UNIT 8 UNCONSTRAINED OPTIMISATION

Structure

8.0 OBJECTIVES

After going through this Unit, you will be able to:

- Explain the concept of a constraint in optimisation exercises;
- State the meaning of total differential;
- Describe the first-order and second-order conditions of optimisation subject to constraints;
- Explain the meaning of quadratic forms; and
- Discuss some economic applications of optimisation subject to constraints.

8.1 INTRODUCTION

In the previous unit, we have concentrated on objective functions with only one choice variable. However, this is a very restrictive assumption. Frequently we come across situations where more than one variable is involved. For example, a multi-product firm has to choose the optimal product mix that will enable the firm to maximise its overall profit. Even if we consider a firm producing a single output – usually the revenue earned by the firm depends not only on the quantity produced, but also on some other factors such as advertising expenditure, price charged by a competing firm, etc. It is more realistic to assume that the revenue (R) accruing to the firm is a function of its quantity (Q), advertising expenditure (A) and price charged by a competing firm (P). Hence it can be expressed as:

$$R = R(Q, A, P)$$

Now the question is: how does one of these variables influence the total revenue? To understand this, we have to take recourse to 'partial differentiation'. The latter enables us to capture the effect of each of these variables on R keeping others as constant.

Before deriving the first and second order conditions of optimisation of objective functions with more than one variable, let us consider the differential version of optimisation condition in one choice variable case. Recall the optimisation conditions laid down in course BECC-102 were in terms of "derivatives" as against "differentials". To prepare the background for solving optimisation problems with multivariate objective functions, it is important to see how these conditions can also be expressed in terms of differentials.

8.2 THE DIFFERENTIAL VERSION OF OPTIMISATION CONDITIONS

8.2.1 First-Order Condition

Consider the following function:

$$z = f(x) \tag{1}$$

At the maximum as well as the minimum points of the function, value of z is stationary. In other words, it is necessary for an extremum of z that $dz = 0$ as x varies. This constitutes the first-order condition for an extremum in the differential form. To verify that this condition is equivalent to the first derivative condition of zero slope, let us differentiate equation (1) totally. Total differentiation of equation (1) yields the following:

$$dz = f'(x)dx \tag{2}$$

Note if $dx = 0$, dz is automatically equal to zero. However, this is not what the first-order condition is all about. The first-order condition requires that at the extremum points, even with infinitesimal changes in *x, dz* should be equal to zero. Now with $dx \neq 0$, *dz* can be zero only if $f'(x) = 0$. Hence, the first-derivative condition $f'(x) = 0$ and the first differential condition "dz = 0 for arbitrary nonzero values of dx", are equivalent.

8.2.2 Second-Order Condition

The sufficient condition for a stationary point to be also a relative maximum is that $dz < 0$ in the immediate neighbourhood of that point. In other words, *z* is decreasing as we move away from this point either to the left or to the right. The fact that $dz = 0$ at the maximum point but $dz < 0$ on the two sides of the point means that *dz* is decreasing as we move away from the former in either direction. Consequently, the sufficient of a stationary value *z* to be a relative maximum is that $d(dz) < 0$ i.e. $d^2z < 0$ for arbitrary non-zero value of *dx*. This constitutes the second-order condition of maximisation in differential form. Again to verify that this is equivalent to the second-order derivative conditions, we totally differentiate equation (2). Total differentiation of equation (2) gives us:

$$d^2z = d(f'(x)dx)$$

Now $dx = constant$ (arbitrary non-zero value), therefore,

$$\begin{aligned} d^2z &= [df'(x)]dx \\ &= [f''(x)dx]dx \\ &= f''(x)(dx)^2 \end{aligned}$$

$$d^2z = f''(x)(dx)^2$$

Note, $(dx)^2 = (constant)^2 > 0$

Therefore, for $d^2z < 0;\ f''(x) < 0$

This again shows that the second-order differential condition is equivalent to the second-order derivative condition of maximisation. Analogously, for a stationary value of *z* to be a relative minimum, it is sufficient that $d(dz) > 0$ i.e. $d^2z > 0$. This is the sufficient condition of minimisation in differential form.

8.3 EXTREMUM VALUES OF A FUNCTION OF TWO VARIABLES

8.3.1 First-Order Condition for Objective Function with Two Variables

Assume that

$$z = f(x, y) \qquad (3)$$

The first-order necessary condition for an extremum (either maximum or minimum) again involves dz = 0. However, now since there are two choice variables, the first-order condition is modified as follows:

dz* = *0* for arbitrary non-zero values of *dx* and *dy

The rationale behind this is similar to the explanation of the condition dz = 0 for the one variable case: an extremum point must necessarily be a stationary point; at a stationary point $dz = 0$ ever for infinitesimal change in the two variables x and y. Totally differentiating equation (3), we get

$$dz = f_x dx + f_y dy$$

where $f_x = {df}/{dx}$ = partial derivative with respect to x and $f_y = {df}/{dy}$ = partial derivative with respect to y.

Now $dx \neq 0; \ dy \neq 0$ (A)

And at the stationary point $dz = 0$ (B)

(A) and (B) can hold simultaneously only if $f_x = f_y = 0$.

Hence the first-order condition for optimisation (location of extremum points) for an objective function with two variables is:

$$f_x = f_y = 0 \text{ or } {\partial z}/{\partial x} = {\partial z}/{\partial y} = 0$$

As in the earlier discussion, the first-order condition is necessary but not sufficient. To develop the sufficiency condition, we must look into the second-order to the differential, which is related to second-order partial derivatives.

