descartes' rule asserts the existence of one positive root to the equation $\boldsymbol{f}(\boldsymbol{j}) = \mathbf{0}$. So, by trial and error, we get $\boldsymbol{f}(\mathbf{0}) = \mathbf{-2000}$ and $\boldsymbol{f}(\mathbf{0.1}) = \mathbf{9322.50}$. Thus, $\boldsymbol{j}$ is between the two values 0 and 0.1. Let $\boldsymbol{j_2} = \frac{0+0.1}{2} = \mathbf{0.05}$. Then $\boldsymbol{f}(\mathbf{0.05}) =$

$\mathbf{1613.16}$ and $\boldsymbol{f(0)f(0.05)} < 0$. Bisecting the interval $[\mathbf{0, 0.05}]$ by the point we get $\boldsymbol{j_3} = \mathbf{0.5(0 + 0.05)} = \mathbf{0.025}$. We now proceed to generate the following Table of *f*(*j*) values.

n	$\boldsymbol{j_n}$	$f(\boldsymbol{j_n})$
0	0	–2000
1	0.1	9322.50
2	0.05	1613.16
3	0.025	–535.44
4	0.0375	436.75
5	0.03125	–72.55
6	0.034375	176.02
7	0.0328125	50.25
8	0.03203125	–11.52
9	0.032421875	19.27
10	0.0322265625	3.852
11	0.0321289063	–3.84
12	0.0321777344	0.005

Thus, $\boldsymbol{j} \approx \mathbf{0.032178}$ accurate to six decimal places as compared to 0.0321 found in the previous example. Hence, $\boldsymbol{i^2} = \mathbf{2_j} = \mathbf{0.06436} = \mathbf{6.436}\%$

8.6 LET US SUM UP

In this unit, we have discussed the time value of money. Different techniques used to determine the value of money transacted are outlined. Value of money problems involve two fundamental techniques: compounding and discounting. These two processes are used to compare money in our pocket today vis-à-vis the money we have to wait to receive at some time in the future. Thus, compounding is about moving money forwards in time. It is the process of determining the future value of an investment made today and/or the future value of a series of equal payments made over time (periodic payments). Discounting, on the other hand is about moving money backwards in time. It is the process of determining the present value of money to be received in the future (as a lump sum and/or as periodic payments). Present value is determined by applying a discount rate to the sums of money to be received in the future.

8.7 KEY WORDS

Periods (*n*)	:	The total number of compounding or discounting periods in the holding period.
Rate (*i*)	:	The periodic interest rate or discount rate used in the analysis, usually in expressed as an *annual* percentage.
Present Value (PV)	:	Represents a single sum of money today.
Payment	:	Represents equal periodic payments received or paid each period. When payments are received they are positive, when payments are made they are negative.
Future Value (FV)	:	A one-time single sum of money to be received or paid in the future.
Cash Flow	:	Money that comes in and goes out of an organisation.
Discounted Cash flow (DCF)	:	This is an application of the time-value-of-money concept. The idea here is that the money to be received or paid at some time in the future should be treated as having less value, than an equal amount actually received or paid today.
Time Value of Money	:	This says that the value of money you have now is greater than a reliable promise to receive the same amount of money at a future date.

8.8 SUGGESTED BOOKS FOR FURTHER READING

1) Finan, Marcel B (2018). A Basic Course in the Theory of Interest and Derivatives Markets (see Internet).
2) Slud Eric V (2001). Actuarial Mathematics and Life-Table Statistics, University of Maryland, College Park (see Internet).
3) Veeh, Jerry Alan (2001). Lecture Notes on Actuarial Mathematics (see Internet).

8.9 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

1) It refers to current worth of a future stream of cash flows given a specified rate of return.
2) $v = \frac{1}{1+i}$ is defined as the 'discount factor' since it discounts the value of an investment at the end of a period to its 'present value' at the

beginning of the period. Here, the investment amount is 1 and it is invested at a r.o.i. 'i'.

3) The NPV accounts for the initial capital outlay required to fund a project, making it a net figure, whereas the PV calculation only accounts for cash inflows.

4) It is a measure of interest paid at the 'beginning of the period'. It is defined as: $d = \frac{i}{1+i}$ where 'i' is the effective rate of interest.

5) Six relationships as in Equations (8.1) to (8.6).

Check Your Progress 2

1) It is because, theoretically, the number of times interest is credited could be very large (like ∞).

2) The accumulation function for the 'force of interest at time t' is defined as $\delta_t = \frac{a'(t)}{a(t)}$.

3) Different principal amounts kept for differing lengths of time cannot be compared till we reduce them to a common date called the 'comparison date'. The equation which accumulates or discounts each payment to the comparison date is called the 'equation of value'.

UNIT 9 ANNUITIES

Structure

9.0 OBJECTIVES

After reading this unit, you will be able to:

- define the term ‘annuity’;
- distinguish between the terms ‘annuity-certain’ and ‘contingent annuities’;
- outline the meaning of the term ‘annuity-immediate’ expressing the relationship between their ‘present and accumulated values’;
- explain the meaning of the term ‘annuity-due’ bringing out the relationship between ‘annuity-immediate’ and ‘annuity-due’;
- state the meaning of the term ‘deferred annuity’;
- differentiate between ‘increasing annuity’ and ‘decreasing annuity’ deriving expressions for their present values;
- discuss the concepts of ‘varying interest annuity’ and ‘varying annuity-due’;
- write a note on ‘perpetuity’;

- indicate how the computation of present value is made under a 'continuous annuity'; and
- describe the significance of 'amortization' with a distinction on the methods adopted for balancing the accumulated loan payments.

9.1 INTRODUCTION

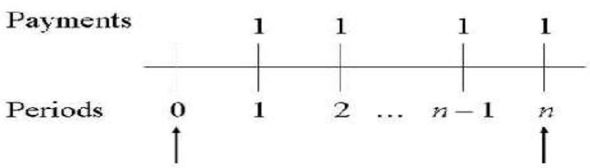
An annuity refers to a series of periodic payments made at equal intervals of time. Examples of annuities are house rent, mortgage payments on homes, installments payments on automobiles, etc. In principle, annuity payments can occur at various times. However, while dealing with standard annuity cases we usually consider equal payments made in equal increments of time. In order to evaluate these payments, we need a set 'points of time' to calculate payments. We use the term 'present time' to refer to the time in which the payment schedule goes into effect. This is not necessarily the time at which the payments start. Most of the standard annuities are *level*, meaning that the payments are equal in amounts. Annuities can be categorised into: (i) regular annuities (or annuity-certain) where payments are guaranteed to occur for a fixed period of time and (ii) contingent annuities, where the beneficiary does not begin receiving payments until a specified event occurs. In case of annuity-certain, the term 'certain' refers to the legal obligation rather than a force of will. If you cease making payments, the lender will recoup the outstanding balance in some other way and/or write it off as a loss i.e. marking the debt as 'paid'.

