UNIT 3

SUBSPACES

Structure

3.1 INTRODUCTION

In BMTC-134, you have seen the concept of a subgroup. We will see an analogous concept, namely that of a subspace. We will also study homomorphisms between vector spaces, quotient spaces which are analogous to the concepts homomorphism between groups and quotient groups, respectively.

Objectives

After studying this unit, you should be able to:

- determine whether a given subset of a vector space is a subspace or not;
- explain what the linear span of a subset of a vector space is;
- differentiate between the sum and the direct sum of subspaces;
- define and give examples of cosets and quotient spaces.

3.2 SUBSPACES

In Exercise 23 in Unit 2, you saw that V, a subset of $\mathbb{C}$, was also a vector space. You also saw, in Example 16 in Unit 2, that the subset

$$V = \{(x, y) \in \mathbb{R}^2 \mid y = 5x\},$$

of the vector space $\mathbb{R}^2$, is itself a vector space under the same operations as those in $\mathbb{R}^2$. In these cases V is a subspace of $\mathbb{R}^2$. Let us see what this means.

Definition 1: Let V be a vector space and $W \subseteq V$. If W is also a vector space under the same operations as those in V, we say that W is a subspace of V.

The following theorem gives the criterion for a subset to be a subspace.

Theorem 1: A non-empty subset W, of a vector space V over field F, is a subspace of V provided

a) $w_1 + w_2 \in W, \ \forall w_1, w_2 \in W$

b) $\alpha w \in W \ \forall \alpha \in F$ and $w \in W$.

c) **0,** the additive identity of V, also belongs to W.

Proof: We have to show that the properties VS1 –VS10 hold for W.

VS1 is true because of (a) given above.

VS2 and VS5 are true for elements of W because they are true for elements of V.

VS3 is true because of (c) above.

VS4 is true because, if $w \in W$ then $(-1)w = -w \in W$, by (b) above.

VS6 is true because of (b) above.

VS7 to VS10 hold true because they are true for V. ■

A non-empty subset of a vector space is a subspace iff it is closed under vector addition and scalar multiplication.

Therefore, W is a vector space in its own right, and hence, it is a subspace of V.

The next theorem says that condition (c) in Theorem 1 is unnecessary.

Theorem 2: A non-empty subset W, of a vector space V over a field F, is a subspace of V if and only if

a) $w_1 \in W, w_2 \in W \Rightarrow w_1 + w_2 \in W$

b) $\alpha \in F, \ w \in W \Rightarrow \ \alpha w \in W$.

Proof: If W is a subspace, then obviously (a) and (b) are satisfied.

Conversely, suppose (a) and (b) are satisfied. To show that W is a subspace of V, Theorem 1 says that we only need to prove that $\mathbf{0} \in W$. Since W is non-empty, there is some $w \in W$. Then, by (b) , $0.w \in W$, i.e., $\mathbf{0} \in W$.

This completes the proof of the theorem. ■

Actually both the condition in Theorem 2 can be merged to give the following compact result.

Theorem 3: A non-empty subset W, of a vector space V over the field F, is a subspace of V if and only if

$$\alpha w_1 + \beta w_2 \in W \, \forall \, \alpha, \beta \in F \text{ and } w_1, w_2 \in W$$

Proof: Firstly, suppose W is a subspace of V. Then, by Theorem 2, for any $\alpha, \beta \in F$ and $w_1, w_2 \in W$, we have $\alpha w_1 \in W$ and $\beta w_2 \in W$, so that $\alpha w_1 + \beta \, w_2 \in W$.

Conversely, suppose $\alpha w_1 + \beta w_2 \in W \, \forall \, \alpha, \beta \in F$ and $w_1, w_2 \in W$. Then, in particular, for $\alpha = 1 = \beta$ (remember $1 \in F$), $w_1 + w_2 \in W$. Also, if we put $\beta = 0$ in $\alpha w_1 + \beta w_2$, we get $\alpha w_1 \in W \, \forall \, \alpha \in F$ and $w_1 \in W$,. By Theorem 2, W is a subspace.

Hence, the theorem is proved. ■

Let us use this theorem to obtain some more examples of vector spaces.

Example 1: Prove that the subset

$$W = \{(x, 2x, 3x) \mid x \in \mathbb{R}\}$$

of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

Solution: If we take $x = 0$, we see that $(0, 0, 0) \in W$, so $W \neq \varnothing$. (Remember $\varnothing$ denotes the empty set.)

Next, $w_1 \in W, w_2 \in W \Rightarrow w_1 = (x, 2x, 3x)$, $w_2 = (y, 2y, 3y)$, where $x \in \mathbb{R}$, $y \in \mathbb{R}$. Thus $\alpha \, w_1 = (\alpha x, \alpha 2x, \alpha 3x)$ and $\beta w_2 = (\beta y, 2\beta y, 3\beta y)$, for $\alpha, \beta \in \mathbb{R}$.

$$\Rightarrow \alpha w_1 + \beta w_2 = (\alpha x + \beta y, 2(\alpha x + \beta y), 3(\alpha x + \beta y))$$
$$\Rightarrow \alpha w_1 + \beta w_2 = (z, 2z, 3z), \text{ where } z = \alpha x + \beta y \in \mathbb{R}$$
$$\Rightarrow \alpha w_1 + \beta w_2 \in W.$$

Hence, by Theorem 3, W is a subspace of $\mathbb{R}^3$.

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Example 2: Which of the following subsets W of $\mathbb{R}^4$ are subspaces of $\mathbb{R}^4$?

The set of all $w = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that

a) $x_1 = 0$,

b) $x_2 = 1$,

c) $x_3 < 0$,

d) $2x_1 + 5x_4 = 0$.

Solution:

a) Here, $W = \{(0, x_2, x_3, x_4) \mid x_2, x_3, x_4 \in \mathbb{R}\}$. In

Obviously, $W \neq \emptyset$as $(0,0,0,0) \in W$.

Next, $w_1, w_2 \in W \Rightarrow w_1 = (0, x_2, x_3, x_4), x_i \in \mathbb{R}$, for $i = 2, 3, 4$ and $w_2 = (0, y_2, y_3, y_4)$, $y_i \in \mathbb{R}$, for $i = 2, 3, 4$

$\Rightarrow \alpha w_1 = (0, \alpha x_2, \alpha x_3, \alpha x_4)$ and $\beta w_2 = (0, \beta y_2, \beta y_3, \beta y_4)$, $\alpha, \beta \in \mathbb{R}$.

$\Rightarrow \alpha w_1 + \beta w_2 = (0, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3, \alpha x_4 + \beta y_4)$

$= (0, u_2, u_3, u_4)$

where $u_2 = \alpha x_2 + \beta y_2$, $u_3 = \alpha x_3 + \beta y_3$ and $u_4 = \alpha x_4 + \beta y_4$. Hence W is a subspace of $\mathbb{R}^4$.

b) Here, $W = \{(x_1, 1, x_3, x_4) \mid x_1, x_2, x_3 \in \mathbb{R}\}$

Again $W \neq \emptyset$ as $(1,1,1,1) \in W$

Now $w_1 \in W$, $w_2 \in W \Rightarrow w_1 = (x_1, 1, x_3, x_4)$, $w_2 = (y_1, 1, y_3, y_4)$

$\Rightarrow w_1 + w_2 = (x_1 + y_1, 2, x_3 + y_3, x_4 + y_4)$

$\Rightarrow w_1 + w_2 \notin W$

since the second component is 2 and not 1. So W is not a subspace of $\mathbb{R}^4$.

Note: An easier proof for (b) would be:

$0 \cdot w = (0,0,0,0) \notin W$; $\therefore$, W is not a subspace.

c) Here, $W = \{(x_1, x_2, x_3, x_4) \mid x_i \in \mathbb{R}, x_3 < 0\}$. As in b), $(0,0,0,0) \notin W$ since $x_3 = 0 \not< 0$

d) Now, $W = \{(x_1, x_2, x_3, x_4) \mid x_i \in \mathbb{R}, 2x_1 + 5x_4 = 0\}$

Obviously $(0,0,0,0) \in W$, so $W \neq \emptyset$. Next,

$w_1 \in W, w_2 \in W \Rightarrow w_1 = (x_1, x_2, x_3, x_4)$ with $2x_1 + 5x_4 = 0$

And $w_2 = (y_1, y_2, y_3, y_4)$ with $2y_1 + 5y_4 = 0$

$\Rightarrow w_1 + w_2 = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)$

with

$$2(x_1 + y_1) + 5(x_4 + y_4) = (2x_1 + 5x_4) + (2y_1 + 5y_4) = 0 + 0 = 0.$$

$\Rightarrow w_1 + w_2 \in W$

Finally,

$\alpha \in \mathbb{R}$, $w \in W \Rightarrow \alpha \in \mathbb{R}$, $w = (x_1, x_2, x_3, x_4)$ with $2x_1 + 5x_4 = 0$

$\Rightarrow \alpha w = (\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4)$ with $2(\alpha x_1) + 5(\alpha x_4) = \alpha(2x_1 + 5x_4) = 0$.

$\Rightarrow \alpha w \in W$.

So W is a subspace of $\mathbb{R}^4$.

Note: We could have also solved (d) by using Theorem 3 as follows:

For $\alpha, \beta \in \mathbb{R}$ and (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) in W we have

$\alpha(x_1, x_2, x_3, x_4) + \beta(y_1, y_2, y_3, y_4) = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3, \alpha x_4 + \beta y_4)$

with $2(\alpha x_1 + \beta y_1) + 5(\alpha x_4 + \beta y_4) = \alpha(2x_1 + 5x_4) + \beta(2y_1 + 5y_4) = 0$.

Thus, $\alpha, \beta \in \mathbb{R}$ and $w_1, w_2 \in W \Rightarrow \alpha w_1 + \beta w_2 \in W$.

This shows that W is a subspace of $\mathbb{R}^4$.

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Example 3: Let V be a vector space over F and $v \in V$.

All scalar multiples of a fixed vector form a subspace.

Show that the subset $F\,v = \{\alpha v \mid \alpha \in F\}$ is a subspace of V.

Solution: $Fv \neq \varnothing$ because $0.v = \mathbf{0} \in Fv$.

Now, if αv and $\beta v \in F\,v$ then $\alpha v + \beta v = (\alpha + \beta) v \in Fv$.

Also, $\alpha \in F$ and $\beta v \in F\,v \Rightarrow \alpha(\beta v) = (\alpha\beta) v \in Fv$, since $\alpha\beta \in F$.

Thus, by Theorem 2, F v is a subspace of V.

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Note: The subspace $\mathbb{R}v$, of $\mathbb{R}^n$, represents a line in n-dimensional space.

E1) Prove that $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 + x_2 - x_3 = 0\}$ is a subspace of $\mathbb{R}^3$.

E2) For each of the following subsets W of $\mathbb{R}^3$, determine whether it is subspace of $\mathbb{R}^3$. W is a set of those vectors (x_1, x_2, x_3) in $\mathbb{R}^3$ such that

a) $x_1 = -x_2$;

b) $x_1^2 \geq 0$;

c) $x_1 x_2 = 0$;

d) $x_1 + x_2 + x_3 = 1$.

E3) Show that $\{\mathbf{0}\}$ is a subspace of the vector space V over F.

In Exercise 28 of Unit 2, we saw that $M_{n,m}(F)$, the set of $n \times m$ matrices over the field F forms a vector space over F. In the next example, we will see some subspaces of $M_n(F)$, the vector space of $n \times n$ matrices over F.