Example*:*

Assuming that the second-order condition is satisfied, find out the profit-maximising values of quantity (Q) and advertising expenditure (A) for a producer with the following profit function (Π):

$$\Pi = 400 - 3Q^2 - 4Q + 2QA - 5A^2 + 48A \qquad (4)$$

The first-order condition for profit-maximisation requires that the partial derivative

$$ {\partial \Pi}/{\partial Q} = {\partial \Pi}/{\partial A} = 0 .$$

Partially differentiating equation (4) with respect to A keeping Q as constant gives us:

$${\partial \Pi}/{\partial A} = 2Q - 10A + 48$$

Partially differentiating equation (4) with respect to Q keeping A as constant gives us:

$\partial\Pi/\partial Q = -6Q - 4 + 2A$

Setting $\partial\Pi/\partial Q = \partial\Pi/\partial A = 0$ as the first-order condition we get:

$$2Q - 10A + 48 = 0 \quad (5)$$

and

$$-6Q - 4 + 2A = 0 \quad (6)$$

Equations (5) and (6) can be written as:

$$Q - 5A = -24 \quad (7)$$

$$-3Q + A = 2 \quad (8)$$

From equation (8), we get $A = 3Q + 2$. Substituting for A in equation (7) and we then solve for Q as follows:

$$Q - 5(2 + 3Q) = -24$$
$$Q - 10 - 15Q = -24$$
$$-14Q = 14$$
$$Q = 1$$

Hence $A = 2 + 3 * 1 = 5$.

So, we obtain that Q =1 and A = 5 are the quantity and advertising expenditure level which maximises profit for the firm using the first-order necessary condition.

8.3.2 Second-Order Partial Derivatives and Total Differentials

Note the partial derivatives are themselves function of the independent variables. Hence, they are capable of generating 'partial derivatives of higher order' through repeated differentiation. In the subsequent section, we shall show how the second-order partial derivatives are being generated. The former plays an important role in formulating second-order condition or the sufficiency condition for optimisation of objective functions with two independent variables.

Second-Order Partial Derivatives:

Once again let us start with the following objective function:

z = f(x,y)

The two first-order partial derivatives are: $f_x = df/dx$ = partial derivative with respect to *x* and $f_y = df/dy$ = partial derivative with respect to *y*.

Now the functions $f_x(x,y)$ and $f_y(x,y)$ are generated by partially differentiating the objective function are themselves functions of '*x*' and '*y*'. Consequently, we can measure the rate of change in f_x with respect to x keeping y constant. This generates second-order partial derivatives, symbolically represented as f_{xx}.

$$f_{xx} = \partial/\partial x (f_x) = \partial/\partial x (\partial z/\partial x) = \partial^2 z/\partial^2 x$$

The notation f_{xx} has a double subscript signifying that the primitive objective function *f(x,y)* has been partially differentiated with respect to x twice. Analogously we can find the second-order partial derivatives with respect to y. Since f_y is a function of y (and also x), we can measure the rate of change f_y with respect to y, keeping x as constant.

$$f_{yy} = \partial/\partial y (f_y) = \partial/\partial y (\partial z/\partial y) = \partial^2 z/\partial^2 y$$

The notation f_{yy} has a double subscript indicating that the primitive objective function *f(x,y)* has been partially differentiated with respect to y twice. This generates two other partial derivatives defined as follows:

$$f_{yx} = \partial/\partial y (f_x) = \partial/\partial y (\partial z/\partial x) = \partial^2 z/\partial y \partial x$$

$$f_{xy} = \partial/\partial x (f_y) = \partial/\partial x (\partial z/\partial y) = \partial^2 z/\partial x \partial y$$

These are also known as "**cross**" (or "**mixed**") partial derivatives.

Two important aspects have to be borne in mind.

- Firstly, even though f_{xy} and f_{yx} has been defined separately, they are identical to each other as long as the two cross partial derivatives are continuous functions (**Young's Theorem**). Hereafter, we shall assume that the cross derivatives are identical unless stated otherwise, i.e. $f_{xy} = f_{yx}$
- Secondly, each of the second order partial derivatives, i.e., f_{xx}, f_{yy} and f_{xy}, like the first partial derivative f_x and f_y, may also be functions of x and y.

Example:

Determine f_8, f_{22} and f_{12} for the following function:

$$f(x_1, x_2) = x_1^2 x_2^2 - x_1 x_2 + 3x_1 - 2x_2$$

$$f_1 = \frac{\partial f}{\partial x_1} = 2x_1 x_2^2 - x_2 + 3 \qquad (9)$$

$$f_2 = \frac{\partial f}{\partial x_2} = 2x_1^2 x_2 - x_1 - 2 \qquad (10)$$

f_8 is obtained by partially differentiating equation (9) with respect to x_1. Hence,

$$f_{11} = \frac{\partial}{\partial x_1}(f_1) = 2x_2^2$$

f_{22} is obtained by partially differentiating equation (10) with respect to x_2. Hence,

$$f_{22} = \frac{\partial}{\partial x_2}(f_1) = 2x_1^2$$

The cross derivative f_{12} is obtained by differentiating equation (10) with respect to x_1. Note that in this case following Young's theorem, same result would have been obtained by differentiating equation (9) with respect to x_2. Hence,

$$f_{12} = \frac{\partial}{\partial x_1}(f_2) = \frac{\partial^2 f}{\partial x_1 x_2} = 4x_1x_2 - 1$$

Therefore the solution:

$$f_{11} = 2x_2^2$$
$$f_{22} = 2x_1^2$$
$$f_{12} = 4x_1x_2 - 1$$

Second-Order Total Differential:

The concept of partial derivatives enables us to write the total differential of a function. Recall for a function $z = f(x,y)$, the total differential can be expressed as:

$$dz = f_x dx + f_y dy \qquad (11)$$

where dx and dy are non-zero arbitrary infinitesimal change in x and y respectively and to be treated as constants. Consequently, dz depends only on f_x and f_y and since f_x and f_y are themselves functions of x and y, dz like z is also a function of the two choice variables x and y.