9.2 TYPES OF ANNUITIES

Broadly, there are two main types of annuities viz. 'annuity – immediate' and 'annuity – due'. There is a third type called 'annuity deferred'. In this section, we will discuss these three types of annuities.

9.2.1 Annuity-Immediate

An annuity-immediate is defined as one under which payments of 1 are made at the end of each period with i as the 'effective rate of interest' for that period. The word 'immediate' conveys that the payments are to be made at the end of the period (which is contrary to a direct meaning of the word immediate). It is also common to assume that the payments cease at some point like n periods (years). The cash stream represented by such an annuity can be visualized on a 'time diagram' as below.



In the figure above, the first arrow shows the beginning of the first period, at the end of which the first payment is due under the annuity. The second arrow indicates the last payment date i.e. just after the last payment has been made. The present value of the annuity-immediate at time $\mathbf{0}$ is denoted by $a_{\overline{n|}\,i}$ or simply $a_{\overline{n|}}$. Using the 'equation of value' with the comparison date kept at time $t = 0$, we can write:

$$a_{\overline{n|}} = v + v^2 + \cdots + v^n \qquad (9.1)$$

i.e. the present value of the annuity is the sum of the present values of each of the n payments. Note that the expression on the right-hand side of (9.1) is a geometric progression. Multiplying both sides of the equation by v we get:

$$va_{\overline{n|}} = v^2 + v^3 \ldots + v^n + v^{n+1} \qquad (9.2)$$

Subtracting (9.2) from (9.1) and simplifying we get:

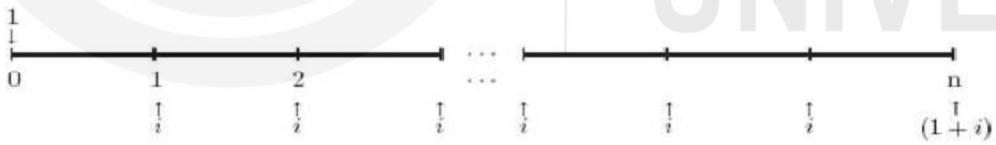
$(1 - v)a_{\overline{n|}} = v(1 - v^n)$. Hence:

$$a_{\overline{n|}} = v.\frac{1-v^n}{1-v} = v.\frac{1-v^n}{iv} = \frac{1-(1+i)^{-n}}{i} \quad \text{(since } v = \frac{1}{1+i}\text{)} \qquad (9.3)$$

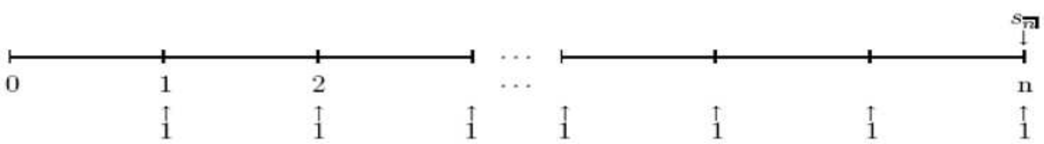
If payments are P instead of 1, then the present value of the annuity (i.e. the present value of all payments) is $Pa_{\overline{n|}}$. It means the certainty of payments in amounts of P made according to the schedule is worth $Pa_{\overline{n|}}$ presently. Note that equation (9.3) is same as:

$$1 = v^n + ia_{\overline{n|}} \qquad (9.4)$$

Equation (9.4) is the equation of value at time $t = 0$ of an investment of 1 for n periods during which an interest of i is received at the end of each period and is reinvested at the same rate i and at the end of the n periods the original investment of 1 is returned. The time diagram of such a transaction is as below.



Let us now determine the accumulated value of an 'annuity-immediate' right after the n^{th} payment is made. It is denoted by $s_{\overline{n|}}$ as in the time diagram below.



Writing the equation of value at the comparison date $t = n$ we find $s_{\overline{n|}} = 1 + (1 + i) + (1 + i)^2 + \cdots + (1 + i)^{n-1}$. That is, $s_{\overline{n|}}$ is the sum of the accumulated value of each of the n payments. We can write $s_{\overline{n|}} = 1 + (1 + i) + (1 + i)^2 + \cdots + (1 + i)^{n-1} = \frac{(1+i)^n-1}{(1+i)-1} = \frac{(1+i)^n-1}{i}$.

The relationships between the Present and the Accumulated values of 'annuity-immediate', for given $a_{\overline{n|}}$ and $s_{\overline{n|}}$, can be expressed as follows:

i) $\boldsymbol{s_{\overline{n|}} = (1+i)^n a_{\overline{n|}}}$

[since $s_{\overline{n|}} = \frac{(1+i)^n-1}{i} = (1+i)^n.\frac{1-(1+i)^{-n}}{i} = (1+i)^n a_{\overline{n|}}$]

(**i. e.** the accumulated value of a principal of $\mathbf{a_{\overline{n|}}}$ after '$\mathbf{n}$' periods is $\mathbf{s_{\overline{n|}}}$.)

ii) $\frac{1}{a_{\overline{n|}}} = \frac{1}{s_{\overline{n|}}} + i$ [since $\frac{1}{s_{\overline{n|}}} + i = \frac{i}{(1+i)^n-1} + i = \frac{+i1+i^n-i}{(1+i)^n-1} = \frac{i}{1-v^n} = \frac{1}{a_{\overline{n|}}}$].

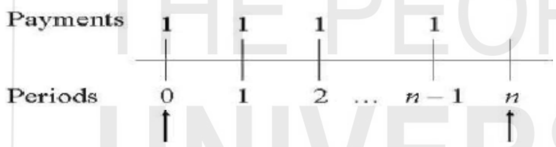
Therefore, we have:

$$s_{\overline{n|}} = (1+i)^n a_{\overline{n|}} \qquad (9.5a)$$

$$\frac{1}{a_{\overline{n|}}} = \frac{1}{s_{\overline{n|}}} + i \qquad (9.5b)$$

9.2.2 Annuity-Due

The word 'due' conveys that payments are made at the beginning of the year. Hence, in case of an annuity-due, payments are made at the beginning of the payment periods. Assume that a series of payments are made in unit amount (Re.1) at the beginning of every period (or year) and that i is the effective rate of interest for that period (or year). Let us also assume that payments cease after some fixed number of years n. The cash stream represented by the annuity can be visualised as in the time diagram below.