Example 4: Show that, the following subsets form a subspace of $M_n(F)$:

a) The set of upper triangular matrices

$$S_1 = \left\{ (a_{ij})_{n,n} \in M_{n,n}(F) \middle| a_{ij} \in F, a_{ij} = 0 \text{ if } i \leq j \right\}$$

b) The set of lower triangular matrices

$$S_2 = \left\{ (a_{ij})_{n,n} \in M_{n,n}(F) \middle| a_{ij} \in F, a_{ij} = 0 \text{ if } i \geq j \right\}$$

c) The set of symmetric matrices

$$S_3 = \left\{ (a_{ij})_{n,n} \in M_{n,n}(F) \middle| a_{ij} \in F, a_{ij} = a_{ji} \ \forall\ 1 \leq i \leq n,\ 1 \leq j \leq n \right\}$$

d) The set of skew-symmetric matrices

$$S_4 = \left\{ (a_{ij})_{n,n} \in M_{n,n}(F) \middle| a_{ij} \in F, a_{ij} = -a_{ji} \ \forall\ 1 \leq i \leq n,\ 1 \leq j \leq n \right\}$$

where $\text{char}(F) \neq 2$. (Note that $a = -a$ for all $a \in F$ if $\text{char}(F) = 2$.)

Solution:

a) Let α, $\beta \in F$ and $(a_{ij})_{n,n}$, $(b_{ij})_{n.n} \in S_1$. We have

$$\alpha (a_{ij})_{n,n} + \beta (b_{ij})_{n,n} = (\alpha a_{ij} + \beta b_{ij})_{n,n}$$

Therefore, the entry in the i^{th} row, jth column of $\alpha (a_{ij})_{n,n} + \beta (b_{ij})_{n,n}$ is $\alpha a_{ij} + \beta b_{ij}$ For each i, j, $1 \le i, j \le n$, we have

$$\alpha a_{ij} + \beta b_{ij} = 0 \text{ if } i \le j$$

since $a_{ij} = 0$, $b_{ij} = 0$ if $i \le j$. It follows that the entry in the i^{th} row, j^{th} column of $\alpha (a_{ij})_{n,n} + \beta (b_{ij})_{n,n}$ is zero if $i \le j$, i.e. $\alpha (a_{ij})_{n,n} + \beta (b_{ij})_{n,n}$ is also an upper triangular matrix.

b) Let α, $\beta \in F$ and $(a_{ij})_{n,n}$, $(b_{ij})_{n.n} \in S_2$. We have

$$\alpha (a_{ij})_{n,n} + \beta (b_{ij})_{n,n} = (\alpha a_{ij} + \beta b_{ij})_{n,n}$$

Therefore, the entry in the i^{th} row, jth column of $\alpha (a_{ij})_{n,n} + \beta (b_{ij})_{n,n}$ is $\alpha a_{ij} + \beta b_{ij}$ For each i, j, $1 \le i, j \le n$, we have

$$\alpha a_{ij} + \beta b_{ij} = 0 \text{ if } i \ge j$$

since $a_{ij} = 0$, $b_{ij} = 0$ if $i \ge j$. It follows that the entry in the i^{th} row, j^{th} column of $\alpha (a_{ij})_{n,n} + \beta (b_{ij})_{n,n}$ is zero if $i \ge j$, i.e. $\alpha (a_{ij})_{n,n} + \beta (b_{ij})_{n,n}$ is also a lower triangular matrix.

c) Let $(a_{ij})_{n,n}$, $(b_{ij}) \in S_3$. Then, we have $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$ for $1 \le i, j \le n$. So, $\alpha a_{ij} + \beta b_{ij} = (\alpha a_{ji} + \beta b_{ji})$ for $1 \le i, j \le n$. So, it follows that $\alpha (a_{ij}) + \beta (b_{ij})$ is a symmetric matrix if $(a_{ij})_{n,n}$ and $(b_{ij})_{n,n}$ are symmetric.

d) Let $(a_{ij})_{n,n}$, $(b_{ij}) \in S_3$. Then, we have $a_{ij} = -a_{ji}$, $b_{ij} = -b_{ji}$ for $1 \le i, j \le n$. So, $\alpha a_{ij} + \beta b_{ij} = -(\alpha a_{ji} + \beta b_{ji})$ for $1 \le i, j \le n$. So, it follows that $\alpha (a_{ij}) + \beta (b_{ij})$ is a skew-symmetric matrix if $(a_{ij})_{n,n}$ and $(b_{ij})_{n,n}$ are skew-symmetric.

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Check your understanding of this example by solving the next exericse.

E4) Recall from Remark 5, Unit 2, that $M_n(\mathbb{C})$ is also a vector space over $\mathbb{R}$. Check that the set of $n \times n$ hermitian and skew-hermitian matrices form subspaces of $M_n(\mathbb{C})$ considered as vector spaces over $\mathbb{R}$.

In Example 3 you saw that an element $v \in V$ gives rise to a subspace of V. In the next section we look at such subspaces of V, which grow out of subsets of V that are much smaller than the concerned subspace.

3.3 LINEAR COMBINATIONS

In Unit 2 you came across the fact that any element of $\mathbb{R}^3$ could be written as $a.\mathbf{i} + b.\mathbf{j} + c.\mathbf{k}$, where $a, b, c \in \mathbb{R}$. In this section we will generalise this. Consider the following definition.

Definition 2: If $v_1, v_2, \ldots, v_n$ are elements of a vector space over F and $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$, then the vector

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

is called a **linear combination** of the vectors $v_1, v_2, \ldots, v_n$, or of the set $\{v_1, v_2, \ldots, v_n\}$.

For instance, since

$$(2, 4, 3) = 2(1, -1, 0) + 3(0, 2, 1), (2, 4, 3)$$

is a linear combination of $(1, -1, 0)$ and $(0, 2, 1)$

We are now ready to generalize the result of Example 3.

Theorem 4: If $v_1, v_2, \ldots, v_n$ belong to a vector space V over a field F, then $W = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \mid \alpha_i \text{ are scalars}\}$ is a subspace of V.

Proof: Firstly, 0 is a scalar and

$$0.v_1 + 0.v_2 + \cdots + 0.v_n = \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0}$$

So $\mathbf{0} \in W$, and $W \neq \varnothing$

Secondly, $w_1 \in W, w_2 \in W$

$$\Rightarrow w_1 = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \sum_{i=1}^{n} \alpha_i v_i, \alpha_i \in F$$

$$\text{and } w_2 = \beta_1 v_1 + \beta_2 v_2 + \beta_n v_n = \sum_{i=1}^{n} \beta_i v_i \in F.$$

$$\Rightarrow w_1 + w_2 = (\alpha_1 + \beta_1)\ v_1 + \cdots + (\alpha_n + \beta_n)\ v_n \Rightarrow w_1 + w_2 \in W$$

Finally, if α is a scalar, and $w \in W$, we have $w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$, where α_i is scalar $\forall\ i = 1, \ldots, n$.

$$\Rightarrow \alpha w = (\alpha\alpha_1) v_1 + (\alpha\alpha_2) v_2 + \cdots + (\alpha\alpha_n) v_n.$$

$$\Rightarrow \alpha w \in W$$

This proves the theorem. ■

We often denote W (in Theorem 4) by $Fv_1 + \cdots + Fv_n$.

Let us look at the vector space $\mathbb{R}^n$, over $\mathbb{R}$. In this, we see that every vector is a linear combination of the n vectors $e_1 = (1, 0 \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 1)$. This is because $(x_1 \ldots, x_n) = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n, x_i \in \mathbb{R}$. In this case we say that the set $\{e_1, \ldots, e_n\}$ spans $\mathbb{R}^n$. Let us see what spanning means.

Definition 3: Let V be a vector space over F, and let $S \subseteq V$. The linear span of S is defined to be the set of all linear combinations of a finite number of elements of S. It is denoted by [S].

Thus,

$$[S] = \left\{ \sum_{i=1}^{n} \alpha_i v_i \,\middle|\, n \in \mathbb{N}, v_i \in S, \alpha_i \text{ scalars} \right\}$$

We also say that S **generates** [S]. Note that S is only a subset of V, and not necessarily a subspace of V. Also note that [S] is the set of finite sums of the form $\alpha_1 v_1 + \cdots + \alpha_n v_n$ where $\alpha_i \in F$ and $v_i \in S$.

Example 5: Suppose $S \subseteq \mathbb{R}^2$, $S = \{(1,0),(0,1)\}$. What is [S]?

Solution: $[S] = \{\alpha(1,0) + \beta(0,1) \mid \alpha, \beta \in \mathbb{R}\}$, i.e.,

$$[S] = \{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\}.$$

In this case, the linear span of S is the whole of $\mathbb{R}^2$. Thus, $\{(1,0),(0,1)\}$ generates $\mathbb{R}^2$.

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Example 6: Suppose $S \subseteq \mathbb{R}^3$, $S = \{(1,-1,0)\}$. What is [S]?

Solution: $[S] = \{\alpha(1,-1,0) \mid \alpha \in \mathbb{R}\}$.

$$[S] = \{(\alpha, -\alpha, 0) \mid \alpha \in \mathbb{R}\}$$

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Example 7: Let P be the vector space of real polynomials, and $S = \{(x, x^2+1, x^3+1)\} \subseteq \mathbf{P}$. What is [S]?

Solution:

$$[S] = \{\alpha x + \beta(x^2+1) + \tau(x^3+1) \mid \alpha, \beta, \tau \in \mathbb{R}\}$$
$$= \{\tau x^3 + \beta x^2 + \alpha x + (\beta + \tau) \mid \alpha, \beta, \tau \in \mathbb{R}\}$$

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E5) Let $S = \{1, x, x^2\}$ be a subset of **P** in the example above. Does $2x + 3x^3 \in [S]$?

In the examples given above you may have noticed that [S] is a subspace of V. We prove this fact now.

Theorem 5: If S is a non-empty subset of a vector space V over F, then [S] is a subspace of V.

Proof: Since $S \neq \emptyset$ and $S \subseteq [S]$, $[S] \neq \emptyset$. Also, since $S \subseteq V$, every linear combination of elements in S is in V. So, $[S] \subseteq V$.

Now, $s_1 \in [S]$, $s_2 \in [S]$

$$\Rightarrow s_1 = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, \quad \text{for } v_i \in S, \alpha_i \in F$$

and

$$s_2 = \beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_m w_m, \text{ for } w_i \in S, \beta_i \in F$$

Thus, for $\alpha, \beta \in F$,

$$\alpha s_1 = \alpha\alpha_1 v_1 + \alpha\alpha_2 v_2 + \cdots + \alpha\alpha_n,$$
$$\beta s_2 = \beta\beta_1 w_1 + \beta\beta_2 w_2 + \cdots + \beta\beta_m w_m$$
$$\Rightarrow \alpha s_1 + \beta s_2 = \alpha\alpha_1 v_1 + \alpha\alpha_2 v_2 + \cdots + \alpha\alpha_n v_n + \beta\beta_1 w_1 + \beta\beta_2 w_2 + \cdots + \beta\beta_m w_m$$

with $v_i, w_i \in S$ and $\alpha\alpha_i \in F$, $\beta\beta_i \in F$ This shows that $\alpha s_1 + \beta s_2$ is a linear combination of a finite number of elements of S. Thus, $\alpha s_1 + \beta s_2 \in [S]$. Therefore, by Theorem 3, [S] is a subspace of V . ■

Theorem 5 shows that the linear span of S is a subspace containing S. In fact, it is the smallest subspace of V containing S, as you will see now.

Theorem 6: If S is subset and T a subspace of the vector space V over F, such that $S \subseteq T$, then $[S] \subseteq T$.

Proof: Let $s \in [S]$, then $s = \sum_{i=l}^{n} a_i v_i$, where $v_i \in S, \alpha_i \in F$

As $S \subseteq T$, we have $v_i \in T \ \forall\, i = 1, \ldots, n$. As T is a subspace and $v_i \in T$ for all i, $\sum_{i=l}^{i} a_i v_i \in T$, i.e., $s \in T$.

We have proved that $s \in [S] \implies s \in T$. Hence, $[S] \subseteq T$. ■

An immediate corollary to Theorem 6 follows.

