UNIT 10 HIGHER-ORDER DERIVATIVES*

Structure

10.0 OBJECTIVES

After going through this unit, you will be able to:

- explain the concept of the derivative of a derivative;
- define Convexity and Concavity along with the Quasi-convexity and Quasi-concavity;
- get an insight into the Taylor Series formula; and
- develop an understanding for the Mean-Value Theorem.

10.1 INTRODUCTION

In the previous unit, we looked at a very important topic in Mathematical methods, namely *differentiation*. We saw it enabled us to speak about the change in the dependent variable as a result of the change in the independent variable. The change in the dependent variable as a result of an infinitesimally small change in the independent variable is called its *derivative*. If y is the dependent variable and x the independent variable, then $\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is the derivative. It is a central concept in Mathematical methods, and is a bedrock of Economic analysis. Economics is full of situations where economic phenomena are explained through relationships among variables through functions, where, in the case of two variables, we say dependent variable y depends on independent variable x. But, it is not enough to just have knowledge about y's dependency on x. We better quantify the change. We better know that if x is increased by a unit (that is an infinitesimal small increase in x), by how much units will y change? Such further insights we get from the derivative. Moreover, the sign of the derivative tells us whether the function is an increasing or decreasing one.

This is all fine, but we should see what happens when we take the derivative of the derivative? When we take the derivative of a function for the first time, it is called the *first-order derivative*, and this is what you studied in the previous unit. You must remember that the derivative of a function is itself a function. In this unit we will study about the processes and outcomes of taking derivatives of derivative. These are called *higher order-derivatives*.

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Higher-order derivatives allow us to see the rate at which the change in y is taking place. The first-order derivative tells us by how much y will change when x changes by a unit, while the second-order derivative tells us if the rate at which y changes as x is changed by a unit itself changes, as x is successively changed. In other words, we want to see whether at higher values of x, as x is changed, y itself changes faster or not.

The next section, section 10.2 of the unit takes up the discussion of the derivative of a derivative (and also even higher-order derivatives) in detail. The section following that, that is, section 10.3 investigates certain geometric properties of curves *i.e.*, of functions, namely convex and concave functions. Finally, the unit discusses two very important ideas utilising higher-order derivatives— the Taylor Series and the Mean Value Theorem.

10.2 DERIVATIVE OF A DERIVATIVE

It is the derivative of the derivative of a given function. For example if $y = f(x)$, then First-order derivative is $\frac{dy}{dx} = f'(x)$ and the second-order derivative will be $\frac{d}{dx} f'(x) = f''(x)$, it can also be denoted by $f_2(x)$ or $\frac{d^2y}{dx^2}$.

For the second-order derivative, the same rules are used as in the case of first-order derivative. Let us now take some examples.

Example 1 If $y = x^5, \frac{dy}{dx} = 5x^4$ and $\frac{d^2y}{dx^2} = 5 \times 4 \times x^3 = 20x^3$

Example 2 If $y = 10x^3, \frac{dy}{dx} = 30x^2$ and $\frac{d^2y}{dx^2} = 30 \times 2 \times x = 60x$

Example 3 If $y = 100, \frac{dy}{dx} = 0, \frac{d^2y}{dx^2} = 0$

Example 4 If $y = ax^5 + bx^4 - cx^2, \frac{dy}{dx} = 5ax^4 + 4bx^3 - 2cx$

and $\frac{d^2y}{dx^2} = \frac{d}{dx} \frac{dy}{dx} = 20ax^3 + 12bx^2 - 2c$

Example 5 If $y = x^2 \log x$, then $\frac{dy}{dx} = x^2 \cdot \frac{1}{x} + \log x \cdot 2x$

$= x + 2x \log x$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}(x + 2x \log x) = 1 + 2x.\frac{1}{x} + 2.\log x$

$= 1 + 2 + 2\log x = 3 + 2\log x$

Example 6 If $y = e^x \cdot \log x$, then $\frac{dy}{dx} = e^x \cdot \frac{1}{x} + \log x \cdot e^x$

and $\frac{d^2y}{dx^2} = \frac{d}{dx} \quad e^x \cdot \frac{1}{x} + \log x \cdot e^x \quad = \frac{d}{dx} \quad e^x \cdot \frac{1}{x} \quad + \frac{d}{dx}\left(\log x \cdot e^x\right)$

$$= \left[e^x.\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}.\frac{d}{dx}(e^x)\right] + \left[\log x.\frac{d}{dx}(e^x) + e^x.\frac{d}{dx}(\log x)\right]$$

$$= -\frac{e^x}{x^2} + \frac{2e^x}{x} + e^x \log x = -e^x \quad \frac{1}{x^2} - \frac{2}{x} - \log x$$

Derivatives of Higher Order

So long as the function is differentiable, we can go in finding third y_3 or $\frac{d^3 y}{dx^3}$, fourth y_4 or $\frac{d^4 y}{dx^4}$, fifth… , n^th^ order derivative y_n or $\frac{d^n y}{dx^n}$.

All these are called higher order-derivatives. In Economics sometimes third-order derivatives are needed. For maximisation, and minimisation problems mostly, only first and second-order derivatives are used.

Check Your Progress 1

1) Find the third-order derivative of $y = 3x^5 + 23x^2 + 19x$.