The second-order total differential, $d^2z \equiv d(dz)$ is a measure of the change of dz itself and is expressed in terms of the second-order partial derivatives defined earlier. To obtain d^2z we need to once again totally differentiated dz, using equation (8).

$$\begin{aligned}
d^2z &\equiv d(dz) \\
&= \frac{\partial(dz)}{\partial x}dx + \frac{\partial(dz)}{\partial y}dy \\
&= \frac{\partial}{\partial x}(f_x dx + f_y dy)dx + \frac{\partial}{\partial y}(f_x dx + f_y dy)dy \\
&= (f_{xx}dx + f_{xy}dy)dx + (f_{yx}dx + f_{yy}dy)dy \\
&= f_{xx}(dx)^2 + f_{xy}dydx + f_{yx}dxdy + f_{yy}(dy)^2 \\
&= f_{xx}(dx)^2 + 2f_{xy}dxdy + f_{yy}(dy)^2,\ as\ f_{yx} = f_{xy}
\end{aligned}$$

In other words, the second-order total differential depends on second-order partial derivatives.

$$d^2z = f_{xx}(dx)^2 + 2f_{xy}dxdy + f_{yy}(dy)^2 \quad \textbf{(A)}$$

Recall, *dz* measures the rate of change in *z*, while d^2z measures the rate of change of *dz*. If $d^2z > 0$ then this implies that *dz* is increasing and if $d^2z < 0$ then this implies that *dz* is decreasing.

Example:

Given $z = 2x^3 + 4xy - y^2$, find *dz* and d^2z.

Step 1: To find *dz*, we need to obtain the first-order partial derivatives f_x and f_y respectively.

Partially differentiating the given equation with respect to *x*, we get

$$f_x = \frac{\partial z}{\partial x} = 6x^2 + 4y = 2(3x^2 + 2y) \quad (12)$$

Partially differentiating the given equation with respect to *y* we get

$$f_y = \frac{\partial z}{\partial y} = 4x - 2y = 2(2x - y) \quad (13)$$

Now

$$dz = f_x dx + f_y dy$$

$$dz = 2(3x^2 + 2y)dx + 2(2x - y)dy$$

Step 2: To obtain d^2z we need to find out the second-order partial derivatives, f_{xx}, f_{xy} and f_{yy}. To find f_{xx}, we partially differentiate equation (12) with respect to *x* to get the following:

$$f_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}[2(3x^2 + 2y)] = 2*6x = 12x$$

To find f_{yy}, we partially differentiate equation (13) with respect to *y* to get the following:

$$f_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}[2(2x - y)] = -2$$

To find f_{xy}, we partially differentiate equation (13) with respect to *x* to get the following:

$$f_{xy} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}[2(2x - y)] = 4$$

Now

$$d^2z = f_{xx}(dx)^2 + 2f_{xy}dxdy + f_{yy}(dy)^2$$

Substituting in the above equation for f_{xx}, f_{xy} and f_{yy}, we get

$$d^2z = 12x(dx)^2 + 2*4dxdy - 2(dy)^2$$
$$d^2z = 12x(dx)^2 + 8dxdy - 2(dy)^2$$

Solution:

$$dz = 2(3x^2 + 2y)dx + 2(2x - y)dy$$

$$d^2z = 12x(dx)^2 + 8dxdy - 2(dy)^2$$

8.3.3 Second-Order Condition for Objective Function with Two Variables

Let us now examine the sufficiency condition for optimisation where the objective function has two decision variables. Using the concept of d^2z, we can state the second-order sufficient condition for a maximum of $z = f(x,y)$ as follows:

$d^2z < 0$ for arbitary non-zero values of *dx* and *dy*.

The rationale behind this is similar to that of the d^2z condition explained in the case where the objective function has one variable. Analogously, the second-order sufficiency condition for a minimum of $z = f(x,y)$ is the following:

$d^2z > 0$ for arbitary non-zero values of *dx* and *dy*.

Note that d^2z is a function of the second order partial derivatives f_{xx}, f_{xy} and f_{yy}. Intuitively, it is clear that the second-order sufficiency condition can be translated in terms of these derivatives. However, the actual translation would require knowledge of quadratic form – the discussion of which is given in section 8.5. Hence, we will <u>state</u> the main results here:

For any values of *dx* and *dy*, not both zero:

$$d^2z < 0 \text{ iff } f_{xx} < 0; f_{yy} < 0 \text{ and } f_{xx}f_{yy} > f_{xy}^2$$

and

$$d^2z > 0 \text{ iff } f_{xx} > 0; f_{yy} > 0 \text{ and } f_{xx}f_{yy} > f_{xy}^2$$

To sum up, the first-order and the second-order condition for optimisation in case of an objective function $z = f(x,y)$ is depicted in the following Table:

Table 8.1: First-Order and Second-Order Condition for Optimisation

	Maximum	**Minimum**
First-Order Necessary Condition	$f_x = f_y = 0$	$f_x = f_y = 0$
Second-Order Sufficiency Condition	$f_{xx}, f_{yy} < 0$ and $f_{xx}f_{yy} > f^2_{xy}$	$f_{xx}, f_{yy} > 0$ and $f_{xx}f_{yy} > f^2_{xy}$

The second order condition is applicable only after the first-order condition has been fulfilled.

In example 1, we had solved for the optimal product (*Q*) and the advertising expenditure (*A*), assuming that second-order condition is satisfied at the optimal point. Now let us examine whether it is actually satisfied or not.

Recall that

$$\Pi = 400 - 3Q^2 - 4Q + 2QA - 5A^2 + 48A$$

$$\Pi_A = \partial\Pi / \partial A = 2Q - 10A + 48 \quad (14)$$

$$\Pi_Q = \partial\Pi / \partial Q = -6Q - 4 + 2A \quad (15)$$

Setting them equal to zero and solving for Q and A yields $Q^* = 1$ and $A^* = 5$, where * denotes the optimal level.