Corollary 3: If S is a subspace containing S, $[S] = S$.

Proof: Since S is a subspace containing S, Theorem 7 gives us $[S] \subseteq S$. But $S \subseteq [S]$ always. Therefore, $[S] = S$. ■

The theorems above say that we can form subspaces from mere subsets of a space. Given a subset S of a vector space V, if S is not a subspace of V, what is the 'minimum' that we must add to S to make it s subspace? The answer is – all the finite linear combinations of vectors of S.

Look at the following examples.

Example 8: Let $S = \{(1,1,0), (2,1,3)\} \subseteq \mathbb{R}^3$. Determine whether the following vectors of $\mathbb{R}^3$ are in [S].

a) $(0,0,0)$;

b) $(1, 2, 3)$;

c) $(4/3, 1, 1)$.

Solution: We have

$$[S] = \{\alpha(1,1,0) + \beta(2,1,3) \mid \alpha, \beta \in \mathbb{R}\}$$
$$= \{(\alpha + 2\beta, \alpha + \beta, 3\beta) \mid \alpha, \beta \in \mathbb{R}\}$$

a) $(0, 0, 0) \in [S]$, since [S] is a subspace and $(0, 0, 0)$ is the additive identity of $\mathbb{R}^3$

b) $(1, 2, 3) \in [S]$ if we can find $\alpha, \beta \in \mathbb{R}$, such that $(\alpha + 2\beta, \alpha + \beta, 3\beta) = (1, 2, 3)$, i.e., $\alpha + 2\beta = 1, \alpha + \beta = 2, 3\beta = 3$. Now $3\beta = 3 \Rightarrow \beta = 1$, and then, $\alpha + \beta = 2 \Rightarrow \alpha = 1$.

But then $\alpha + 2\beta = 1 + 2 = 3 \neq 1$. Hence, $(1, 2, 3) \notin [S]$.

c) $(4/3, 1, 1) \in [S]$ if $\alpha + 2\beta = 4/3$, $\alpha + \beta = 1$, $3\beta = 1$ for some $\alpha, \beta \in \mathbb{R}$. These equations are satisfied if $\beta = 1/3$, $\alpha = 2/3$.
So $(4/3, 1, 1) \in [S]$.

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E6) If $S = \{(1, 2, 1), (2, 1, 0)\} \subseteq \mathbb{R}^3$, determine whether the following vectors of $\mathbb{R}^3$ are in [S].

a) $(5, 3, 1)$,

b) $(2, 1, 0)$,

c) $(4, 5, 2)$

E7) Let P be the vector space of polynomials over $\mathbb{R}$ and $S = \{x, x^2 + 1, x^3 + 1\}$. Determine whether the following polynomials are in [S].

a) $x^2 + x + 1$,

b) $2x^3 + x^2 + 3x + 2$.

Now that you have got used to the concept of subspaces we go on to construct new vector spaces from existing ones.

3.4 ALGEBRA OF SUBSPACES

In this section we will consider the union, intersection, sum and direct sum of vector spaces.

3.4.1 Intersection

If U and W are subspaces of a vector space V over a field **F**, then the set $U \cap W$ is a subset of a V. We will prove that it is actually a subspace of V.

Theorem 7: The intersection of two subspaces is a subspace.

Proof: Let U and W be two subspaces of a vector space V. Then $\mathbf{0} \in U$ and $\mathbf{0} \in W$ Therefore, $\mathbf{0} \in U \cap W$; hence $U \cap W \neq \emptyset$.

Next, if $v_1 \in U \cap W$, and $v_2 \in U \cap W$, then $v_1 \in U$, $v_2 \in U$, $v_1 \in W, v_2 \in W$.

Thus, for any $\alpha, \beta \in F, \alpha v_1 + \beta v_2 \in U$, $\alpha v_1 + \beta v_2 \in W$ (as U and W are subspaces).

$\therefore \alpha v_1 + \beta v_2 \in U \cap W$.

This proves that $U \cap W$ is a subspaces of V ■

Example 9: $U = \{(x, 2x, 3x) \mid x \in \mathbb{R}\}$ and $W = \{(0, y, (3/2)\, y \mid y \in \mathbb{R}\}$ are subspaces of $\mathbb{R}^3$. What is $U \cap W$?

Solution: Any element of $U \cap W$ is of the form $(x, 2x, 3x)$ and of the form $(0, y, (3/2)\, y)$. If $v \in U \cap W$, there are $x, y \in \mathbb{R}$ such that $(x, 2x, 3x) = (0, y, (3/2)y)$. It follows that $x = 0$, $2x = y$ and $3x = (3/2)y$. From $x = 0$ and $2x = y$ we get $(x=0)$, $y = 0$. Thus, the only possibility is $(0, 0, 0)$. Therefore, $U \cap W = \{(0, 0, 0)\}$. By Exercise 3 you know that this is a vector space.

* * *

Example 10: $U = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ and $W = \{(0, a, b) \mid a, b \in \mathbb{R}\}$ are subspaces of $\mathbb{R}^3$. What is $U \cap W$?

Solution: If $v \in U \cap W$, there are $x, y, a, b \in \mathbb{R}$ such that $v = (x, y, 0)$ and $v = (0, a, b)$. So, we have $x = 0$, $y = a$ and $0 = b$. From $x = 0$ and $b = 0$, it follows that the first and third coordinates of every element in $U \cap W$ is zero. Since a takes all possible real values, y also takes all possible real values. Therefore, $U \cap W$ is the vector space $\{(0, y, 0) \mid y \in \mathbb{R}\}$

In this example note that U is the xy-plane, W is the yz-plane and $U \cap W$ is the y-axis.

* * *

Example 11: Let U be the vector space of $n \times n$ symmetric matrices over $\mathbb{R}$ and W be the vector space of $n \times n$ skew-symmetric matrices over $\mathbb{R}$. Then, U and W are subspaces of $M_n(\mathbb{R})$, the vector space of $n \times n$ real matrices. What is $U \cap W$?

Solution: Let $A = (a_{ij})$. Suppose $A \in U \cap W$. Since A is symmetric, we have $a_{ij} = a_{ji}$. Since it is skew-symmetric also, we have $a_{ij} = -a_{ji}$. So, $a_{ji} = -a_{ji}$ for all $1 \leq i, j \leq n$. Therefore $2a_{ji} = 0$ or $a_{ji} = 0$ for all $1 \leq i, j \leq n$. So, A is the zero matrix. Therefore, $U \cap W = \{\mathbf{0}\}$.

* * *

Here are some exercises for you to try.

E8) If $U = \{(x, y, 2x) \mid x, y \in \mathbb{R}\}$ and $W = \{(x, 2x, y) \mid x, y \in \mathbb{R}\}$, what is $U \cap W$?

E9) Let U be the vector space of $n \times n$ complex hermitian matrices and W be the vector space of $n \times n$ complex skew-hermitian matrices. Then, U and W are subspaces of $M_n(\mathbb{C})$ considered as a vector space over $\mathbb{R}$. What is $U \cap W$?

Note: It can be shown that the intersection f any finite or infinite family of subspaces is a subspace. In particular, if $V_1, V_2, \ldots, V_n$ are all subsapces of V, then $V_1 \cap V_2 \cap \ldots \cap V_n$ is a subspaces of V. Let us now look at such an example.

Example 12: Consider the system of equations

$$\left.\begin{array}{rl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots & \vdots \; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{array}\right\} \qquad \ldots(1)$$

Show that the set

$$V = \{a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \mid a \text{ satisfies Eqn. (1)}\}$$

is a subspace of $\mathbb{R}^n$.

Solution: We give a sketch of the solution. One way to prove this directly. Another way is to prove that, for $1 \le i \le m$,

$$V_i = \{a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \mid a \text{ satisfies } a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0\}$$

is a subspace of $\mathbb{R}^n$. The proof is similar to the one in Exercise 26. It is easy to see that $V = V_1 \cap V_2 \cap \cdots V_{m-1} \cap V_m$. It follows that V is a also subspace of $\mathbb{R}^n$.

* * *

Let us now look at what happens to the union of two or more subspaces.

3.4.2 Sum

Consider the subspaces U and W of $\mathbb{R}^3$ given in Example 15. Here $v_1 = (1, 2, 0) \in U$ and $v_2 = (0, 2, 3) \in W$. Therefore, v_1 and v_2 belong to $U \cup W$. But $v_1 + v_2 = (1, 4, 3)$ is neither in U nor in W, and hence, not in $U \cup W$. So $U \cup W$ is not a subspace of $\mathbb{R}^3$. Thus, we see that, while the intersection of two subspaces is a subspace, the union of two subspaces may not a subspace. However, if we take two subspaces U and W, of a vector space V, then $[U \cup W]$, the linear span of $U \cup W$, is a subspace of V.

What are the elements of $[U \cup W]$? They are linear combinations of elements of $U \cup W$. So, far each $v \in [U \cup W]$, there are vectors $v_1, v_2, \ldots, v_n \in U \cup W$ of which v is linear combination. Now some (or all) of all $v_1, \ldots, v_n$ are in U and the rest in W. We rename those that are in U as $u_1, u_2, \dots, u_j$ and those in W as $w_1, w_2, \dots, w_k$ $(j \ge 0,\ k \ge 0,\ j + k = n)$.

Then, there are scalars $\alpha_1, \dots, \alpha_j, \beta_1, \dots, \beta_k$ such that

$$\begin{aligned} v &= \alpha_1 u_1 + \cdots + \alpha_j u_j + \beta_1 w_1 + \cdots + \beta_k w_k \\ &= u + w, \end{aligned}$$

where $u = \alpha_1 u_1 + \cdots + \alpha_j u_j \in U$, since each $u_i \in U$, and $w = \beta_1 w_1 + \cdots + \beta_k w_k \in W$, since each $w_i \in W$. (If $j = 0$, we take $u = 0$; if $k = 0$, we take $w = 0$). So what we have proved is that every element of $[U \cup W]$ is of the type $u + w,\ u \in U,\ w \in W$. This motivates the following definition.

Definition 4: If A and B are subsets of a vector space, we define the set $A + B$ by $A + B = \{a + b \mid a \in A,\ b \in B\}$.

Thus, each element of $A + B$ is the sum of an element of A and an element of B.

Example 13: If $A = \{(0, 0), (1, 1), (2, -3)\}$ and $B = \{(-3, 1)\}$ are subsets of $\mathbb{R}^2$, find $A + B$.

Solution: $A + B = \{(-3, 1), (-2, 2), (-1, -2)\}$ because, for example, $(0, 0) + (-3, 1) = (-3, 1)$, $(1, 1) + (-3, 1) = (-2, 2)$, etc.

* * *

Example 14: Let $A = \{(0, y, z) \mid y, z \in \mathbb{R}\}$ and $B = \{(x, 0, z) \mid x, z \in \mathbb{R}\}$. Prove that $A + B = \mathbb{R}^3$.

Solution: $A \subseteq \mathbb{R}^3, B \subseteq \mathbb{R}^3$, so $A + B \subseteq \mathbb{R}^3$. It is, therefore, enough to prove that $\mathbb{R}^3 \subseteq A + B$. Let $(a, b, c) \in \mathbb{R}^3$. Then $(a, b, c) = (0, b, c/2) + (a, 0, c/2)$, where $(0, b, c/2) \in A$ and $(a, 0, c/2) \in B$. So $(a, b, c) \in A + B$. Thus, $\mathbb{R}^3 \subseteq A + B$. Hence, $A + B \subseteq \mathbb{R}^3$.

* * *

Here is another example.

Example 15: Let S_1 be the set of all $n \times n$ real symmetric matrices and S_2 be the set of all $n \times n$ skew-symmetric matrices. We have seen in Example 4 we have seen that S_1 and S_2 are subspaces of $M_n(\mathbb{R})$. Show that $S_1 + S_2 = M_n(\mathbb{R})$.