In the time diagram above, the first arrow shows the beginning of the first period at which the first payment is made under the annuity. The second arrow shows n^{th} period i.e. beginning of the last payment period. Let i denote the interest rate per period. The present value of the 'annuity-due' at time '0' will be denoted by $\ddot{a}_{\overline{n|}}$ · To determine $\ddot{a}_{\overline{n|}}$, we consider the equation of value at time $t = 0$ as: $\ddot{a}_{\overline{n|}} = 1 + v + v^2 ... + v^{n-1}$. That is, $\ddot{a}_{\overline{n|}}$ is equal to the sum of the present values of each of the n payments. Hence, if payments are P instead of Re.1, then the present value of the annuity is $P\ddot{a}_{\overline{n|}}$ · The right-hand side in $\ddot{a}_{\overline{n|}}$. is a geometric progression. Multiplying both sides by v we obtain, $v\ddot{a}_{\overline{n|}}. = 1 + v + v^2 + v^3 ... + v^{n-1} + v^n$. Subtracting $v\ddot{a}_{\overline{n|}}.$ from the equation for $\ddot{a}_{\overline{n|}}$ we get $(1-v)\ddot{a}_{\overline{n|}}. = (1-v^n)$. Hence:

$$\ddot{a}_{\overline{n|}} = \frac{1-v^n}{1-v} \qquad (9.6)$$

Since $1 - v = d$, we have $\ddot{a}_{\overline{n|}} = \frac{1-v^n}{d} = \frac{1-(1+i)^{-n}}{d} = \frac{1-(1-d)^n}{d}$ (9.6a)

We, therefore, have the relationship between the present and the accumulated values of annuity-due as:

a) $\ddot{s}_{\overline{n}|} = \frac{(1+i)^n-1}{d} = (1+i)^n + \frac{1-(1+i)^{-n}}{d} = (1+i)^n \ddot{a}_{\overline{n}|}$ (9.7a)

b) $\frac{1}{\ddot{s}_{\overline{n}|}} + d = \frac{d}{(1+i)^{n-}} + d\frac{(1+i)^n-1}{(1+i)^n-1} = \frac{d+d[(1+i)^n-1]}{(1+i)^n-1} = \frac{d^{(1+i)^n}}{(1+i)^n-1} = \frac{d}{1-(1+i)^{-n}} = \frac{1}{\ddot{a}_{\overline{n}|}}$

(9.7b)

Equation (9.7), in particular, states that if the present value at time 0, $\ddot{a}_{\overline{n}|}$, is accumulated forward to time *n,* then we will have its future value, $\ddot{s}_{\overline{n}|}$. Now, four relationships between 'annuity-immediate' and 'annuity-due' can be easily arrived at as follows.

a) Since $d = \frac{i}{i+1}$, we have $\ddot{a}_{\overline{n}|} = \frac{1-(1+i)^{-n}}{d} = (1+i).\frac{1-(1+i)^{-n}}{i} = (1+i)a_{\overline{n}|}$.

b) Since $d = \frac{i}{i+1}$, we have: $\ddot{s}_{\overline{n}|} = \frac{(1+i)^n-1}{d} = (1+i).\frac{(1+i)^n-1}{i} = (1+i)s_{\overline{n}|}$

c) Since $\ddot{a}_{\overline{n}|} = \frac{1-(1+i)^{-n}}{d} = \frac{i+}{i}[1-(1+i)^{-n}] = \frac{1-(1+i)^{-n+1}+i}{i} = 1 + a_{\overline{n-1}|}$.

d) Since $\ddot{s}_{\overline{n-1}|} = \frac{(1+i)^{-n}-1}{d} = \frac{i+1}{i}[(1+i)^{n-1}-1] = \frac{(1+i)^{-n}-1-i}{i} = s_{\overline{n}|}$ -1

Thus, the four relationships are:

a) $\ddot{a}_{\overline{n}|} = (1+i)\, a_{\overline{n}|}$ (9.8a)

b) $\ddot{s}_{\overline{n}|} = (1+i)\, s_{\overline{n}|}$ (9.8b)

c) $\ddot{a}_{\overline{n}|} = 1+a_{\overline{n-1}|}$. (9.8c)

d) $s_{\overline{n}|} = \ddot{s}_{\overline{n-1}|}+1$. (9.8d)

9.2.3 Deferred Annuity

A 'deferred annuity' is an annuity in which the payments start at some future time. A standard deferred annuity immediate is one in which payments are deferred for k periods i.e. its first payment of 1 made at time $k+1$. More precisely, since all payments are made at the end of the time period, the first payment of 1 is also made at the end of year $k+1$. The fact that an annuity has been deferred m years is conveyed by having the symbol '$m|$' appear in the lower left of an annuity symbol. For instance, the n-year term annuity, of which the first *k*-year are deferred for payment, is denoted by $_{k|}\ddot{a}_{\overline{n}|}$. Note that the present value of a *k*-year deferred annuity is just v^k times the present value of the annuity where the payments begin at present. Hence, $_{k|}\ddot{a}_{\overline{n}|} = v^k\ddot{a}_{\overline{n}|}$. From the perspective of a person standing at time k, this deferred annuity-immediate looks like a standard n period annuity-immediate.

9.2.4 Illustrations

Let us now consider some illustrations which will help you to understand the application of above equations in practice.

a) *For a given interest rate i, $a_{\overline{n}|}$= 8.3064 and $s_{\overline{n}|}$ = 14.2068. Calculate (i) i and (ii) n.*

i) $i = \frac{1}{8.3064} - \frac{1}{14.2068} = 5\%$

ii) $n = \frac{1}{\ln(1+i)} \ln\left(\frac{s_{\overline{n}|}}{a_{\overline{n}|}}\right) = 11$

b) *Calculate the future value of an annuity-immediate of amount Rs. 100 paid annually for 5 years at the rate of interest of 9%.*

$$100 s_{\overline{5}|} = 100 \times \frac{(1.09)^5 - 1}{0.09} \approx 598.47$$

c) *Show that* $a_{\overline{m+n}|} = a_{\overline{m}|} + v^m a_{\overline{n}|} = a_{\overline{n}|} + v^n a_{\overline{m}|}$. *Interpret the result.*

$$a_{\overline{m}|} + v^m a_{\overline{n}|} = \frac{1 - v^m}{i} + v^m . \frac{1 - v^n}{i} = \frac{1 - v^m + v^m - v^{m+n}}{i}$$

$$= \frac{1 - v^{m+n}}{i} = a_{\overline{m+.n}|}$$

The present value of the first m payments of an $(m+n)$-year annuity-immediate of 1 is $a_{\overline{m}|}$. The remaining n payments have value $a_{\overline{n}|}$ at time $t = m$. Discounted to the present, it is $v^m a_{\overline{n}|}$ at time $t = 0$.

d) *Calculate the present value of an annuity-immediate of amount Rs. 100 paid annually for 5 years at the rate of interest of 9%.*

$100 a_{\overline{5}|} = 100 \frac{1-(1.09)^{-5}}{0.09} \approx 388.97$.