For the second-order condition we need to derive the following partial derivatives:

$$\frac{\partial\Pi^2}{\partial A^2}, \frac{\partial\Pi^2}{\partial Q^2} \text{ and } \frac{\partial\Pi^2}{\partial A\partial Q}$$

Partial differentiation of equation (14) with respect to A gives us:

$$\Pi_{AA} = \frac{\partial\Pi^2}{\partial A^2} = -10 < 0 \quad (16)$$

Partial differentiation of equation (15) with respect to Q gives us:

$$\Pi_{QQ} = \frac{\partial\Pi^2}{\partial Q^2} = -6 < 0 \quad (17)$$

Partial differentiation of equation (15) with respect to A gives us:

$$\Pi_{AQ} = \frac{\partial\Pi^2}{\partial A\partial Q} = 2$$

Now

$\Pi_{AA} * \Pi_{QQ}$ = -10 * -6 = 60

and $(\Pi_{AQ})^2 = (2)^2 = 4$

Hence $\Pi_{AA} * \Pi_{QQ} > (\Pi_{AQ})^2$ (18)

From (16), (17) and (18), it follows that the second-order condition is satisfied for the output level (*Q*) equal to 1 and the advertising expenditure (*A*) equal to 5.

Example:

Find the extreme values of the following function and that determine whether it is a maxima or a minima.

$$z = -x^2 + xy - y^2 + 2x + y \qquad (19)$$

For a point (x_0, y_0) to be an extrema, it is necessary that the first-order condition is satisfied at that point.

First-order condition: $f_x = f_y = 0$ or $\partial z/\partial x = \partial z/\partial y = 0$

Partially differentiating equation (19) with respect to x we get,

$$f_x = \frac{\partial z}{\partial x} = -2x + y + 2 \qquad (20)$$

Partially differentiating equation (19) with respect to y we get,

$$f_y = \frac{\partial z}{\partial y} = x - 2y + 1 \qquad (21)$$

Using the first-order necessary condition, set equation (20) and (21) to zero and solve for x and y to locate the extrema. Hence,

$$-2x + y + 2 = 0 \qquad (22)$$

$$x - 2y + 1 = 0 \qquad (23)$$

Multiplying equation (23) with 2 we get:

$$2x - 4y + 2 = 0 \qquad (24)$$

Adding equation (22) and equation (24) we get:

$$-2x + y + 2 + 2x - 4y + 2 = 0$$

$$-3y + 4 = 0$$

$$y = 4/3$$

Substituting $y = 4/3$ in equation (23) we can obtain the value for x:

$$x - 8/3 + 1 = 0$$

$$x = 11/3$$

Hence $x = 8/3$ and $y = 4/3$ satisfies the first-order necessary condition for an extrema.

To verify whether the second order condition is satisfied at this point and examine whether it is a maximum or a minimum, we need to determine the following second order partial derivatives: f_{xx}, f_{xy} and f_{yy}.

Partially differentiating equation (20) with respect to x we get,

$$f_{xx} = \frac{\partial^2 z}{\partial x^2} = -2 < 0 \qquad (25)$$

Partially differentiating equation (21) with respect to y we get,

$$f_{yy} = \frac{\partial^2 z}{\partial y^2} = -2 < 0 \qquad (26)$$

Partially differentiating equation (21) with respect to *x* we get,

$$f_{xy} = \frac{\partial^2 z}{\partial x \partial y} = 1$$

Now using these values, we see that

$$f_{xx} f_{yy} = -2 * -2 = 4; f_{xy}^2 = 1 * 1 = 1; \text{this implies } f_{xx} f_{yy} > f_{xy}^2 \qquad (27)$$

From, (25), (26) and (27), it follows that the second-order condition for a maximum is satisfied at *x =8/3* and *y = 4/3*. The corresponding stationary values of z (which is also a relative maxima of the function!) is thus solved by substituting the optimal values of *x* and *y* in the objective function given by equation (19). We get *z = -17/9*.

Solution: *x = 8/3*, *y = 4/3* and *z = -17/9* is relative maximum.

Check Your Progress 1

1) Consider the following utility function of a consumer:

   a) $U = q_1^2 + q_2^2$

   b) $U = q_1 + q_2 + 2q_1q_2 - 0.01(q_1^2 + q_2^2)$

   c) $U = Aq_1^{\alpha} q_2$ where $A > 0; \alpha > 0$

   Find the four second-order partial derivatives, determine their signs and interpret the results economically.

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2) Find the first-order (dU) and the second-order total differentials (d^2U) of the three functions given in Problem 1.

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8.4 QUADRATIC FORMS

The expression for d^2z in equation (A) some pages earlier exemplifies what is known as 'quadratic forms'. A polynomial expression in which each of the components is of second degree (sum of exponents in each term equals 2) constitutes a quadratic form. In this section, we will go on to express quadratic equation in matrix form. Thereafter we shall state the conditions of "positive definiteness" and "negative definiteness". The latter plays an important role in locating extreme values of objective functions with multiple numbers of decision variables.

The quadratic equation in general form with '*n*' number of variables can be expressed as:

$$\begin{aligned} Q = & a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \ldots\ldots + a_{1n}x_1x_n \\ & + a_{21}x_1x_2 + a_{22}x_2^2 + a_{23}x_2x_3 + \ldots\ldots + a_{2n}x_2x_n \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + a_{n1}x_1x_n + a_{n2}x_2x_n + a_{n3}x_3x_n + \ldots\ldots + a_{nn}x_n^2 \end{aligned}$$

Assuming that $a_{ij} = a_{ji}$, we get

$$\begin{aligned} Q = & a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \ldots\ldots + a_{1n}x_1x_n \\ & + a_{12}x_1x_2 + a_{22}x_2^2 + a_{23}x_2x_3 + \ldots\ldots + a_{2n}x_2x_n \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + a_{1n}x_1x_n + a_{2n}x_2x_n + a_{3n}x_3x_n + \ldots\ldots + a_{nn}x_n^2 \end{aligned}$$

Suppose X is a $(n \times 1)$ column vector comprising of the n variables and A is $(n \times n)$ is a square and symmetric matrix comprising of the coefficients a_{ij}'s, i.e.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}_{n\times 1} \quad \text{and } A = \begin{bmatrix} a_{11} & a_{12} & . & a_{1n} \\ a_{12} & a_{22} & . & a_{2n} \\ . & . & . & . \\ a_{1n} & a_{2n} & . & a_{nn} \end{bmatrix}_{n\times n}$$

Then Q can be expressed as a product of these matrices, i.e.