Solution: Let $A \in M_n(\mathbb{R})$. We have $A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$. We have to prove that $\frac{1}{2}(A + A^t)$ is a symmetric matrix and $\frac{1}{2}(A - A^t)$ is skew-symmetric. From Exercise 7 in Unit 1, we have

Note that $(\alpha A)^t = \alpha A^t$ for $\alpha \in \mathbb{R}$.

$$\begin{aligned} \left(\frac{1}{2}(A + A^t)\right)^t &= \frac{1}{2}((A + A^t))^t = \frac{1}{2}\left(A^t + (A^t)^t\right) \\ &= \frac{1}{2}(A + A^t) \end{aligned}$$

since $(A^t)^t = A$ by Exercise 8 in Unit 1. So, $\frac{1}{2}(A + A^t)$ is a symmetric matrix.

Again, we have

$$\left(\frac{1}{2}(A - A^t)\right)^t = \frac{1}{2}((A - A^t))^t = \frac{1}{2}\left(A^t - (A^t)^t\right)$$
$$= \frac{1}{2}(A^t - A) = -\frac{1}{2}(A - A^t)$$

So, $\frac{1}{2}(A - A^t)$ is a skew-symmetric matrix.

* * *

Here is an exercise to test your understanding of Example 15.

E10) Let V_1 be the set of all $n \times n$ hermitian matrices and let V_2 be the set of all $n \times n$ skew-hermitian matrices. In Exercise 4, we have seen that V_1 and V_2 are subspaces of $M_n(\mathbb{C})$. Show that $M_n(\mathbb{C}) = V_1 + V_2$.

Note that in the discussion preceding the definition of a sum of subsets, we have actually proved that if U and W are subspaces of a vector space V, then $[U \cup W] \subseteq U + W$. Indeed, we have the following theorem.

Theorem 8: If A and B are subspaces of a vector space V, then$[A \cup B] = A + B$.

Proof: We have already proved (See above) that $[A \cup B] = A + B$. So it only remains to prove that $A + B \subseteq [A \cup B]$.

Let $v \in A + B$, then $v = a + b, a \in A, b \in B$. Now
$a \in A \Rightarrow a \in A \cup B \Rightarrow a \in [A \cup B]$.

Similarly, $b \in B \Rightarrow b \in A \cup B \Rightarrow b \in [A \cup B]$. As $[A \cup B]$ is a vector space and $a, b \in [A \cup B]$, we see that $a + b \in [A \cup B]$, i.e., $v \in [A \cup B]$. This completes the proof of the theorem.

Since $[A \cup B]$ is the smallest subspace containing $A \cup B$, we see, from Theorem 9 that $A + B$ is the smallest subspace of V containing both A and B. ■

E11) For the subspaces $A = \{(x, 0, 0) \mid x \in \mathbb{R}\}$ and $B = \{(0, y, 0) \mid y \in \mathbb{R}\}$ of $\mathbb{R}^3$, find $[A \cup B]$.

We consider a special kind of sum of subsets now.

3.4.3 Direct Sum

If A and B are subspaces of a vector space, you know that every vector v in $A + B$ is of the form $a + b$, where $a \in A,\ b \in B$. But in how many ways can a given $v \in A + B$ be expressed in the form, $a + b$?

In Example 14 we have expressed $(a, b, c) = (0, b, c/2) + (a, 0, c/2)$.

But we could also write

$$(a, b, c) = (0, b, 0) + (a, 0, c)$$

or $(a, b, c) = (0, b, c) + (a, 0, 0)$

or $(a, b, c) = (0, b, c/3) + (a, 0, 2c/3)$.

Indeed, for any real number δ we can write $(a, b, c) = (0, b, \delta) + (a, 0, c - \delta)$. Note that, in each case, we have expressed (a, b, c) as a sum of a vector form A and a vector from B. So, in this case, there are infinitely many ways of writing $v \in A + B$ in the form $a + b$, with $a \in A,\ b \in B$.

But there are some cases in which every vector $v \in A + B$ can be written in one and only one way as $a + b, a \in A, b \in B$. For example, suppose $A = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ and $B = \{(0, 0, z) \mid z \in \mathbb{R}\}$.

Then, for any $(p, q, r) \in \mathbb{R}^3$ we can write

$$(p, q, r) = (p, q, 0) + (0, 0, r) \in A + B$$

It follows that $A + B = \mathbb{R}^3$. But here (p, q, r) can be written in only one way as $a + b$ namely $(p, q, 0) + (0, 0, r)$, because, if we write $(p, q, r) = (x, y, 0) + (0, 0, z)$, then $(p, q, r) = (x, y, z)$, so that $x = p, y = q, z = r$. This means that $(x, y, 0) = (p, q, 0)$ and $(0, 0, z) = (0, 0, r)$.

Now, note that in this case $A \cap B = \{(0, 0, 0)\}$, whereas in the earliear example

$$A \cap B = \{(0, 0, z) \mid z \in \mathbb{R}\} \neq \{(0, 0, 0)\}$$

It is this difference in $A \cap B$ that is reflected in a unique or a multiple representation of v in the form $a + b$.

Definition 5: Let A and B be subpsaces of a vector space. The sum $A + B$ is said to be the direct sum of A and B (and is denoted by $A \oplus B$) if $A \cap B = \{\mathbf{0}\}$.

We have the following result.

Theorem 9: A sum $A + B$, of subspaces A and B, is a direct sum $A \oplus B$ if and only if every $v \in A + B$ is uniquely expressible in the form $a + b,\ a \in A,\ b \in B$.

Proof: First suppose $A + B$ is a direct sum i.e., $A \cap B = \{0\}$. If possible, suppose v has two representations,

$v = a_1 + b_1$ and $v = a_2 + b_2,\ a_i \in A,\ b_i \in B$.

Then, $a_1 + b_1 = a_2 + b_2$, i.e., $a_1 - a_2 = b_2 - b_1$

Now, $a_1, a_2 \in A \Rightarrow a_1 - a_2 \in A$. Similarly, $b_2 - b_1 \in B$, that is,

$a_1 - a_2 \in B$ (since $a_1 - a_2 = b_2 - b_1$).

Thus, $a_1 - a_2 \in A \cap B \Rightarrow a_1 - a_2 = 0 \Rightarrow a_1 = a_2$.

And then, $b_1 = b_2$.

This means that $a_1 + b_1$ and $a_2 + b_2$ are the same representation of v as a + b.

Conversely, suppose every $v \in A + B$ has exactly one representation as a + b. We must prove that $A \cap B = \{\mathbf{0}\}$.

Since A and B are subspaces, $0 \in A$, $0 \in B \ldots \therefore, \{\mathbf{0}\} \in A \cap B$.

If $A \cap B \neq \{0\}$, then there must be some $v \neq 0$ such that $v \in A \cap B$.

Then, v has two distinct representations as a + b, namely, $v + 0\,(v \in A, 0 \in B)$ and $0 + v\,(0 \in A,\ v \in B)$. This is a contradiction so $A \cap B = \{\mathbf{0}\}$. Hence A + B is a direct sum. ■

Example 16: Let A and B be subspaces of $\mathbb{R}^3$ defined by

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\},\ B = \{(0, y, z) \mid y, z \in \mathbb{R}\}.$$

Prove that $\mathbb{R}^3 = A + B$.

Solution: First note that $A + B \subseteq \mathbb{R}^3$. Secondly, if $(a, b, c) \in A \cap B$, then $a = b = c$, and $a = 0$; so $a = 0 = b = c$, i.e., $(a, b, c) = (0, 0, 0)$. Hence, the sum A + B is the direct sum $A \oplus B$. Next given any $(a, b, c) \in \mathbb{R}^3$, we have $(a, b, c) = (a, a, a) + (0, b - a, c - a)$, where $(a, a, a) \in A$ and $(0, b - a, c - a) \in B$; this proves that $\mathbb{R}^3 \subseteq A \oplus B$. Therefore, $\mathbb{R}^3 = A \oplus B$.

* * *

A function $f : \mathbb{R} \to \mathbb{R}$ is called an even function if $f(x) = f(-x)\ \forall\, x \in \mathbb{R}$, and an odd function if $f(-x) = -f(x)\ \forall\, x \in \mathbb{R}$.

Example 17: Let V be the space of all functions from **R** to **R**, and A and B be the subspaces of V defined by

$$A = \{f \mid f(x) = f(-x),\ \forall x\}$$
$$B = \{f \mid f(-x) = -f(x),\ \forall x\}$$

i.e., A is the subspace of all even functions and B is the subspace of all odd functions. Show that $V = A \oplus B$.

Solution: First, suppose $f \in A \cap B$, then $\forall x \in \mathbb{R}$, $f(-x) = f(x)$ and $f(-x) = -f(x)$. So, $\forall x,\ f(x) = -f(x)$, i.e., $\forall x,\ f(x) = 0$. Thus, f is the zero function, and $A \cap B = \{\mathbf{0}\}$.

Next, let $f \in V$, define

$$g(x) = \frac{1}{2}\{f(x) + f(-x)\} \text{ and}$$
$$h(x) = \frac{1}{2}\{f(x) - f(-x)\}$$

Then,

i) $f(x) = g(x) + h(x)\ \ \forall x \in \mathbb{R}$, i.e., $f = g + h$,

ii) $g(-x) = \frac{1}{2}\{f(x) + f(-x)\} = g(x),\ \ldots g \in A$.

iii) $h(-x) = \frac{1}{2}\{f(-x) - f(x)\} = -h(x),\ \ldots h \in B$.

Thus, for each $f \in V, f = g + h$, for some $g \in A$, $h \in B$.

$\Longrightarrow V = A + B$, and, as $A \cap B = \{\mathbf{0}\}$, we get

$V = A \oplus B$

* * *

Example 18: Let S_1 be the vector subspace of $M_n(\mathbb{R})$ symmetric matrices and let S_2 be the vector subspace of $M_n(\mathbb{R})$ of skew-symmetric matrices. Show that $M_n(\mathbb{R}) = S_1 \oplus S_2$.

Solution: We have done most of the work already. In Example 15 we proved that $S_1 + S_2 = M_n(\mathbb{R})$. We have proved $S_1 \cap S_2 = \varnothing$ in Example 3. It follows that $M_n(\mathbb{R}) = S_1 \oplus S_2$.

* * *

Note: Example 17 says that every function from $\mathbb{R}$ to $\mathbb{R}$ can be uniquely written as the sum of an even function and an odd function.

E12) Let A, B, C be the subspaces of $\mathbb{R}^3$ given by

$A = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$

$B = \{(x, y, z) \in \mathbb{R}^3 \mid x = z\}$, $C = \{(0, 0, z) \; z \in \mathbb{R}\}$

Prove that $\mathbb{R}^3 = A + C$ and $\mathbb{R}^3 = B + C$.
Which of these sums is/are direct?

E13) Consider the real vector space C, of all complex numbers (Example 3). If A and B are the subspaces of C given by $A = \{a + i.0 \mid a \in \mathbb{R}\}$, $B = \{ib \mid b \in \mathbb{R}\}$, prove that $C = A \oplus B$.

E14) Let U be the vector subspace of $M_n(\mathbb{C})$ of hermitian matrices and let V be the vector subspace of $M_n(\mathbb{C})$ of skew-hermitian matrices. Show that $M_n(\mathbb{C}) = U \oplus V$.

Now, we will look at vector spaces that are obtained by "taking the quotient" of a vector space by a subspace.

3.5 QUOTIENT SPACES

From a vector space V, and its subspace W, we will now create a new vector space. For this, we first define the concept of a coset.

3.5.1 Cosets

Definition 6: Let W be a subspace of V. If $v \in V$, the set $v + W$, defined by $v + w = \{v + w \mid w \in W\}$ is called a **coset** of W in V.