e) *Calculate the present value of an annuity-due paying annual payments of 1200 for 12 years with the first payment two years from now. The annual effective interest rate is 6%.*

$1200(1.06)^{-2} \ddot{a}_{\overline{12}|} = 1200 \left(\ddot{a}_{\overline{14}|} - \ddot{a}_{\overline{2}|}\right) = 1200(9.8527 - 1.9434) \approx\ 9{,}491.16$

f) *For four years, an annuity pays Rs. 200 at the end of each year with an effective 8% rate of interest. Find the accumulated value of the annuity 3 years after the last payment.*

$$200(1+0.08)^3 s_{\overline{4}|} = 200\left(s_{\overline{7}|} - s_{\overline{3}}\right) = 200(8.9228 - 3.2464)$$
$$= Rs. 1135.28$$

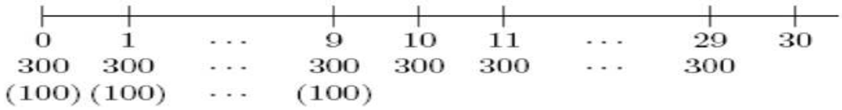
It is also possible to work with annuities-due instead of annuities-immediate. You may verify that $(1+i)^m \ddot{s}_{\overline{n}|} = \ddot{s}_{\overline{m+n}|} - \ddot{s}_{\overline{m}|}$.

g) *A monthly annuity-due pays 100 per month for 12 months. Calculate the accumulated value 24 months after the first payment using a nominal rate of 4% compounded monthly.*

$$100\left(1 + \frac{0.04}{12}\right)^{12} \ddot{s}_{\overline{12}|\frac{0.04}{12}} = 1{,}276.28$$

h) *Over the next 30 years, you deposit money into a retirement account at the beginning of each year. The first 10 payments are 200 each. The remaining 20 payments are 300 each. The effective annual rate of interest is 9%. (i) Find the present value of these payments. (ii) Find* $\ddot{a}_{\overline{n|}}$ *if the effective rate of discount is 10%.*

i) The time diagram of this situation is as follows.



$\therefore 300\ddot{a}_{\overline{30|}} - 100\ddot{a}_{\overline{10|}} = \mathbf{1.09}\left(\mathbf{300}a_{\overline{30|}} - \mathbf{100}a_{\overline{10|}}\right) = \mathbf{Rs. 2659.96}$

ii) Here, d = 0.10. Therefore, $\mathbf{1 = v - d = 0.9}$ **Hence,** $\ddot{a}_{\overline{8|}} = \frac{1-(0.9)^8}{0.1} = \mathbf{5.6953279.}$

j) *Estimate the amount you must invest today at 6% interest rate compounded annually so that you can withdraw Rs.5,000 at the beginning of each year for the next 5 years?*

$5000\ddot{a}_{\overline{n|}} = 5000.\frac{1-(1.06)^{-5}}{0.06(1.06)^{-1}} = \mathbf{22,325.53}.$

Check Your Progress 1 [answer within the space given in about 50-100 words]

1) What is an 'annuity'? Give illustrations.

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2) Distinguish between 'annuity-certain' and 'contingent annuities'.

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3) What does $a_{\overline{n|}}$ and $s_{\overline{n|}}$ denote? How are they related?

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4) What is the expression for the calculation of $a_{\overline{n|}}$?

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5) How is an ‘annuity-due’ different from that of an ‘annuity-immediate’?

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6) What do the expressions $\ddot{a}_{\overline{n|}}$ and $\ddot{s}_{\overline{n|}}$ represent?

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7) Write the expression for calculating $\ddot{a}_{\overline{n|}}$ in terms of ‘i’.

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8) Indicate the expressions for the relationship between the present and accumulated values of ‘annuity-due’.

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9) State the relationships between 'annuity-immediate' and 'annuity-due'.

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10) What is meant by a 'deferred annuity'?

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9.3 INCREASING AND DECREASING ANNUITY

An 'increasing annuity immediate' with a term of n periods pays 1 at the end of the first period, 2 at the end of the second period, 3 at the end of the third period, & so on, so that it pays 'n' at the end of the n^{th} period. The present value of such an annuity is $(Ia)_{\overline{n|}} = \sum_{j=1}^{n} jv^j$. Although this is not a geometric series, using the same technique as used before, we get:

$$(Ia)_{\overline{n|}} - v(Ia)_{\overline{n|}} = v + v^2 + \ldots + v^n - nv^{n+1}.$$

$$\therefore\ (Ia)_{\overline{n|}} = \frac{\left(a_{\overline{n|}} - nv^{n+1}\right)}{1-v} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i}.$$

Following the method of increasing annuity above, a decreasing annuity immediate can also be similarly worked out. For instance, a decreasing annuity immediate with a term of 'n' periods pays n at the end of the first period, $n-1$ at the end of the second period, $n-2$ at the end of the third period, & so on, so that, it pays 1 at the end of the n^{th} period. To find out the present value of such an annuity, $(Da)_{\overline{n|}}$, we take $(Ia)_{\overline{n|}} + (Da)_{\overline{n|}} = (n+1)a_{\overline{n|}}$. This gives, $(Da)_{\overline{n|}} = \frac{(n - a_{\overline{n|}})}{i}$.

Likewise, an annuity immediate with $2n-1$ payments pays 1 at the end of the first period, 2 at the end of the second period, & so on. It pays 'n' at the end of the n^{th} period, $n-1$ at the end of the $(n+1)^{st}$ period, . . . , 1 at the end of the $(2n-1)^{st}$ period. Its present value is therefore:

$$(Ia)_{\overline{n|}} + v^n (Da)_{\overline{n-1|}} = \frac{\left(1 + a_{\overline{n-1|}} - v^n - v^n a_{\overline{n-1|}}\right)}{i} = (1-v^n)\frac{\left(1 + a_{\overline{n-1|}}\right)}{i} = \ddot{a}_{\overline{n|}} a_{\overline{n|}}.$$

9.3.1 Varying Interest Annuity

Let us consider situations in which interest can vary each period with compound interest in effect. Let us denote $\boldsymbol{i_k}$ as the rate of interest applicable from time $\boldsymbol{k-1}$ to time $\boldsymbol{k}$. We first consider the present value of an annuity-immediate for '$\boldsymbol{n}$ −period' with two variations: the first is when $\boldsymbol{i_k}$ is applicable only for period $\boldsymbol{k}$ regardless of when the payment is made i.e. the rate $\boldsymbol{i_k}$ is used only in period $\boldsymbol{k}$ for discounting all payments. In this case, the present value, $\boldsymbol{a_{\overline{n|}}}$, is given by:

$$(1+i_1)^{-1} + (1+i_1)^{-1}(1+i_2)^{-1} + \cdots + (1+i_1)^{-1}(1+i_2)^{-1} \ldots (1+i_n)^{-1}.$$