$Q = X'AX$ where X' indicates transpose of X.

Example:

Consider the quadratic equation $Q = x_1^2 + x_2^2 - 2x_1x_2$. Express it in matrix form.

$$Q = x_1^2 + x_2^2 - 2x_1x_2 \qquad (28)$$

Rearranging the individual components in Q, equation (28) can be written as:

$$\begin{aligned} Q = & x_1^2 - x_1x_2 \\ & - x_1x_2 + x_2^2 \end{aligned}$$

In his case $a_8 = 1, a_{12} = a_{21} = -1$ *and* $a_{22} = 1$

Hence $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{2\times2}$ and $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times1}$

Consequently $Q = X' \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} X$

Once we have defined a quadratic form, we can go on to lay down the conditions that need to be satisfied for positive definiteness and negative definiteness.

Fact 1 : The quadratic form $Q = X'AX$ in n variables is positive definite when it takes a positive value for any values of the variables (not all zero).

Fact 2 : The quadratic form $Q = X'AX$ in n variables is negative definite when it takes a negative value for any values of the variables (not all zero).

Fact 3 : The quadratic form $Q = X'AX$ in n variables is positive definite if and only if the principal minors of determinant of A are all positive. Note A by definition is a square, symmetric and non-singular matrix. The principal minor is obtained by deleting the last *(n-i)* rows and *(n-i)* columns of A.

Fact 4 : The quadratic form $Q = X'AX$ is negative definite if and only if the principal minors of the determinant of A is alternate in signs, with the first principal minor being negative.

Example: Determine $Q = x_1^2 + x_2^2$ is positive or negative definite.

$$Q = x_1^2 + 0 \times x_1 x_2 \\ + 0 \times x_1 x_2 + x_2^2$$

Hence $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$ and $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times1}$

Therefore, $|A_{11}| = 1 > 0$ and $|A_{22}| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 > 0$

This implies that Q is positive definite.

Example: Determine whether $Q = x_1^2 + 6x_2^2 + 3x_3^2 - 2x_1x_2 - 4x_2x_3$ is positive or negative definite

The equation can be rewritten as:

$$Q = x_1^2 - x_1x_2 + 0 \times x_1x_3$$
$$- x_1x_2 + 6x_2^2 - 2x_2x_3$$
$$+ 0 \times x_1x_3 - 2x_2x_3 + 3x_3^2$$

In this case, $a_8 = 1$, $a_{22} = 6$, $a_{33} = 3$, $a_{13} = a_{31} = 0$, $a_{12} = a_{21} = -1$, $a_{23} = a_{32} = -2$

So,

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 3 \end{bmatrix}_{3\times3}$$

The principal minors of $|A|$ are as follows:

$$|A_{11}| = 1 > 0$$

$$|A_{22}| = \begin{vmatrix} 1 & -1 \\ -1 & 6 \end{vmatrix} = 6 - 1 = 5 > 0$$

$$|A_{33}| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 3 \end{vmatrix} = 1(18-4) + 1(-3) + 0(2-0) = 11 > 0$$

Therefore, the quadratic form is positive definite.

8.5 SECOND-ORDER TOTAL DIFFERENTIAL AS A QUADRATIC FORM

Recall the second-order condition in differential form for an objective function with two variables, i.e. $z = f(x,y)$ is

$$d^2z = f_{xx}(dx)^2 + 2f_{xy}dxdy + f_{yy}(dy)^2$$

The above can be expressed as:

$$d^2z = f_{xx}(dx)^2 + f_{xy}dxdy$$
$$+ f_{xy}dxdy + f_{yy}(dy)^2$$

Expressing d^2z in matrix form, we get

$$d^2z = X'AX \text{ where } X = \begin{bmatrix} dx \\ dy \end{bmatrix}_{2\times1} \text{ and } A = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}_{2\times2}$$

The second-order sufficiency condition for extremum requires d^2z to be positive definite (for a minimum) and negative definite (for a maximum) regardless what values dx and dy take (as long as they both are not equal to zero).

Hence for a minimum,

$d^2z > 0 \Leftrightarrow$ principal minors of $A = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}_{2\times 2}$ are all positive. In other words, $|A_{11}| : f_{xx}; f_{yy} > 0$ and $|A_{22}|$, i.e. $f_{xx}f_{yy} - (f_{xy})^2 > 0 \Rightarrow f_{xx}f_{yy} > (f_{xy})^2$ (C)

And for a maximum

$d^2z < 0 \Leftrightarrow$ principal minors of $A = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}_{2\times 2}$ are alternate in sign. In other words, $|A_{11}| : f_{xx}; f_{yy} < 0$ and $|A_{22}|$, i.e. $f_{xx}f_{yy} - (f_{xy})^2 > 0 \Rightarrow f_{xx}f_{yy} > (f_{xy})^2$ (D)

We have already depicted conditions (C) and (D) in Table 1.

8.6 OBJECTIVE FUNCTIONS WITH MORE THAN TWO VARIABLES

8.6.1 First-Order Condition for Extremum if Objective Function has More than Two Variables

Let us consider a function with three variables

$$z = f(x_1, x_2, x_3) \quad (29)$$

with first-order partial derivatives f_1, f_2 and f_3 and second-order partial derivatives $f_{ij} (\equiv \frac{\partial^2 z}{\partial x_i \partial x_j})$ with $j = 1, 2, 3$. On the basis of Young's theorem we get $f_{ij} = f_{ji}$ for all $i \neq j$.

As mentioned earlier, an extremum point (maximum or minimum) corresponds to a stationary value of 'z'. In other words, to have an extremum of z, it is necessary that

$dz = 0$ for arbitrary values of dx_1, dx_2 and dx_3, not all zero. Totally differentiating equation (29) we get:

$$dz = f_1dx_1 + f_2dx_2 + f_3dx_3 \quad (30)$$

where $f_1 = \frac{\partial f}{\partial x_1}, f_2 = \frac{\partial f}{\partial x_2}$ and $f_3 = \frac{\partial f}{\partial x_3}$. Since, dx_1, dx_2 and dx_3 are arbitrary (infinitesimal) changes in the independent variables, not all zero, the only way to ensure a zero dz is to have $f_1 = f_2 = f_3 = 0$. Once again we see that the necessary condition for an extremum is that all the first-order partial derivatives are equal to zero.