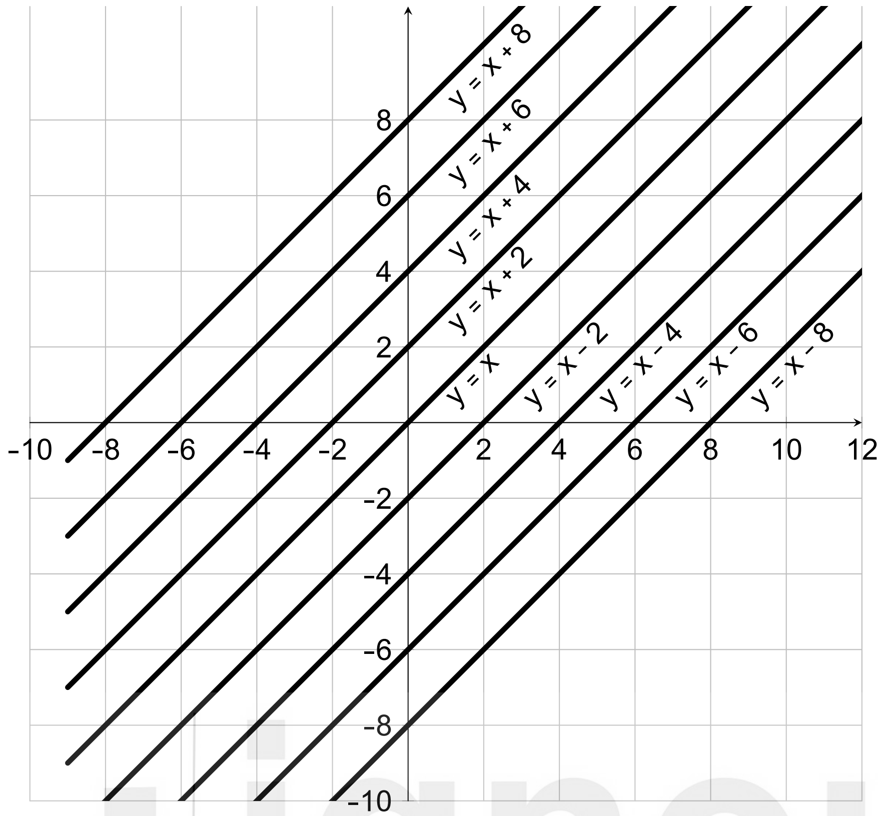

Fig. 1: Some cosets of the subspace $\{(x, y) \in \mathbb{R}^2 | y = x\}$

Example 19: Consider the vector subspace $W = \{(x, y) \in \mathbb{R}^2 | x = y\}$ of $\mathbb{R}^2$. What are it cosets?

Solution: Note that, W is the straight line $x = y$ passing through the origin. First let us look at a few examples. Consider the vector $v_1 = (2, 2) \in W$. Then,

$$\begin{aligned} v_1 + W &= \{(x, y) + (2, 2) | (x, y) \in W\} \\ &= \{(x, y) + (2, 2) | x = y\} \\ &= \{(a, a) + (2, 2) | a \in \mathbb{R}\} \\ &= \{(a + 2, a + 2) | a \in \mathbb{R}^2\} \end{aligned}$$

As a varies over all of $\mathbb{R}$, $a + 2$ varies over over all of $\mathbb{R}$. So,

$$\begin{aligned} v_1 + W &= \{(a, a) | a \in \mathbb{R}\} \\ &= \{(x, y) \in \mathbb{R}^2 | x = y \in \mathbb{R}^2\} = W \end{aligned}$$

This is true in general, as we will see in Theorem 10. If W is a subspace of V and $v \in V$, then $v + W$ is a subspace of V if and only if $v + W = W$.

Suppose, we take $v_2 = (0, 2)$. Then, we have

$$\begin{aligned} v_2 + W &= \{(x, y) + (0, 2) | (x, y) \in W\} \\ &= \{(x, y) + (0, 2) | x = y\} \\ &= \{(a, a) + (2, 2) | a \in \mathbb{R}\} \\ &= \{(a, a + 2) | a \in \mathbb{R}^2\} \\ &= \{(x, y) \in \mathbb{R}^2 | y = x + 2 \in \mathbb{R}^2\} \end{aligned}$$

So, the coset $v_2 + W$ is the straight line $y = x + 2$ in the plane. Notice that, this line doesn't pass through the origin (See Fig. 1). In other words, the $v_2 + W$ doesn't contain the element $\mathbf{0} = (0, 0)$. So it is not a subspace, which is to be expected since $v_2 \notin W$.

Let us now find out the general form of a coset of W. Let $u = (r, s) \in \mathbb{R}^2$. Then, we have

$$\begin{aligned} u + W &= \{(x, y) + (r, s) \mid (x, y) \in W\} \\ &= \{(x, y) + (r, s) \mid x = y\} \\ &= \{(a + r, a + s) \mid a \in \mathbb{R}\} \\ &= \{(x, y) \in \mathbb{R}^2 \mid y - x = s - r \in \mathbb{R}^2\} \end{aligned}$$

So, this is the line $y = x + s - r$. For example, if we take $v = (3, 2)$, $v + W$ is the line $y = x - 1$. Note that, if $s = r$, we get the line $\{(x, y) \in \mathbb{R}^2 \mid y = x\}$.

* * *

Observe that each element v of V yields a coset $v + W$ of W. Every coset of W in V is a subset of V, but it may not be a subspace of V, as you have seen in Example 19. Theorem 10 tells us when is coset a subspace.

Theorem 10: Let V be a vector space over a field F and W be it subspace. Suppose $v \in V$. The following statements are equivalent:

i) $v + W$ is a subspace of V.

ii) $v + W = W$

iii) $v \in W$

Proof: i) ⇒ ii) We have $v + \mathbf{0} = v \in v + W$ since $\mathbf{0} \in W$. Let $v + w$ be any element in $v + W$. Then $v + w \in W$ since $v, w \in W$ and W is a subspace. So, $v + W \subset W$.

On other hand, given any element $w \in W$, $w - v \in W$ since $v, w \in W$. Therefore, $w = v + (w - v) \in v + W$ and $W \subseteq v + W$. So, $v + W = W$.

ii) ⇒ iii) Since $v + 0 \in v + W$ and $v + W = W$, it follows that $v \in W$.

iii) ⇒ i) Let $v + w_1, v + w_2 \in v + w$. Then

$$v + w_1 + v + w_2 = v + (v + w_1 + w_2) \in v + W$$

We have $v + w_1 + w_2 \in W$ since v, w_1 and $w_2 \in W$ and W is a subspace of W. Therefore $v + w_1 + v + w_2 \in v + W$.

Let $\alpha \in F$ and $v + w \in v + W$. We have

$$\alpha(v + w) = \alpha v + \alpha w = v + (\alpha - 1)v + \alpha w$$

We have $(\alpha - 1)v \in W$ since $v \in W$ and W is a subspace. Also, $\alpha w \in W$. Therefore, $(\alpha - 1)v + w \in W$, $(\alpha - 1)v + w = w' \in W$, say. It follows that $\alpha(v + w) = v + w' \in v + W$ and therefore $v + W$ is a subspace of V.

■

E15) Let $W = \left\{(x, y) \in \mathbb{R}^2 \middle| y = \frac{2}{3}x\right\}$. Describe the cosets of W. Show geometrically the cosets $(0, 2) + W$, $(0, 4) + W$, $(0, 6) + W$, $(0, 8) + W$, $(0, -2) + W$, $(0, -4) + W$, $(0, -6) + W$ and $(0, -8) + W$ as straight lines in the plane.

E16) Consider the subspace $W = \{(a, 0, 0) \mid a \in \mathbb{R}\}$ of $\mathbb{R}^3$.
Let $v = (1, 0, 2)$. Find the coset $v + w$. Is it a subspace of $\mathbb{R}^3$?.

E17) With W as in Exercise 16 and $v = (2, 0, 0)$, prove that $v + w$ is a subspace and, in fact, $v + w = w$.

E18) In the proof above, we have essentially proved that

$$v_1 - v_2 \in W \Rightarrow v_1 + W = v_2 + W.$$

The converse of this is also true. Prove it.

Now we ask : Given a vector space V and a subspace W, can we get V back if we know all the cosets of W? The answer is given in the following theorem.

Theorem 11: If W is a subspace of V, the union of all the cosets of W in V is V.

Proof: Since every coset of W in V is a subset of V the union is certainly a subset of V. Conversely, given $v \in V$, $v = v + 0 \in v + W$ (as $0 \in W$). Thus, every $v \in V$ belongs to some coset of W in V. Hence, V is contained in the union of all the cosets of W in V. Hence, the theorem is established. ■

We may write the statement of Theorem 11 as

$$v = \bigcup_{v \in V} (v + w)$$

A very special property of cosets is given in the following theorem.

Theorem 12: . Two cosets $v_1 + W = v_2 + W$ in V are either equal or disjoint. In fact, $v_1 + W = v_2 + W$ if $v_1 - v_2 \in W$, for $v_1, v_2 \in V$.

Proof: We have to prove that, for $v_1, v_2 \in V$ either $(v_1 + W) \cap (v_2 + W) = \{\mathbf{0}\}$ or $v_1 + W = v_2 + W$. Now, suppose $(v_1 + W) \cap (v_2 + W) \neq \{0\}$. Then they have a common non-zero element v, say. That is, $v = v_1 + w_1 = v_2 + w_2$, for some $w_1, w_2 \in W$. Then,

$$v_1 - v_2 = w_1 - w_2 \in W \qquad \ldots(2)$$

We want to prove that $v_1 + W = v_2 + W$. For this we prove that $v_1 + W \subseteq v_2 + W$ and $v_2 + W \subseteq v_1 + W$.

Now,

$$u \in v_1 + W \Rightarrow u = v_1 + w_3, \text{ where } w_3 \in W$$
$$\Rightarrow u = v_2 + (w_2 - w_1) + w_3, \text{ by Eqn. (2)}$$
$$= v_2 + w_4, \text{ where } w_4 \in W$$
$$\Rightarrow u \in v_2 + W.$$

Hence,

$$v_1 + W \subseteq v_2 + W$$

We can similarly show that $v_2 + W \subseteq v_1 + W$.

Hence, $v_1 + W = v_2 + W$. Note that we have shown that

$$v_1 - v_2 \in W \Rightarrow v_1 + W = v_2 + W.$$

■

$v_1 + W = v_2 + W \Leftrightarrow v_1 - v_2 \in W$

Here is an example that illustrates the use of Theorem 12.

Example 20: Let **P** be the vector space of all polynomials over R and let W be the subspace

$$W = \{f(x) \in \mathbf{P} | f(1) = 0\}$$

What are the cosets W in V? Which of the following pairs of elements in **P** are in the same coset of W:

i) $x^3 + x^2 + 2x - 2, x^3 + 2x - 1$ ii) $x^4 + x^3 + x^2 + 2x - 1, x^2 + 9x + 1$

Solution: Let f(x) be in **P**. Using division algorithm, we get $f(x) = (x - 1)g(x) + r$ for some polynomial $g(x) \in \mathbf{P}$ and $r \in \mathbb{R}$. Then, writing $h(x) = f(x) - r = (x - 1)g(x)$, we get $h(0) = 0$, so $h(x) \in W$. So, by Theorem 12, $f(x) + W = r + W$. So, for any coset $f(x) + W$, there is an $r \in \mathbb{R}$ such that $f(x) + W = r + W$ where r is the reminder on dividing f(x) by x. So, every coset of W is of the form $\{r + W | r \in \mathbb{R}\}$.

i) Let $g(x) = (x^3 + x^2 + 2x - 2) - (x^3 + 2x - 1) = x^2 - 1$. Then $g(1) = 0$. So, g(1) is in W and this pair of polynomials are in the same coset of W.

ii) Let $g(x) = (x^4 + x^3 + 2x - 1) - (x^2 + 9x + 1) = x^4 + x^3 - 7x - 2$ and $g(1) = -7 \neq 0$. So, this pair of polyomials are in different cosets of W.