The second variation we consider is when a payment is made at time $\boldsymbol{k}$ with the rate $\boldsymbol{i_k}$ used as the effective rate of interest for each period $\boldsymbol{i \le k}$. In this case, the present value is:

$$a_{\overline{n|}} = (1+i_1)^{-1}(1+i_2)^{-1} + \cdots + (1+i_n)^{-n}.$$

The present value of annuity-due can be obtained from the present value of annuity immediate by using $\boldsymbol{\ddot{a}_{\overline{n|}} = 1 + a_{\overline{n-1|}}}$. Let us now turn to accumulated values where we will consider an annuity-due. Again, we consider two different situations. If $\boldsymbol{i_k}$ is applicable only for period $\boldsymbol{k}$ regardless of when the payment is made, then the 'accumulated value' is given by:

$$\ddot{s}_{\overline{n|}} = (1+i_1)(1+i_2) \ldots (1+i_n) + \cdots + (1+i_{n-1})(1+i_n) + (1+i_n).$$

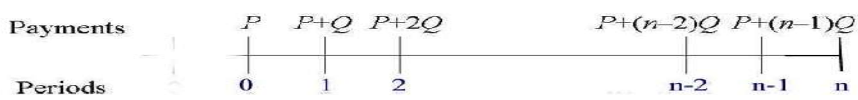
For $\boldsymbol{i_k}$ (applicable for all periods $\boldsymbol{i \le k}$), the 'accumulated value' is:

$$\boldsymbol{\ddot{s}_{\overline{n|}} = (1+i_1)^n + (1+i_2)^{n-1} + \cdots + (1+i_n)}.$$

Now, accumulated values of annuity-immediate can be obtained from the accumulated values of annuity-due by using the relationship: $\boldsymbol{s_{\overline{n+1|}} = \ddot{s}_{\overline{n|}} + 1}$

9.3.2 Varying Annuity-Due

Let us consider the case of an increasing annuity-due where an annuity with the first payment is $\boldsymbol{P}$ at the beginning of year 1 and the payments increase by $\boldsymbol{Q}$ thereafter, continuing for $\boldsymbol{n}$ years. A time diagram of this situation is as below.



The present value for this annuity-due is:

$$\boldsymbol{PV = P + (P+Q)v + (P+2Q)v^2 + \cdots + [P+(n-1)Q]v^{n-1}} \quad \textbf{(9.9a)}$$

Multiplying by $\boldsymbol{v}$, we get

$$\boldsymbol{vPV = Pv + (P+Q)v^2 + (P+2Q)v^3 + \cdots + [P+(n-1)Q]v^{n}} \quad \textbf{(9.9b)}$$

Subtracting (9.9b) from (9.9a) we get:

$$(1-v)PV = P(1-v^n) + (v + v^2 + \cdots + v^n)Q - nv^nQ$$

$$\text{i.e.}\ PV = P\ddot{a}_{\overline{n}|} + Q\frac{\left[a_{\overline{n}|} - nv^n\right]}{d} \qquad (9.9c)$$

The 'accumulated value' of these payments at time n is:

$$AV = (1+i)^n PV = P\ddot{s}_{\overline{n}|} + Q\frac{\left[s_{\overline{n}|} - n\right]}{d} \qquad (9.10a)$$

In the special case when $P = Q = 1$ we have:

$$(I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d} \qquad (9.10b)$$

$$\text{and}\ (I\ddot{s})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d} = \frac{s_{\overline{n+1}|} - (n+1)}{d} \qquad (9.10c)$$

9.3.3 Illustrations

Let us now consider some empirical illustrations as before.

a) *Find the accumulated value of a 12-year annuity-immediate of Rs.500 per year, if the effective rate of interest (for all money) is 8% for the first 3 years, 6% for the following 5 years, and 4% for the last 4 years.*

The accumulated value of the first 3 payments to the end of year 3 is:

$$500s_{\overline{3}|0.08} = 500(3.2464) = Rs.1623.20$$

The accumulated value of the first 3 payments to the end of year 8 at 6% and then to the end of year 12 at 4% is:

$$\mathbf{1623.20(1.06)^5(1.04)^4 = Rs.2541.18.}$$

The accumulated value of payments 4, 5, 6, 7, and 8 at 6% to the end of year 8 is:

$$500s_{\overline{5}|0.06} = 500(5.6371) = Rs.2818.55$$

The accumulated value of payments 4, 5, 6, 7, and 8 to the end of year 12 at 4% is:

$$\mathbf{2818.55(1.04)^4(1.04)^4 = Rs.3297.30.}$$

The accumulated value of payments 9, 10, 11, and 12 to the end of year 12 at 4% is:

$$500s_{\overline{4}|0.04} = 500(4.2464) = Rs.2123.23$$

The accumulated value of the 12-year annuity immediate is:

$$\mathbf{2541.18 + 3297.30 + 2123.23 = Rs.7961.71.}$$

b) *How much must a person deposit now into a special account in order to withdraw Rs.1,000 at the end of each year for the next fifteen years, if the effective rate of interest is equal to 7% for the first five years, and equal to 9% for the last ten years?*

$$PV = 1000\left(a_{\overline{5}|0.07}\right) + a_{\overline{10}|0.09}(1.07)^{-5} = 1000(4.1002 + 4.5757)$$
$$= 8675.90.$$

c) *Determine the present value and the future value of payments of Rs.75 at time 0, Rs.80 at time 1, Rs.85 at time 2, and so on up to Rs.175 at time 20 years. The annual effective rate is 4%.*

The present value is $\boldsymbol{70\ddot{a}_{\overline{21}|} + 5(I\ddot{a})_{\overline{21}|} = Rs.1,720.05}$. The future value is

$$\boldsymbol{(1.04)^{21}(1,720.05) = Rs.3919.60.}$$

Note that in the case of a decreasing annuity-due (where $\boldsymbol{P = n}$ **and** $\boldsymbol{Q = -1}$), the present value at time 0 is $\boldsymbol{(D\ddot{a})_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{d}}$ and the accumulated value at time $\boldsymbol{n}$ is $\boldsymbol{(D\ddot{s})_{\overline{n}|} = (1+i)^n (D\ddot{a})_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{d}}$.

d) *Determine the present value at time 0 of payments of Rs.10 paid at time 0, Rs.20 paid at time 1 year, Rs.30 paid at time 2 years, and so on, assuming an annual effective rate of 5%.*

$$10(I\ddot{a})_{\overline{\infty}|} = \frac{10}{d^2} = 10\left(\frac{1.05}{0.05}\right)^2 = Rs.4{,}410.00.$$

Check Your Progress 2 [answer within the space given in about 50-100 words]

1) Write the expression for obtaining the 'present value' of an 'increasing annuity immediate' and 'decreasing annuity immediate'.