8.6.2 Second-Order Condition for Extremum if Objective Function has More than Two Variables:

If the first-order condition for an extremum is fulfilled then the sufficiency condition that needs to be satisfied is as follows: at a stationary value of z if we find that d^2z is positive definite then this will suffice to establish z as a minimum. Analogously, at a stationary value of z if we find that d^2z is negative definite then this is sufficient to establish z as a maximum.

As before, the expression for d^2z can be obtained by totally differentiating equation (30). Recall $f_i = f_i(x_1, x_2, x_3)$; and dx_i measures arbitrary non-zero constant change for all

$i = 1, 2, 3$. Hence, $dz = \Phi(x_1, x_2, x_3)$.

To get the exact form, total differentiation of equation (30) gives us the following:

$$\begin{aligned} d^2z &= d(dz) \\ &= \frac{\partial(dz)}{\partial x_1}dx_1 + \frac{\partial(dz)}{\partial x_2}dx_2 + \frac{\partial(dz)}{\partial x_3}dx_3 \\ &= \frac{\partial}{\partial x_1}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_1 \\ &\quad + \frac{\partial}{\partial x_2}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_2 \\ &\quad + \frac{\partial}{\partial x_3}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_3 \\ &= (f_{11}dx_1 + f_{12}dx_2 + f_{13}dx_3)dx_1 \\ &\quad + (f_{12}dx_1 + f_{22}dx_2 + f_{23}dx_3)dx_2 \\ &\quad + (f_{13}dx_1 + f_{23}dx_2 + f_{33}dx_3)dx_3 \end{aligned}$$

Assuming $f_{ij} = f_{ji}$

Or

$$\begin{aligned} d^2z = {} & f_{11}dx_1^2 + f_{12}dx_1dx_2 + f_{13}dx_1dx_3 \\ & + f_{12}dx_1dx_2 + f_{22}dx_2^2 + f_{23}dx_2dx_3 \\ & + f_{13}dx_1dx_3 + f_{23}dx_2dx_3 + f_{33}dx_3^2 \end{aligned} \qquad (31)$$

Note that this is in the form of a quadratic equation with three variables dx_1, dx_2 and dx_3 and the coefficients expressed in terms of the second-order partial derivatives. We can thus express equation (31) in matrix form as shown below:

Let $X = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}_{3\times1}$ and $A = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{bmatrix}_{3\times3}$

Here the matrix consisting of second-order partial derivatives as elements is called a Hessian matrix.

Then

$$d^2z = X'AX \qquad (32)$$

where A is by definition a square symmetric matrix. As stated earlier the second-order sufficiency condition requires that d^2z is positive definite if z is a minimum. Using Fact 3, recall that d^2z is positive definite if and only if the principal minor determinants of the Hessian A are all positive, i.e. $f_{11}, f_{22} > 0, \begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix} > 0 \text{ and } |A| > 0$

Using Fact 4, the second-order condition for maximization of an objective function with three decision variables reduces to d^2z is a maximum (negative definite) if and only if the principal minor determinants of the Bordered-Hessian A are alternate in sign with the first principal minor being negative, i.e. $|A_{11}| < 0, |A_{22}| > 0, |A_{33}| < 0 \Rightarrow f_{11} < 0, \begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix} > 0 \text{ and } |A| < 0$

The requisite necessary and sufficient condition for optimisation of an objective function with three decision variables is presented in tabular form in Table 2.

Table 8.2: Condition for Extremum: $z = f(x_1, x_2, x_3)$

Condition	Maximum	Minimum
First-Order	$f_1 = f_2 = f_3 = 0$	$f_1 = f_2 = f_3 = 0$
Second-Order	$f_{11} < 0$ and $f_{11}f_{22} > (f_{12})^2$ and $\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{vmatrix} < 0$	$f_{11} > 0$ and $f_{11}f_{22} > (f_{12})^2$ and $\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{vmatrix} > 0$

Example :

Find the extreme values of $z = -x_1^3 + 3x_1x_3 + 2x_2 - x_2^2 - 3x_3^2$ (33)

For the first-order condition, we need to partially differentiate equation (33) with respect to x_1, x_2 and x_3 and set it to be equal to zero.

$$\frac{\partial z}{\partial x_1} = -3x_1^2 + 3x_3 = 0$$

$$\frac{\partial z}{\partial x_2} = 2 - 2x_2 = 0$$

$$\frac{\partial z}{\partial x_3} = 3x_1 - 6x_3 = 0$$

Solving these three equations yields us the following solution:

$$(x_1^*, x_2^*, x_3^*) = \begin{cases} (0,1,0) \text{ implying } z^* = 1 \\ (\frac{1}{2}, 1, \frac{1}{4} \text{ implying } z^* = \frac{17}{16} \end{cases}$$

The second-order partial derivatives can be rearranged to get the following determinant:

$$|A| = \begin{vmatrix} -6x_1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{vmatrix}$$

The principal minors of $|A|$ are:

$$|A_{11}| = |-6x_1|; |A_{22}| = \begin{vmatrix} -2 & 0 \\ 0 & -6 \end{vmatrix}; \text{and } |A_{33}| = |A|$$

At $x_1 = 0$, $|A_{11}| = 0$ which does not satisfy the second-order condition as stated earlier. Hence (0,1,0) is ruled out as a possible extremum.

At $x_1 = 1/2$, $|A_{11}| = -3; |A_{22}| = 12$ and $|A_{33}| = -18$. This duly alternates in sign. Consequently, $z^* = \frac{17}{16}$ is a maximum.