* * *

Note: Theorem 11 and Theorem 12 tells us that if W is a subspace of V, then W partitions V into mutually disjoint subsets (namely, the cosets of W in V).

Consider the following example in which we show how a vector space can be partitioned by cosets of a subspace.

Example 21: Consider the subspace $W = \{\alpha(1, 0, 0) \alpha \in \mathbb{R}\}$ of $\mathbb{R}^3$. How can you write $\mathbb{R}^3$ as the union of disjoint cosets of W?

Solution: Note that W is just the x-axis in 3 dimensional space. Any coset of W is of the form.

$$(a, b, c) + W = \{(a, b, c) + (\alpha, 0, 0) \mid \alpha \in \mathbb{R}\}$$
$$= \{(a + \alpha + a, b, c) \mid \alpha \in \mathbb{R}\}$$

Now, for any $(a, b, c) \in \mathbb{R}^3$, $(a.b, c) - (0, b, c) = (a, 0, 0) \in W$.

Therefore, $(a, b, c) + W = (0, b, c) + W$. Also, the cosets

$(0, b, c) + W$ and $(0, b', c') + W$ are the same iff $b = b'$ and $c = c'$.

Thus, $\{(0, b, c) + W \mid b, c \in \mathbb{R}\}$ is the set of disjoint cosets of W in $\mathbb{R}^3$.

And $\mathbb{R}^3 = \cup\{(0, b, c) + W \mid b, c \in \mathbb{R}\}$.

Geometrically, the coset $(0, b, c) + W$ represents a line (in the plane determined by the point $(0, b, c)$ and the x-axis) which is parallel to the x-axis and passes through the point $(0, b, c)$. Thus, $\mathbb{R}^3$ is the union of all such distinct lines.

* * *

E19) Write $\mathbb{R}^3$ as a disjoint union of the cosets of

a) the subspace $\{(0, 0)\}$,

b) the subspace $\mathbb{R}^2$.

E20) Let $W = \{f(x) \in \mathbf{P} \mid f(1) = 0\}$ be a subspace of **P**, the real vector space of all polynomials in one variable x.

a) If $v = (x - 1)(x^2 + 1)$, what is $v + W$?

b) If $v = (x - 2)(x^2 + 1)$, what is $v + W$?

Before we proceed, let us stress that our notation for a coset of W in V has a peculiarty. A coset $v_1 + W$ can also be written as $v_2 + W$ provided $v_1 - v_2 \in W$. **So the same coset can be written in many different ways.** Indeed, if C is a coset of W in V, then $C = v + W$, for any vector v in C.

Let us now see how the set of all cosets of W in V can form a vector space.

3.5.2 The Quotient Space

We have pointed out that generally a coset $v + W$ of a subspace W of a vector space V is not itself a subspace of V. We shall now prove that if we take the set of all cosets of W in V, this set can be made into a vector space by defining addition and scalar multiplication suitably.

Notation: Let W be a subspace of the vector space V. We denote the set of all cosets of W in V by V/W. Thus, $V/W = \{v + W \mid v \in V\}$.

Consider the following example.

Example 22: Let P be the vector space of real polynomials in x and $W = \{f \mid f \in P,\ f(0) = 0\}$ be the subspace of P consisting of all those polynomials whose constant term in zero. Show that $\mathbf{P/W} = \{a + W \mid a \in \mathbb{R}\}$.

Solution: Since $a \in \mathbf{P}\ \forall\ a \in \mathbb{R}$, certainly $a + W$ is coset of W in **P**. So $a + W \in \mathbf{P/W}\ \forall\ a \in \mathbb{R}$. Conversely, take an element of **P/W,**

Say $f(x) + W$, where $f(x)$ is a polynomial. Suppose

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, a_i \in \mathbb{R}.$$

Then $f(x) = a_0 + g(x) = a_i x + a_2 x^2 + \cdots + a_n x^n$.

Since $g(0) = 0,\ g \in W$.

Hence, $f = a_0 + g$, where $g \in W$.

Thus, $f + W = a_0 + W$ (Theorem 13).

Hence, $f + W \in \{a + W \mid a \in \mathbb{R}\}$.

This completes the proof that $\mathbf{P/W} = \{a + \mathbf{W} \mid a \in \mathbb{R}\}$.

* * *

E21) If $\mathbf{P}_n$ denotes the vector space of all polynomials of degree $\le n$, prove that $\mathbf{P}_3/\mathbf{P}_2 = \{ax^3 + \mathbf{P}_2 \mid a \in \mathbb{R}\}$.

Hint: For any $f(x) \in \mathbf{P}_3$, $\exists a \in \mathbb{R}$ such that $f(x) - ax^3 \in \mathbf{P}_2$).

We now proceed to introduce two operations on the set V/W, namely, addition and scalar multiplication.

Definition 7: Let W be a subspace of V. We define addition on V/W by $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$.

If $\alpha \in \mathbb{R}$, $v + W \in V/W$, then we define scalar multiplication on V/W by $\alpha \bullet (v + W) = (\alpha v) + W$.

Note that our definitions of addition and scalar multiplication seem to depend on the way in which we write a coset. Let us explain this. Suppose C_1 and C_2 are two cosets. What is $C_1 + C_2$? To find $C_1 + C_2$ we must express C_1 as $v_1 + W$ and C_2 as $v_2 + W$.

Having done this we can then say that

$$C_1 + C_2 = (v_1 + v_2) + W.$$

But C_1 can be written in the form $v + W$ in many ways and the same is true for C_2. So the question arises: Is $C_1 + C_2$ dependent on the particular way of writing C_1 and C_2, or is it independent of it? In other words, suppose $C_1 = v_2 + W = v_2' + W$. Then we may say that

$$C_1 + C_2 = (v_1 + W) + (v_2 + W) = (v_1 + v_2) + W;$$

But we may also say that

$$C_1 + C_2 = (v_1' + W) + (v_2' + W) = (v_1' + v_2') + W$$

Are these two answers the same? If they are not, which one is to be $C_1 + C_2$? A similar question can arise in the case of αC where α is scalar and C a coset. These are important questions. Fortunately, they have simple answers as shown by the following theorem.

Theorem 13: Let W be a subspace of a vector space V. If $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$, then

a) $(v_1 + v_2) + W = (v_1' + v_2') + W$ Also, if α is any scalar, then

b) $(\alpha v_1) + W = (\alpha v_1') + W$

Proof: a) For $v_1,\ v_1', v_2,\ v_2' \in V,\ v_1 + W = v_1' + W, v_2 + W = v_2' + W$

$$\Longrightarrow v_1 - v_1' \in W,\ v_2 - v_2' \in W \text{ (by E 23)}$$
$$\Longrightarrow (v_1 - v_1') + (v_2 + v_2') \in W$$
$$\Longrightarrow (v_1 + v_2) - (v_1' + v_2') \in W$$
$$\Longrightarrow (v_1 + v_2) + W = (v_1' + v_2') + W \text{ (by Theorem 13).}$$

Thus, (a) is true.

b) For any scalar α and $v_1, v_1' \in V,\ v_1 + W = v_1' + W \Rightarrow v_1 - v_1' \in W$

$$\Longrightarrow \alpha(v_1 - v_1') \in W$$
$$\Longrightarrow \alpha v_1 - \alpha v_1' \in W$$
$$\Longrightarrow \alpha v_1 + W = \alpha v_1' + W$$

Thus (b) is also proved.

■

Theorem 14 assures us that the sum and scalar multiplication of cosets is independent of the particular way in which a coset is written. We express this by saying that addition and scalar multiplication of cosets are well defined by the way we have defined them.

This also means that when adding two cosets or when multiplying a scalar and a coset we are free to use any representation for the cosets involved.

We now come to the actual proof that V/W is a vector space.

Theorem 14: Let V be a vector space over a field **F,** and W be a subspace. Then V/W is a vector space over.

Proof: We will show that VS1-VS10 hold for V/W where the operations are addition and scalar multiplication as defined above.

i) VS1 is true since the sum of two cosets is a coset.

ii) For v_1, v_2, v_3 in V we know that $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$.

Therefore,

$$\{(v_1 + W) + (v_2 + W)\} + (v_3 + W) = \{(v_1 + v_2) + W\} + (v_3 + W)$$

$$= \{(v_1 + v_2) + v_3\} + W = \{v_1 + (v_2 + v_3)\} + W$$
$$= (v_1 + W) + \{(v_2 + v_3) + W\}$$
$$= (v_1 + W) + \{(v_2 + W) + (v_3 + W)\}$$

Thus, VS2 is true.

iii) We claim that the coset $\mathbf{0} + W = W$(since $\mathbf{0} \in \mathbf{W}$ is the identity element for V/W

For

$$v \in V,\ W + (v + W) = (\mathbf{0} + W) + (v + W) = (\mathbf{0} + v) + W = v + W.$$

Similarly,

$$(v + W) + W = (v + W) + (\mathbf{0} + W) = v + W.$$

Hence W is the ‘zero’ of V/W, and VS3 is true.

iv) The additive inverse of $v + W$ is $(-v) + W$, because

$$(v + W) + \{(-v) + W\} = (v + (-v) + W) + W = \mathbf{0} + W = W$$

and

$$(-v) + W + (v + W) = (-v + v) + W = \mathbf{0} + W = W.$$

This proves that VS4 is true.

v) We note that addition in V is already commutative because V is a vector space. So

$$\forall v_1, v_2 \in V,\ v_1 + v_2 = v_2 + v_1.$$

Hence,

$$\begin{aligned}(v_1 + W) + (v_2 + W) &= (v_1 + v_2) + W \\ &= (v_2 + v_1 + W) \\ &= (v_2 + W) + (v_1 + W)\end{aligned}$$

Thus VS5 holds for V/W.

vi) VS6 is true because, for $\alpha \in F$ and $v + W \in V/W, \alpha(v + W) = \alpha v + W \in V/W$

vii) To prove that VS7 holds, let $\alpha \in F$ and $u, v \in V$. Then

$$\begin{aligned}\alpha\{(u+W)+(v+W)\} &= \alpha\{(u+v)+W\}\\ &= \alpha(u+v)+W\\ &= (\alpha u+\alpha v)+W\\ &= (\alpha u+W)+(\alpha v+W)\\ &= \alpha(u+W)+\alpha(v+W).\end{aligned}$$

viii) For any $\alpha, \beta \in F$ and $v \in V$ you can show, as above, that

$$(\alpha+\beta)(v+W) = \alpha(v+W)+\beta(v+W).$$

Thus, VS8 holds.

ix) For any $\alpha.\beta \in F$ and $v \in V$ we have

$$\begin{aligned}\alpha(\beta(u+W)) &= \alpha(\beta u+W)\\ &= (\alpha\beta)\ u+W\\ &= (\alpha\beta)(u+W)\end{aligned}$$

Thus VS9 is true for V/W

x) For $u \in V$, we have $1.\ u+W = (1 \bullet u)+W = u+W$.

Thus, VS10 holds for V/W.

■

The vector space we have just obtained has a name.

Definition 8: If W is a subspace of V, then the vector space V/W is called the quotient space of V by W.

The name quotient is very apt because, in a sense, we quotient out the elements of W from those of V.

Example 23: Let V be a vector space over **F** and $W = \{\mathbf{0}\}$. What is V/W?

Solution:

$$\begin{aligned}V/W &= \{v+W \mid v \in V\} = \{v+\{\mathbf{0}\} \mid v \in V\}\\ &= \{v \mid v \in V\} = V\end{aligned}$$

* * *

E22) Let $W = \{\alpha(0,1) \mid \alpha \in \mathbb{R}\}$. What is $\mathbb{R}^2/W$?