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2) What is the expression for calculating the 'present value' of an 'varying interest annuity' when a rate i_k is applied in period k for discounting all payments? What is the corresponding expression for 'present value' if the rate i_k used as the effective rate of interest for each period $i \leq k$?

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3) Write the expression for the 'present value' and the 'accumulated value' of a 'varying annuity due' where the first payment is P and the successive payments increase by Q for the next n years.

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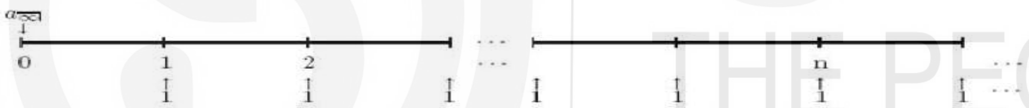
9.4 PERPETUITY

A perpetuity is an annuity whose term is infinite. In other words, it is an annuity whose payments continue forever. The first payment can occur either immediately (perpetuity-due) or one period from now (perpetuity-immediate). The accumulated values of perpetuities do not exist. Let us determine the present value of a perpetuity-immediate at the time of one period before the first payment. Let us assume a payment of 1 is made at the end of each period. The present value, denoted by $\boldsymbol{a_{\overline{\infty}|}}$, bears a time diagram as below.

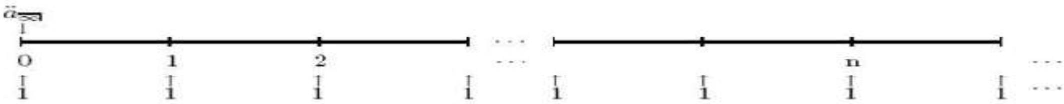


Using the equation of value at time $t = 0$ we find that:

$$\boldsymbol{a_{\overline{\infty}|} = v + v^2 + \ldots} \qquad (9.11)$$

(9.11) is an infinite geometric progression (with $\boldsymbol{v} < 1$). It is equal to $\frac{v}{1-v} = \frac{v}{iv} = \frac{1}{i}$). The verbal interpretation of Equation (9.11) is therefore as follows. If the periodic effective rate of interest is $\boldsymbol{i}$, then one can invest a principal of $\frac{1}{i}$ for one period and obtain a balance of $\boldsymbol{1 + \frac{1}{i}}$ at the end of the first period. A payment of Rs.1 is made and the remaining balance of $\frac{1}{i}$ is reinvested for the next period. This process continues forever. Now, since $\boldsymbol{a_{\overline{n}|}} = \frac{1-v^n}{i}$ and $\boldsymbol{\lim_{n\to\infty} v^n = 0}$ for $0 < \boldsymbol{v} < 1$ we have $\boldsymbol{a_{\overline{\infty}|} = \lim_{n\to\infty} a_{\overline{n}|}} = \frac{1}{i}$.

Analogous to perpetuity-immediate, we can define a perpetuity-due to be an infinite sequence of equal payments where each payment is made at the beginning of the period. If $\boldsymbol{\ddot{a}_{\overline{\infty}|}}$ denotes the present value of such a perpetuity-due at the time of first payment, a time diagram describing the case would be as follows.



The equation of value at time $t = 0$ is:

$$\ddot{\boldsymbol{a}}_{\overline{\infty}|} = 1+\boldsymbol{v}+\boldsymbol{v^2}+\cdots = \frac{\mathbf{1}}{\mathbf{1}-\boldsymbol{v}} = \frac{\mathbf{1}}{\boldsymbol{d}} = \lim_{\boldsymbol{n}\to\infty}\ddot{\boldsymbol{a}}_{\overline{\boldsymbol{n}}|} \qquad (9.12)$$

9.4.1 Continuing Annuity

Theoretically, an annuity could be paid continuously i.e. the annuitant can receive money at a constant rate of 1 rupee per unit of time for ever. The present value of such an annuity that pays 1 per unit time for n time periods can be denoted by $\overline{a}_{\overline{n}|}$. Its value at time 0 can be computed as follows. The value of dt rupees, in the time interval 't to $t + dt$', is $v^t dt = e^{-\delta t}dt$. Hence:

$$\bar{a}_{\overline{n}|} = \int_0^n e^{-\delta t}dt = \frac{1-v^n}{\delta} \qquad (9.13)$$

Annuity payments can be made either more or less often than interest is compounded. In such cases, the equivalent rate of interest can be used to compute the value of the annuity. The symbol $a_{\overline{n}|i}^{(m)}$ denotes the present value of an 'annuity-immediate' that pays $\frac{1}{m}$ at the end of each m^{th} part of a period for n periods under the assumption that the effective interest rate is i per period. For instance, if $m = 12$ and the period is a year, payments of 1/12 are made at the end of each month. We need an expression (a formula) for $a_{\overline{n}|}^{(m)}$ assuming the effective rate of interest as i per period. Notice that the payments are made more frequently than interest is compounded. Hence, using the equivalent rate $i^{(m)}$ makes the computations easy. Using geometric series, we have:

$$a_{\overline{n}|}^{(m)} = \frac{1}{m}\sum_j^{nm}\left(1+\frac{i^{(m)}}{m}\right)^{-j} = \left(\frac{1-v^n}{i^{(m)}}\right) = \frac{ia_{\overline{n}|}}{i^{(m)}} \qquad (9.14)$$

The symbol $\ddot{a}_{\overline{n}|i}^{(m)}$ can be used to denote the present value of an 'annuity-due' that pays $\frac{1}{m}$ at the beginning of each m^{th} part of a period for n periods when the effective periodic interest rate is i. For this, we have $\ddot{a}_{\overline{n}|}^{(m)} = \frac{(1-v^n)}{d^m} = d\ddot{a}_{\overline{n}|}$ as the expression for $\ddot{a}_{\overline{n}|}^{(m)}$ if the effective periodic rate of interest is i. The symbol $(Ia)_{\overline{n}|}^{(m)}$ is the present value of an annuity that pays $\frac{1}{m}$ at the end of each m^{th} part of the first period, $\frac{2}{m}$ at the end of each m^{th} part of the second period, . . . , $\frac{n}{m}$ at the end of each m^{th} part of the nth period, & so on. For $(Ia)_{\overline{n}|}^{(m)}$ the equation is:

$$(Ia)_{\overline{n}|}^{(m)} = \frac{\ddot{a}_{\overline{n}|}^{(m)} - nv^n}{i^{(m)}} \qquad (9.15)$$

The LHS of (9.15) is same as $(I^{(m)}a)_{\overline{n}|}^{(m)}$ and is the present value of the annuity. The annuity pays $1 - m^2$ at the end of the first m^{th} of the first period, $2 - m^2$ at the end of the second m^{th} of the first period,. . . , $\frac{nm}{m^2}$ at the end of last m^{th} of the first period. A computational equation for this is equation (9.15). So far, the value of an annuity has been computed at time 0. Another common time point at which the value of an annuity, consisting of n payments of 1, is computed is time n. Denoting by $s_{\overline{n|}}$ the value of an annuity-immediate at time *n* (i.e. payable immediately after the n^{th} payment), we have $s_{\overline{n|}} = (1 + i)^n_{a_{\overline{n|}}}$.