Example:

Find the extreme value of the following function:

$$z = 29 - (x_1^2 + x_2^2 + x_3^2)$$

The first-order condition for extremum requires that the simultaneous satisfaction of the following three equations based on the first-order partial derivatives:

$$f_1 = \frac{\partial z}{\partial x_1} = -2x_1 = 0$$

$$f_2 = \frac{\partial z}{\partial x_2} = -2x_2 = 0$$

$$f_3 = \frac{\partial z}{\partial x_3} = -2x_3 = 0$$

There exists only a unique solution $x_1^* = x_2^* = x_3^* = 0$. This means that there is only one stationary value of z i.e. $z^* = 29$.

To evaluate whether this is a relative extremum the second-order condition must be fulfilled. The Hessian determinant (defined earlier) of this function is:

$$|A| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{vmatrix} = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix}$$

To verify the second-order condition, we need to see the sign of the principal minors of the Hessian.

Note that

$$|H_{11}| = -2 < 0, |H_{22}| = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0 \text{ and } |H_{33}| = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -8 < 0 .$$

This duly alternates in sign with the first principal minor being negative. Thus we conclude that $z^* = 29$ is a maximum.

Check Your Progress 2

1) Express each quadratic form given below as a matrix products involving a square symmetric coefficient matrix:

   a) $z = 3x_1^2 - 4x_1x_2 + 7x_2^2$

   b) $z = 6q_1q_2 - 2q_1^2 - 5q_2^2$

   c) $z = x_1^2 + x_2^2 + x_3^2 - x_1x_3 - x_2x_3$

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2) Ascertain whether the quadratic forms given in Problem 1 are positive definite or negative definite.

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3) Obtain the extreme values, if any of the following functions (indicate whether it is maximum or minimum)

   a) $z = 5 - (x_1^2 + x_2^2 + x_3^2)$

   b) $z = e^{2x} - e^{y} + e^{w} - 2(x + e^{w}) + y$

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8.7 FUNCTIONS WITH n VARIABLES

We have already defined the first-order and second-order conditions for optimisation of functions with three variables. The concept can easily be extended to generate the necessary and sufficient conditions of optimisation of functions with *n – variables* defined below:

$$z = f(x_1, x_2, ..., x_n) \qquad (34)$$

First-Order Condition:

The first-order (necessary) condition for extremum is that *z* is stationary at that point, i.e., $dz = 0$ for arbitrary constant change in $x_1, x_2, ..., x_n$, not all equal to zero.

Totally differentiating equation (34) we get,

$$dz = f_1 dx_1 + f_2 dx_2 + ... + f_n dx_n$$

For first-order condition to hold, we need $f_1 = f_2 = ... = f_n = 0$

Second-Order Condition:

The second-order differential, as seen earlier, can be expressed in Quadratic form. The relevant Hessian determinant is:

$$|A| = \begin{vmatrix} f_{11} & f_{12} & f_{13} & . & f_{1n} \\ f_{12} & f_{22} & f_{23} & . & f_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ f_{1n} & f_{2n} & f_{3n} & . & f_{nn} \end{vmatrix}_{n \times n}$$

where f_{ij}'s are the second-order partial derivatives and following Young's theorem $f_{ij} = f_{ji}$ for all $i \neq j$. The n-principal minors $|A_{11}|, |A_{22}|, ..., |A_{nn}|$, as defined before, are formed by deleting the last (*n-i*) rows and (*n-i*) columns.

8.8 ECONOMIC APPLICATION OF OPTIMISATION PROBLEM

8.8.1 Multi-plant Monopolist

In this case, we shall examine the case of a monopolist who produces a homogenous product in two different plants. The analysis has been restricted to two plants for simplicity but can easily be generalized to '*n*' plants.

Suppose the aggregate market demand (*Q*) faced by the monopolist as a function of price (*P*) alone as follows:

$$Q = 100 - 2P$$
$$P = 100 - \tfrac{1}{2}Q \qquad (35)$$

Assume that the cost of producing the output in two different plants is the following:

$$C_1 = 10Q_1 \text{ and } C_2 = \frac{1}{4}Q_2^2 \tag{36}$$

where C_1 and C_2 is the total cost of producing the product in plant 1 and in plant 2 respectively. Q_1 and Q_2 is the amount of the homogenous product produced in plant 1 and in plant 2 respectively. Obviously,

$$Q = Q_1 + Q_2 \tag{37}$$

The total profit function faced by the monopolist is:

$$\Pi = R - C_1 - C_2$$

where R is the total revenue.

Now,

$$\begin{aligned} R &= PQ \\ &= (100 - \tfrac{1}{2}Q)Q \\ &= (100 - \tfrac{1}{2}(Q_1 + Q_2))(Q_1 + Q_2) \\ &= 100(Q_1 + Q_2) - \tfrac{1}{2}(Q_1 + Q_2)^2 \\ &= 100Q_1 + 100Q_2 - \tfrac{1}{2}(Q_1^2 + Q_2^2 + 2Q_1Q_2) \end{aligned}$$

Hence,

$$R = -\frac{1}{2}Q_1^2 - \frac{1}{2}Q_2^2 + 100Q_1 + 100Q_2 - Q_1Q_2 \tag{38}$$

Substituting the value of R from equation (38) into the profit function, we get

$$\Pi = -\frac{1}{2}Q_1^2 - \frac{1}{2}Q_2^2 + 100Q_1 + 100Q_2 - Q_1Q_2 - C_1 - C_2$$

Substituting for C_1 and C_2 on the basis of equation (36), we get

$$\Pi = -\frac{1}{2}Q_1^2 - \frac{1}{2}Q_2^2 + 100Q_1 + 100Q_2 - Q_1Q_2 - 10Q_1 - \frac{1}{4}Q_2^2$$

$$\Pi = -\frac{1}{2}Q_1^2 - \frac{3}{4}Q_2^2 + 90Q_1 + 100Q_2 - Q_1Q_2$$

To solve for Q_1 and Q_2 we set the first-order partial derivatives

$\dfrac{\partial \Pi}{\partial Q_1} = \dfrac{\partial \Pi}{\partial Q_2} = 0$ and we get

$$\Pi_1 = \frac{\partial \Pi}{\partial Q_1} = -Q_1 + 90 - Q_2 = 0$$

$$Q_1 + Q_2 = 90 \tag{39}$$

$$\Pi_2 = \frac{\partial \Pi}{\partial Q_2} = -\frac{3}{2}Q_2 + 100 - Q_1 = 0$$

$$2Q_1 + 3Q_2 = 200 \tag{40}$$

Solving equations (39) and (40) simultaneously yields the following output combinations:

$Q_1^* = 70$ and $Q_2^* = 20$

Our next step is to verify that the second-order condition is satisfied at this point. The relevant Hessian determinant is:

$$|A| = \begin{vmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -1 & -3/2 \end{vmatrix}$$

where Π_{ij}'s are the second-order partial derivatives. The sufficiency condition for maximization requires that the principal minors of the Hessian alternates in sign with the first principal minor being negative, i.e.

$|A_{11}| = \Pi_{11} < 0$ and

$|A_{22}| = |A| > 0$

Note that

$|\Pi_{11}| = -1 < 0$ and

$$|A| = \begin{vmatrix} -1 & -1 \\ -1 & -3/2 \end{vmatrix} = 1/2 > 0$$

Thus the necessary and sufficient conditions for profit maximization are both satisfied for $Q_1^* = 70$ and $Q_2^* = 20$.

The monopolist's profit at this output combination i.e. $\Pi^* = 3525$ is the maximum.

Solution: Output produced in first plant = 70 units

Output produced in second plant = 20 units

Maximum Profit = 3525 units

8.8.2 Price Discriminating Monopolist

A monopolist need not sell his entire output in one market. In some situations he is able to sell his output in two or more different market, at different price and thereby increase his aggregate profit. A fundamental requirement for price discrimination is that the buyers cannot buy the product from one market and resell it into another. Price discrimination is often possible in markets that are regionally separated for instance 'home' and 'abroad'. It is also often encountered for products like 'electricity' where resale of such commodities is not feasible.

Assume, for simplicity that a monopolist, sells his product in two markets whose demand functions are as follows:

In Market 1: $p_1 = 80 - 5q_1$ where p_1, q_1 are price charged and quantity sold in the first market.

In Market 2: $p_2 = 180 - 20q_2$ where p_2, q_2 are price charged and quantity sold in the second market.

The aggregate cost function of the monopolist is: $C = 50 + 20\ (q_1 + q_2)$

The revenue earned by the monopolist from Market 1 is the following:

$$R_1 = p_1 q_1 = (80 - 5q_1)q_1 = 80q_1 - 5q_1^2$$

The revenue earned by the monopolist from Market 2 is the following:

$$R_2 = p_2 q_2 = (180 - 20q_2)q_2 = 180q_2 - 20q_2^2$$

The aggregate profit accruing to the monopolist is

$\prod = R_1 + R_2 - C$

Substituting for R_1, R_2 and C we get:

$$\Pi = 80q_1 - 5q_1^2 + 180q_2 - 20q_2^2 - 50 - 20q_1 - 20q_2 \qquad \text{(E)}$$

The First Order Condition for maximisation requires the simulataneous solution of the following equations:

$$\Pi_1 = \frac{\partial \Pi}{\partial q_1} = 80 - 10q_1 - 20 = 0$$

$$or \quad q_1^* = 6$$

$$\Pi_2 = \frac{\partial \Pi}{\partial q_2} = 180 - 40q_2 - 20 = 0$$

$$or \quad \text{q}_2^* = 4$$

The next step is to see whether the Second Order Condition of maximisation is fulfilled at (q_1^*, q_2^*).

For this we require the principal minors of the requisite Hessian Determinant to alternate in signs, with the first being negative.

The Hessian in this case is the following:

$$|A| = \begin{vmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{vmatrix} = \begin{vmatrix} -10 & 0 \\ 0 & -40 \end{vmatrix}$$

Note $|A_8| = -10 < 0$ and $|A_{22}| = |A| = 400 > 0$

This implies that the second order condition is satisfied for this output combination.

The maximum profit earned by the price discriminating monopolist can be calculated by substituting for $q_1^* = 6$ and $q_2^* = 4$ in equation (E).

Hence

II = 80(6) – 5(36) + 180(4) – 20(16) – 50 – 20(6) – 20(4) = 450

Solution: Output sold in Market 1 = 6 units

Output sold in Market 2 = 4 units

Maximum Profit = 450 units

Check Your Progress 3

1) A monopolist uses one input, *X*, which he purchases at the fixed price = *5* to produce his output, *q*. The demand and production functions are $p = 85 - 2q$ and $q = 2\sqrt{x}$ respectively. Determine the values of *p*, *q*, and *x* at which the monopolist maximises his profits.

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2) Let the demand and cost functions of a monopolist be $p = 100 - 3q + 4\sqrt{A}$ and $C = 4q^2 + 10q + A$, where p denotes the price, q the quantity and A the level of advertising expenditure. Find the values of A, q and p that maximises his profit.

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8.9 LET US SUM UP

In this unit, we extended the discussion that was carried out the in relevant unit of the previous course which discussed unconstrained optimisation in the case of only one dependent variable. The analytical process of locating 'extremal points' or in other words the 'optimisation process' was explored in more general terms in this unit. The entire analysis was based on unconstrained optimisation of functions. The restrictive assumption of objective function with only one decision variable was relaxed in this unit and the former was allowed to be a multivariate function. The first and second order conditions that need to be satisfied to classify a point either as a relative 'maximum' or as a 'minimum' for an objective function with multiple decision variables was presented. Finally, we applied these mathematical tools in the context of economics, with examples from the case of a multi-plant monopolist and discriminating monopolist.

8.10 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) Read sections 8.2 and 8.3

2) Read section 8.3

Check Your Progress 2

1) Read section 8.4

2) Read section 8.6

3) Read subsection 8.6.2

Check Your Progress 3

1) Read section 8.8

2) Read section 8.8