E23) For any vector space V, show that V/V has only 1 element, namely, the coset V.

And now, let us see what we have done in this unit.

3.6 SUMMARY

et us conclude the unit by summarizing what we have covered in it.

In this unit we have

1. defined a general vector space.
2. given several examples of vector spaces.
3. proved some important properties of a general vector space.
4. defined the notion of a subspace and given criteria to identify subspaces.
5. introduced the idea of the linear span of a set of vectors.
6. shown that the intersection of subspaces of a vector space is a subspace.
7. defined the sum and direct sum of subspaces of a vector space and shown that they are subspaces also.
8. defined cosets and a quotient space.

3.7 SOLUTIONS/ANSWERS

E1) This is a particular case of the vector space in E6 (with $a = 2,\ b = 1, c = -1$).

If $\alpha, \beta \in \mathbb{R}$ and $(x_1, x_2, x_3), (y_1, y_2, y_3) \in W$, then

$$\alpha(x_1, x_2, x_3) + \beta(y_1, y_2, y_3) = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)$$

Also,

$$2(\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) - (\alpha x_3 + \beta y_3)$$

$= \alpha(2x_1 + x_2 - x_3) + \beta(2y_1 + y_2 - y_3) = 0$, since

$2x_1 + x_2 - x_3 = 0$ and $2y_1 + y_2 - y_3 = 0$.

Thus, $\alpha(x_1, x_2, x_3) + \beta(y_1, y_2, y_3) \in W$.

Hence, W is a subspace of $\mathbb{R}^3$.

E2) a) $W = \{(x_1, -x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$, $W \neq \varnothing$, since $(0, 0, 0) \in W$.

For $\alpha, \beta \in \mathbb{R}$ and $= (x_1, -x_1, x_2)$, $v = (y_1, -y_1, y_2) \in W$, we have

$\alpha u + \beta v$

$$= (\alpha x_1 + \beta y_1, -(\alpha x_1 + \beta y_1), \alpha x_2 + \beta y_2) \in W$$

Writing $w_1 = \alpha x_1 + \beta y_1$, $w_2 = \alpha x_2 + \beta y_2$, we get

$$\alpha u + \beta v = (w_1, -w_1, w_2) \in W$$

$\therefore$ W is a vector space.

b) $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 \geq 0\}$
Since $x_1^2 \geq 0 \; \forall x_1 \in \mathbb{R}$, we see that $W = \mathbb{R}^3$, and hence is a vector space.

c) $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 x_2 = 0\}$
$W \neq \emptyset$, since $(0, 0, 0) \in W$.
Now, $(1, 0, 0) \in W$ and $(0, 1, 0) \in W$, but
$(1, 0, 0) + (0, 1, 0) = (1, 1, 0) \notin W$.
$\therefore$, W is not a subspace of $\mathbb{R}^3$.

d) $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1\}$.
Now, $(1, 0, 0)$ and $(0, 1, 0) \in W$, but $(1, 0, 0) + (0, 1, 0) = (1, 1, 0) \notin W$, since $1 + 1 + 0 = 2 \neq 1$. $\therefore$,W is not a subspace of $\mathbb{R}^3$.

E3) Firstly, $\{\mathbf{0}\}$ is non-empty. Secondly, $\mathbf{0} + \mathbf{0} = \mathbf{0} \in \{\mathbf{0}\}$ and $\alpha \bullet \mathbf{0} = \mathbf{0} \in \{\mathbf{0}\}$, for any $\alpha \in F$. Thus, by Theorem 3, $\{\mathbf{0}\}$ is a subspace of V.

E4) Let $A = (a_{ij})$, $B = (b_{ij})$ be two $n \times n$ hermitian matrices and $\alpha, \beta \in \mathbb{R}$. Then, we have $a_{ij} = \overline{a_{ji}}$ and $b_{ij} = \overline{b_{ji}}$. Let $C = \alpha A + \beta B$. Then $C = (c_{ij}) = (\alpha a_{ij} + \beta b_{ij})$. We have $\overline{c_{ji}} = \overline{\alpha a_{ji} + \beta b_{ji}} = \overline{\alpha}\,\overline{a_{ji}} + \overline{\beta}\,\overline{b_{ji}} = \overline{\alpha}\, a_{ij} + \overline{\beta}\, b_{ij}$. Since $\alpha, \beta \in \mathbb{R}$, we have $\overline{\alpha} = \alpha$ and $\overline{\beta} = \beta$. Therefore $\overline{c_{ji}} = \overline{\alpha}\, a_{ij} + \overline{\beta}\, b_{ij} = \alpha\, a_{ij} + \beta\, b_{ij} = c_{ij}$. Therefore $C^* = (\overline{c_{ji}}) = (c_{ij}) = C$. It follows that $\alpha A + \beta B = C$ is hermitian.

Let $A = (a_{ij})$, $B = (b_{ij})$ be two $n \times n$ hermitian matrices and $\alpha, \beta \in \mathbb{R}$. Then, we have $a_{ij} = -\overline{a_{ji}}$ and $b_{ij} = -\overline{b_{ji}}$. Let $C = \alpha A + \beta B$. Then $C = (c_{ij}) = (\alpha a_{ij} + \beta b_{ij})$. We have $\overline{c_{ji}} = \overline{\alpha a_{ji} + \beta b_{ji}} = \overline{\alpha}\,\overline{a_{ji}} + \overline{\beta}\,\overline{b_{ji}} = -\overline{\alpha}\, a_{ij} - \overline{\beta}\, b_{ij}$. Since $\alpha, \beta \in \mathbb{R}$, we have $\overline{\alpha} = \alpha$ and $\overline{\beta} = \beta$. Therefore $\overline{c_{ji}} = -\overline{\alpha}\, a_{ij} - \overline{\beta}\, b_{ij} = -c_{ij}$. Therefore $C^* = (\overline{c_{ji}}) = (-c_{ij}) = -C$. It follows that $\alpha A + \beta B = C$ is skew-hermitian.

E5) $[S] = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$. So, [S] contains polynomials of degree at most two. Therefore, $2x + 3x^3 \notin [S]$

E6) $[S] = \{a(1, 2, 1) + \beta(2, 1, 0) \mid \alpha, \beta \in \mathbb{R}\}$
$= \{(\alpha + 2\beta, 2\alpha + \beta, \alpha) \mid \alpha, \beta \in \mathbb{R}\}$

a) $(5, 3, 1) \in [S] \iff \exists \alpha, \beta \in \mathbb{R}$ such that

$$\alpha + 2\beta = 5, 2\alpha + \beta = 3, \alpha = 1.$$

Now, $\alpha = 1$ and $\alpha + 2\beta = 5 \Rightarrow \beta = 2$. But then

$$2\alpha + \beta = 2 + 2 = 4 \neq 3.$$

$$\therefore (5, 3, 1) \notin [S].$$

b) $(2, 1, 0) \in S \subseteq [S]$. $\therefore$, $(2, 1, 0) \in [S]$

c) $(4,5,2) \in [S] \iff \exists \alpha, \beta \in \mathbb{R}$ such that

$$\alpha + 2\beta = 4, 2\alpha + \beta = 5, \alpha = 2.$$

. Since $\alpha = 2$, from $2\alpha + \beta = 2$, we get $\beta = 1$. The other equation $\alpha + 2\beta = 4$ is also satisfied by $\alpha = 2$, $\beta = 1$. So, $(4,5,2) \in [S]$.

E7) $[S] = \{ax + b(x^2+1) + c(x^3-1) \mid a,b,c \in \mathbb{R}\}$
$= \{cx^3 + bx^2 + ax + (b-c) \mid a,b,c \in \mathbb{R}\}$

a) Comparing the coefficients of $x^2 + x + 1$ and taking $c = 0, b = 1, a = 1$, $b - c = 1$. So, $x^2 + x + 1 = 1 \cdot x + 1 \cdot (x^2 + 1)$ and $x^2 + x + 1 \in [S]$

b) Comparing the coefficients of $2x^3 + x^2 + 3x + 2$ and $cx^3 + bx^2 + ax + (b - c)$, we get $c = 2$, $b = 1$, $a = 3$ and $b - c = 2$. From $c = 2$ and $b - c = 2$, we get $b = 4$ which is not consistent with the equation $b = 1$. So, $2x^3 + x^2 + 3x + 2 \notin [S]$.

E8) If $(x,y,z) \in U \cap W$ then $(x,y,z) \in U$ and $(x,y,z) \in W$.

Now,

$(x,y,z) \in U \Rightarrow z = 2x$, and
$(x,y,z) \in W \Rightarrow y = 2x$

$\therefore$, any element of $U \cap W$ is of the form $(x, 2x, 2x), x \in \mathbb{R}$. That is, $U \cap W = \{(x, 2x, 2x)\ x \in \mathbb{R}\}$.

E9) Let $A = (a_{ij})$ and suppose that $A \in U \cap W$. Since U is hermitian, we have $a_{ij} = \overline{a_{ij}}$. Since A is skew-hermitian, we have $a_{ij} = -\overline{a_{ji}}$. So, we have $\overline{a_{ij}} = -\overline{a_{ij}}$. Taking complex conjugates both sides, it follows that $a_{ji} = -a_{ji}$ or $2a_{ji} = 0$ for all $1 \le i,j \le n$. Therefore $a_{ij} = 0$ for all $1 \le i,j \le n$, so $(a_{ij}) = 0$.

E10) Let $A \in M_n(\mathbb{C})$. We have $A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*)$. We have to prove that $\frac{1}{2}(A + A^t)$ is a hermitian and $\frac{1}{2}(A - A^t)$ is a skew-hermitian. From Exercise 9 in Unit 1, we have

Note that $(\alpha A)^* = \alpha A^*$ for $\alpha \in \mathbb{R}$.

$$\left(\frac{1}{2}(A + A^*)\right)^* = \frac{1}{2}((A + A^*))^* = \frac{1}{2}(A^* + (A^*)^*)$$
$$= \frac{1}{2}(A + A^*)$$

since $(A^t)^* = A$ by Exercise 8 in Unit 1. So, $\frac{1}{2}(A + A^*)$ is a hermitian matrix.

Again, we have

$$\left(\frac{1}{2}(A - A^*)\right)^* = \frac{1}{2}((A - A^*))^* = \frac{1}{2}(A^* - (A^*)^*)$$
$$= \frac{1}{2}(A^* - A) = -\frac{1}{2}(A - A^*)$$

So, $\frac{1}{2}(A - A^*)$ is a skew-hermitian matrix.

E11) $[A \cup B] = A + B = \{(x,0,0) + (0,y,0) \mid x,y \in \mathbb{R}\}$
$= \{(x,y,0) \mid x,y \in \mathbb{R}\}$

E12) Each of A + C and B + C are subspace of $\mathbb{R}^3$. Now, for any $(a, b, c) \in \mathbb{R}^3$,

$$(a, b, c) = (a, b, -a - b) + (0, 0, a + b + c) \in A + C, \text{ and,}$$
$$(a, b, c) = (a, b, a) + (0, 0, c - a) \in B + C.$$

Therefore, $\mathbb{R}^3 = A + C$ and $\mathbb{R}^3 = B + C$

Now

$$A \cap C = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0 \text{ and } x = 0 = y\}$$
$$= \{(0, 0, 0)\}, \therefore A + C \text{ is a direct sum.}$$

Also

$$B \cap C = \{(x, y, z) \in \mathbb{R}^3 \mid x = z \text{ and } z = 0 = y\}$$
$$= \{(0, 0, 0)\}, \therefore B + C \text{ is also a direct sum.}$$

E13) Firstly, $A + B \subseteq C$. Secondly, $A \cap B = \{x + iy \mid y = 0 \text{ and } x = 0\} = \{\mathbf{0}\}$. This means that the sum, A + B, is a direct sum. Finally, take any element $x + iy \in C$.