9.4.2 Amortization

The term 'amortization' indicates the paying off of a debt with a *fixed repayment schedule* in regular installments over a period of time. For instance, you are going to buy a house for which the purchase price is Rs. 100,000 and the down payment is Rs.20,000. You will get the Rs. 80,000 financed by borrowing this amount from a bank at 10% interest with a 30 year term. What is your monthly payment? Typically, such a loan is amortized i.e. you will make equal monthly payments for the life of the loan with each payment consisting partially of interest and partially of principal. From the banks point of view, this transaction represents the purchase by the bank of an annuity-immediate. The monthly payment, p, is thus the solution of the equation $80000 = pa_{\overline{360|}\frac{0.10}{12}}$. In this setting, the quoted interest rate on the loan is assumed to be compounded at the same frequency as the payment period. We find the monthly payment and the total amount of the payment made by $pa_{\overline{360|}\frac{0.10}{12}} = 113.95$ from which we get $p = 702.06$. Thus, the total amount of the payments is $360p = 2{,}52{,}740.60$.

An 'amortization table' is a table which lists the principal and interest portions of each payment for a loan which is being amortized. Such a table can be constructed as follows. Let us denote by b_k the loan balance immediately after the k^{th} payment and by b_0 the original loan amount. Now, the interest part of the k^{th} payment is $p - ib_{k-1}$ and the principal amount of the k^{th} payment is $(P - ib_{k-1})$ where P is the periodic payment amount. Notice that $b_{k+1} = (1 + i)b_k - P$. These relations allow the rows of the amortization table to be constructed sequentially.

A single row of the amortization table that is desired can be arrived at without constructing the whole table. It is called as the 'prospective method' of calculating the loan balance. The loan balance at any point in time is the present value of the remaining loan payments. The 'prospective method' gives the loan balance immediately after the k^{th} payment as $b_k = Pa_{\overline{n-k|}}$. In the 'retrospective method' the loan balance at any point in time is the accumulated original loan amount minus the accumulated value of the past loan payments. The retrospective method gives the loan balance immediately after the k^{th} payment as $b_k = b_0(1 + i)^k - Ps_{\overline{k|}}$. Either method can be used to find the loan balance at an arbitrary time point. Note that for the retrospective method, $b_0(1 + i)^k - Ps_{\overline{k|}} = b_0 + (ib_0 - P)s_{\overline{k|}}$ is a simple identity of $v^{-n} = 1 + i\, s_{\overline{n|}}$. We may check that the prospective and retrospective methods give the same value as follows. Direct computation using the formula gives $a_{\overline{n-k|}} = v^{-k}\left(a_{\overline{n|}} - a_{\overline{k|}}\right)$ and $P = \frac{b_0}{a_{\overline{n|}}}$ gives $Pa_{\overline{n-k|}} = b_0 v^{-k} \frac{\left(a_{\overline{n|}} - a_{\overline{k|}}\right)}{a_{\overline{n|}}} = b_0(1+i)^k - Ps_{\overline{k|}}$. A further bit of insight is obtained by examining the case in which the loan amount is $a_{\overline{n|}}$ so that each loan payment is 1. In this case, the interest part of the k^{th} payment is $ia_{\overline{n-k+1|}} = 1 - v^{n-k+1}$ and the principal part of the k^{th} payment is v^{n-k+1}. This shows that the principal payments form a geometric series. We may hence note that the prospective and retrospective method can be applied to any series of loan payments.

Sinking Fund: A second way of paying off a loan is by means of a 'sinking fund'. Again, consider Rs. 80,000 is borrowed at 10% annual interest. But now, only the interest is required to be paid each month. The principal amount is to be repaid in full at the end of 30 years. Here, the borrower would accumulate a separate fund, called a sinking fund, which will accumulate to Rs. 80,000 in 30 years. The borrower may earn less than 10% interest (say 5%) compounded monthly. In this scenario, the monthly interest payment is $80000(\frac{0.10}{12}) = 666.67$. The contribution c each month into the sinking fund must satisfy $cs_{\overline{360|}\frac{0.05}{12}} = 80000$ from which we get $c = 96.12$. As expected, the combined payment is higher, since the interest rate earned on the sinking fund is lower than 10%.

9.4.3 Illustrations

Let us consider some empirical examples as before.

a) *Suppose a company issues a stock that pays a dividend at the end of each year of Rs. 10 indefinitely. The company's cost of capital is 6%. What is the value of the stock at the beginning of the year?*

$10.\boldsymbol{a}_{\overline{\infty|}} = 10.\frac{\mathbf{1}}{\mathbf{0.06}}$ = Rs.166.67.

b) *What would you be willing to pay for an infinite stream of Rs. 37 annual payments (cash inflows) beginning now if the interest rate is 8% per annum?*

$37\ddot{\boldsymbol{a}}_{\overline{\infty|}} = \frac{\mathbf{37}}{\mathbf{0.08(1.08)^{-1}}}$ = Rs.499.50.

c) *The present value of a perpetuity paying 1 at the end of every 3 years is* $\frac{\mathbf{125}}{\mathbf{91}}$. *Find i.*

$\frac{\mathbf{125}}{\mathbf{91}} = \frac{\mathbf{1}}{\boldsymbol{is}_{\overline{3|}}} = \frac{\mathbf{1}}{(\mathbf{1}+\boldsymbol{i})^{3}-\mathbf{1}}$. So, $(\mathbf{1}+\boldsymbol{i})^{3} = \frac{\mathbf{91}}{\mathbf{125}} + \mathbf{1} = \frac{\mathbf{216}}{\mathbf{125}}$ Hence, $\boldsymbol{i} = \mathbf{0.20}$.

d) *You are receiving an annuity with payments made continuously at a rate of 1000 per year. The annuity is for 10 years. Calculate the present value of this annuity at an annual effective interest rate of 6%.*

This is a level continuous annuity with $\boldsymbol{i = 6\%}$ *and* $\boldsymbol{\delta = ln\,(1.06)}$. The present value is $\mathbf{1000}\left(\frac{\mathbf{1-v^{10}}}{\boldsymbol{\delta}}\right) = \mathbf{7578.75}$.

e) *A loan of 1000 is being repaid by equal annual installments of 100 together with a smaller final payment at the end of 10 years. If the interest rate is 4%, show that the balance immediately after the fifth payment is* $\mathbf{1000 - 60s_{\overline{5}|}}$.

The retrospective method gives the balance as $1000\boldsymbol{v^{-5}} - 100\boldsymbol{s_{\overline{5}|.04}}$, which re-arranges to the stated quantity using the identity $\boldsymbol{v^{-n} = 1 + is_{\overline{n}|}}$.

f) *A loan of 1200 is to be repaid over 20 years. The borrower is to make annual payments of 100 at the end of each year. The lender receives 5% on the loan for the first 10 years and 6% on the loan balance for the remaining years. After accounting for the interest to be paid, the remainder of the payment of 100 is deposited in a sinking fund earning 3%. What is the loan balance still due at the end of 20 years?*

The amount in the sinking fund at the end of 20 years is $\mathbf{40s_{\overline{10}|}(1.03)^{10} + 28s_{\overline{10}|} = 937.25.}$ Hence, the loan balance is $\mathbf{1200 - 937.25 = 262.75}$.

Check Your Progress 3 [answer within the space given in about 50-100 words]

1) State the expression for the computation of 'present value' in a continuously paid annuity.

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2) State what do the symbols $(Is)_{\overline{n}|}$ and $(I\ddot{s})_{\overline{n}|}$ represent?

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3) What is an amortization Table?

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4) What is meant by a 'sinking fund'?

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5) Differentiate between 'prospective and retrospective methods' of loan balance.

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9.5 LET US SUM UP

In this unit, we have discussed the concept of an 'annuity'. Annuity is a financial product that pays out a fixed stream of payments to an individual. There are different types of annuities based on the basis of payments. Annuity-immediate is paid at the end of payment periods whereas annuity-due is paid at the beginning of payment periods. These are the two broad types of annuities. Deferred annuities are those whose payments start at some future time. For all these types of annuities, the computation of their present and accumulated values are discussed in the unit. We have also discussed the 'increasing and decreasing annuities' by varying interest rates. Perpetuities, which are a constant stream of identical cash flows with no end, and continuous annuities (where the annuitant receives payment at a constant rate per unit time), are also discussed in the unit. The concept of amortization, which refers to a plan of paying off a debt with a fixed repayment schedule, is explained. The use of a sinking fund, where an amount deposited periodically accumulates to help pay off the principal amount, is outlined.

Two methods of balancing a loan account viz. the prospective and retrospective methods are explained.

9.6 KEY WORDS

Annuity	:	A financial product that pays out a fixed stream of payments to an individual.
Annuity-Certain	:	Payments guaranteed to occur for a fixed period of time.
Annuity-Due	:	Payments made at the beginning of payment periods.
Annuity-Immediate	:	Payments made at the end of payment periods, so that the interest accrues between the issue of the annuity and the first payment.
Contingent Annuities	:	This is an arrangement in which the beneficiary does not begin receiving payments until a specified event occurs.
Continuous Perpetuity	:	This is an annuity where the Annuitant receives payment at a constant rate of 1 per unit time.
Deferred Annuity	:	An annuity in which the payments start at some future time.
Level	:	A plan paying interest and principal in such a way that the total is same for each payment.
Perpetuity	:	Payment of a constant stream of identical cash flows with no end.
Prospective Method	:	It calculates the loan balance as the present value of all future payments to be made.
Retrospective Method	:	Calculates the loan balance as the accumulated value of the loan at the time of evaluation minus the accumulated value of all instalments paid up to the time of evaluation.
Sinking Fund	:	An amount deposited periodically so as to accumulate to the principal over the duration of the loan period.

9.7 SUGGESTED BOOKS FOR FURTHER READING

1) Marcel B Finan (2017). A Basic Course in the Theory of Interest and Derivatives Markets (internet).

2) Jerry Alan Veeh (2006). Lecture Notes on the Mathematics of Finance.

9.8 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) Annuity is a series of periodic payments at equal intervals of time. They can be payments received or made. Examples of annuities are house rent, mortgage payments, instalments.

2) In the former, payments are guaranteed in the sense of a 'legal obligation' to pay. In the latter, it begins after the occurrence of an event (e.g. attainment of 60 years, retirement from service).

3) $a_{\overline{n}|}$ denotes the 'present value' of an 'annuity-immediate' at time t = 0. $s_{\overline{n}|}$ denotes the 'accumulated value' of an 'annuity-immediate' after n-payments are made. They are related in terms of the expressions: $\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$ and $s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}$.

4) $a_{\overline{n}|} = \frac{1-(1+i)^{-n}}{i}$ where i is the r.o.i. paid.

5) In the former, payments are made at the beginning of the year. In the latter, it is made at the end of the year.

6) They represent the 'present value' at time t = 0 and the 'accumulated value' after n-payments are made of an 'annuity-due' respectively. Note that having 'double dot' is only for distinction and has no special significance.

7) $\ddot{a}_{\overline{n}|} = \frac{1-(1-d)^n}{d}$ where $d = \frac{i}{i+1}$.

8) $\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d} = (1+i)^n \ddot{a}_{\overline{n}|}$.

9) They are given by Equations (9.8a) to (9.8d).

10) A 'deferred annuity' is an annuity in which the payments start at some future time.

Check Your Progress 2

1) $(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$ and $(Da)_{\overline{n}|} = \frac{(n - a_{\overline{n}|})}{i}$.

2) $a_{\overline{n}|} = (1+i_1)^{-1} + (1+i_1)^{-1}(1+i_2)^{-1} + \cdots + (1+i_1)^{-1}(1+i_2)^{-1} \ldots (1+i_n)^{-1}$ and $a_{\overline{n}|} = (1+i_1)^{-1}(1+i_2)^{-1} + \cdots + (1+i_n)^{-n}$.

3) $PV = P\ddot{a}_{\overline{n}|} + Q\frac{[a_{\overline{n}|} - nv^n]}{d}$ and $AV = (1+i)^n PV = P\ddot{s}_{\overline{n}|} + Q\frac{[s_{\overline{n}|} - n]}{d}$.

Check Your Progress 3

1) Paid at a constant rate of 1 rupee per unit time, the present value is $\int_0^n e^{-\delta t} dt = \frac{1-v^n}{\partial}$.

2) The symbol $(Is)_{\overline{n|}}$ is the value of an increasing annuity immediate computed at time n and $(I\ddot{s})_{\overline{n|}}$ is the value of an increasing annuity due at time n.

3) It is a Table which lists the principal and interest portions of each payment for a loan which is being amortized.

4) It is a fund into which periodic payments are made which, with compound interest earned, will ultimately be sufficient to meet a known future capital commitment or discharge a liability.

5) Retrospective Method for loan balancing is based on payments already made. Contrary to such a method, the prospective method is based on 'looking into the future' i.e. evaluating the value of remaining payments.