Then $x + iy = (x + i0) + iy \in A + B$

Therefore, $C = A \oplus B$.

E14) We have seen that $M_n(\mathbb{C}) = U + V$ in Exercise 10. We have also seen that $U \cap V =$ emptyset in Exercise 9. It follows that $M_n(\mathbb{C}) = U \oplus V$.

E15) $v_1 + W = v_2 + W \Rightarrow v_1 \in v_1 + W = v_2 + W$

$$\Longrightarrow v_1 \in v_2 + W \Rightarrow v_1 = v_2 + w, \text{ for some } w \in W$$
$$\Longrightarrow v_1 - v_2 = w \in W \Rightarrow v_1 - v_2 \in W.$$

E16) Let $(r, s) \in \mathbb{R}^2$. Then,

$$v + W = \left\{(x, y) + (r, s) \,\middle|\, x, y \in \mathbb{R} y = \frac{2}{3}x\right\} = \left\{(x + r, y + s) \,\middle|\, y = \frac{2}{3}x, x, y \in \mathbb{R}\right\}$$
$$= \left\{(x, y) \,\middle|\, y = \frac{2}{3}x + s - r\right\}$$

E17) $v + w = \{v + w \mid w \in W\}$

$$= \{(1, 0, 2) + (a, 0, 0) \mid a \in \mathbb{R}\}$$
$$= \{(a + 1, 0, 2) \mid a \in \mathbb{R}\}$$

Thus, $v + w = \{(a, 0, 2) \mid a \in \mathbb{R}\}$.
(because, as a takes all the real values, a + 1 also takes all the real values, so that we may replace a + 1 by a).
$v + w$ is not a subspace of $\mathbb{R}^3$ as $(0, 0, 0) \notin v + W$.

E18) Here

$$v + w = \{(2, 0, 0) + (a, 0, 0) \mid a \in \mathbb{R}\}$$
$$= \{(a + 2, 0, 0) \mid a \in \mathbb{R}\}$$
$$= \{(p, 0, 0) \mid p \in \mathbb{R}\}$$
$$= W$$

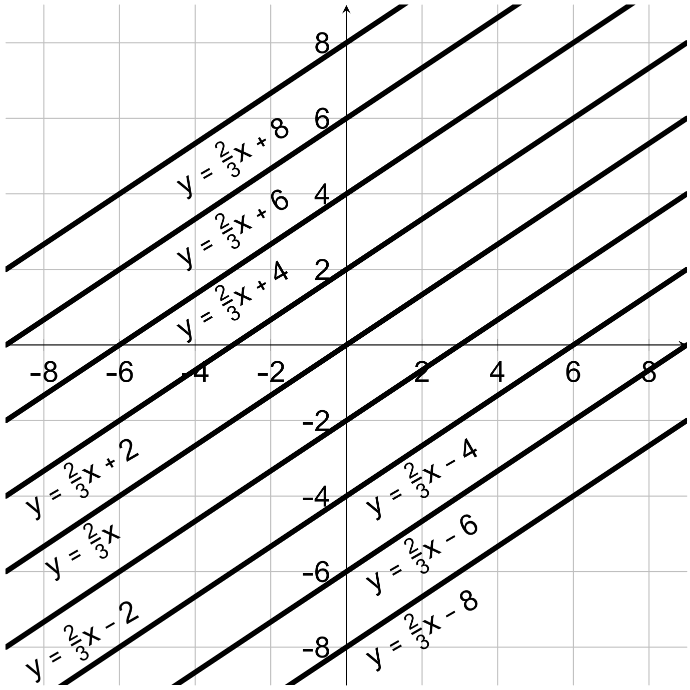

Fig. 2: Some cosets of the subspace $W = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x = \frac{2}{3}y\right\}$

E19) a) Any coset of $\{(0,0)\}$ in $\mathbb{R}^2$ is $(a,b) + \{(0,0)\} = \{(a,b)\}$. Thus two cosets $\{(a,b)\}$ and $\{(c,d)\}$ are disjoint, iff $(a,b) \neq (c,d)$, i.e.,iff (a,b) and (c,d) are distinct elements of $\mathbb{R}^3$. Thus,

$$\mathbb{R}^2 = \cup\left\{(a,b) + \{(0,0)\} \,\middle|\, a, b \in \mathbb{R}\right\} = \cup\left\{(a,b) \,\middle|\, a, b \in \mathbb{R}\right\}$$

b) Any coset $(a,b) + \mathbb{R}^2 = \mathbb{R}^2$, since $(a,b) \in \mathbb{R}^2$. Thus, the only coset of $\mathbb{R}^2$ in $\mathbb{R}^2$ is $\mathbb{R}^2$ itself. So the disjoint union is only $\mathbb{R}^2$.

E20) a) Since $v \in W, v + W = W$

b) $v + W = \left\{(x-2)\left(x^2+1\right) + f(x) \in \mathbf{P} \text{ and } f(1) = 0\right\}$

E21) $\mathbf{P_3}/\mathbf{P_2} = \left\{\left(ax^3 + bx^2 + cx + d\right) + \mathbf{P}_2 \,\middle|\, a, b, c, d \in \mathbb{R}\right\}$.

Now, $\left\{ax^3 + \mathbf{P}_2 \,|\, a \in R\right\} \subseteq \mathbf{P_3}/\mathbf{P_2}$. Conversely, any element of $\mathbf{P_3}/\mathbf{P_2}$ is $\left(ax^3 + bx^2 + cx + d\right) + \mathbf{P}_2 = ax^2 + \mathbf{P}_2$(by Theorem 13)

$$\in \left\{ax^3 + \mathbf{P}_2 a \in \mathbb{R}\right\}.$$

Thus, $\mathbf{P_3}/\mathbf{P_2} = \left\{ax^3 + \mathbf{P}_2 \mid a \in \mathbb{R}\right\}$

E22) Firstly, note that W is a subspace of $\mathbb{R}^2$, and hence $\mathbb{R}^2/W$ is meaningful.

Now $\mathbb{R}^2/W = \left\{(a,b) + W \,\middle|\, a, b \in \mathbb{R}\right\}$.

For any $(a,b) - (a,0) = (0,b) \in W$

$$\therefore, \ (a,b) + W = (a,0) + W$$

Therefore, $\mathbb{R}^2/W = \{(a,0) + W \,|\, a \in \mathbb{R}\}$

E23) $V/V = \{v + V \mid v \in V\}$. But $v + V = V \; \forall v \in V, \therefore, V/V = V$.

3.8 MISCELLANEOUS EXERCISES

Example 1: Which of the following statements are true and which are false? Justify your answer with a short proof or a counter-example.

1) Product of any two symmetric matrices is symmetric.
2) If AB = BA, $A^tB^t = B^tA^t$.
3) If A and B are skew-symmetric matrices with AB = BA then AB is symmetric.
4) If A and B are skew-symmetric matrices, then A + B is skew-symmetric.
5) If A is a hermitian matrix, –A is skew-hermitian. (*IGNOU, Feb. 2021*)
6) There are two subspaces U and W of $\mathbb{R}^3$ such that $U \cap W = \varnothing$. (*IGNOU, Jun. 2019*)
7) $\mathbb{R}^2$ has infinitely many non-zero proper vector subspaces. (*IGNOU, Jun. 2021*)

Solution:

1) False. If $A = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and $B = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$, the A and B are symmetric, but $AB = \begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}$ is not symmetric.
2) True. We have $(AB)^t = B^tA^t = (-B)(-A) = BA = AB$
3) True. Taking transpose both sides, we get $(AB)^t = (BA)^t$ or $B^tA^t = A^tB^t$.
4) False. Let $A = \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$. Then, $A^* = A$. We have $-A = \begin{bmatrix}-1 & -2\\ -2 & -1\end{bmatrix}$ ad $(-A)^* = \begin{bmatrix}\overline{-1} & \overline{-2}\\ \overline{-2} & \overline{-2}\end{bmatrix} = \begin{bmatrix}-1 & -2\\ -2 & -1\end{bmatrix}$ $-(-A) = A$. We have $(-A)^* \neq A$. So, A is not skew-hermitian.
5) False. We have $\mathbf{0} \in U$ and $\mathbf{0} \in W$ since U and W are subspaces of R^3. So, $\mathbf{0} \in U \cap W$, so $U \cap W \neq \varnothing$.
6) True. For each $\alpha \in \mathbb{R}$, we have a subspace $W_\alpha = \{\}$

* * *

Here are some exercises for you to try.

E1) If A and B are n × n symmetric matrices with AB = BA, then AB is also a symmetric matrix.

E2) If A and B are skew-symmetric matrices, then A + B is skew-symmetric.

E3) The set of hermitian matrices form a subspace of $M_n(\mathbb{C})$ considered as a vector space over $\mathbb{C}$.

Example 2: Let $U = \{(a_{ij}) \in M_n(\mathbb{R}) | \, a_{ij} = 0 \text{ if } i > j\}$ the set of upper triangular mmatrices, and $L = \{(a_{ij}) \in M_n(\mathbb{R}) | \, a_{ij} = 0 \text{ if } i \le j\}$, the set of strictly lower triangular matrices. Show that U and L are subspaces of $M_n(\mathbb{R})$ and $M_n(\mathbb{R}) = U \oplus L$

Solution: We can visualise the result from the following equation:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ a_{21} & 0 & 0 & \cdots & 0 \\ a_{31} & a_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix}$$

In the above equation, the first matrix in the RHS is in U and the second matrix is in L. For example, we have

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & 3 & 1 \\ -1 & 3 & 1 & 1 \\ -3 & 4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ -3 & 4 & 1 & 0 \end{bmatrix}$$

We show that U is a subspace of $M_n(\mathbb{R})$. The proof for L is similar. Let $A + (a_{ij})$ and $B = (b_{ij})$ be in U. We have

$$a_{ij} = 0,\ b_{ij} = 0 \text{ for } i > j,\ 0 \le i, j \le n \qquad \ldots(3)$$

Suppose $C = (c_{ij})$ and $C = \alpha A + \beta B$. Then, from Eqn. (3), we have $c_{ij} = \alpha a_{ij} + \beta b_{ij} = 0$ for $i > j$, $1 \le i, j \le n$.

If $A = (a_{ij}) \in U \cap L$. Since $a_{ij} = 0$ for both $i > j$ as well as $i \le j$, $1 \le i, j \le n$, it follows that $a_{ij} = 0$ for all i, j, $1 \le i, j \le n$, i.e. $A = \mathbf{0}$. So, $U \cap L = \{\mathbf{0}\}$.

Let $A = (a_{ij}) \in M_n(\mathbb{R})$. Let $A' = (a'_{ij})$ and $A'' = (a''_{ij})$ where

$$a'_{ij} = \begin{cases} a_{ij} & \text{if } i \le j \\ 0 & \text{if } i > j \end{cases} \qquad \ldots(4)$$

and

$$a''_{ij} = \begin{cases} 0 & \text{if } i \le j \\ a_{ij} & \text{if } i > j \end{cases} \qquad \ldots(5)$$

Then, $A' \in U$ and $A'' \in L$. Further, check that $a_{ij} = a'_{ij} + a''_{ij}$ for all i, j, $1 \le i, j \le n$. It follows that $A = A' + A''$.

* * *

3.9 SOLUTIONS/ANSWERS

E1) True. We have $(AB)^t = B^tA^t = BA = AB$ since A and B commute.

E2) True We have $(A + B)^t = A^t + B^t = (-A) + (-B) = -(A + B)$.

E3) False. Let $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$. Then, $A^* = \begin{bmatrix} 1 & \overline{-i} \\ \overline{i} & 1 \end{bmatrix} = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} = A$. But, we have

$iA = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$ and $(iA)^* = \begin{bmatrix} \overline{i} & \overline{1} \\ \overline{-1} & \overline{i} \end{bmatrix} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \neq iA$.

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