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2) Find the second order derivative of $y = e^{3x+2}$

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10.3 CONCAVITY AND CONVEXITY

In this section we will discuss some geometric properties of functions that are second-order derivatives of some function. As you know, the derivative of a function is also a function. And since a second-order derivative is also a derivative, it is also a function. In other words, if *y* is a function of *x*, then so are $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$.

A rising or a falling function, with constant, increasing or decreasing rates of rising or falling can be either,

a) Linear

or b) Non-linear

We present below graphic representation of such functions (See Figure 10.1 – A, B, C, D, E and F)

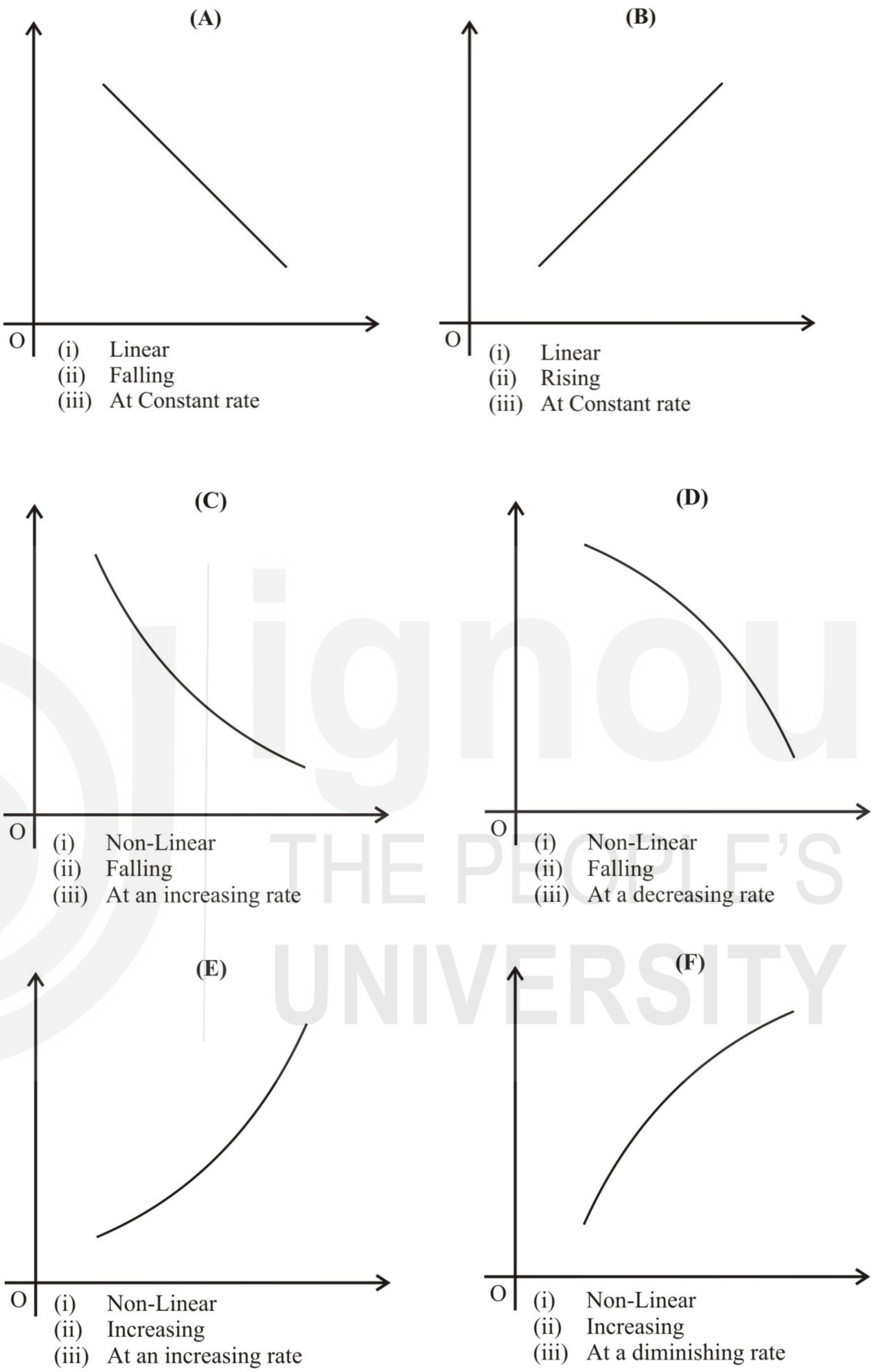

Figure 10.1

Sign of second-order derivative along with that of the first, helps us to identify different types of situations useful in Economic Analysis. They include:

a) A convex function such as indifference curve;

b) A concave function such as Production Possibility curve;

c) Maxima of a function such as maximisation of Total Product;

d) Point of inflexion, etc.

Note: You will learn in detail about the convex and concave functions in the next unit.

Let us now take up combinations of signs of first and second-order derivatives in detail.

Consider a function, $y = f(x)$, which is differentiable twice at least. The following cases depict the characteristics and the graph of function $f(x)$ on the basis of the signs of its first and second-order derivatives.

Case I: $f'(x) > 0$, $f''(x) > 0$ Convex to x-axis

i) $f'(x) > 0$ implies that the function is *increasing*.

ii) $f''(x) > 0$ implies that the function is increasing at an *increasing rate* $(\theta_2 > \theta_1)$.

Together, it shows that the function is *increasing at an increasing rate*. (refer Figure 10.2 A)

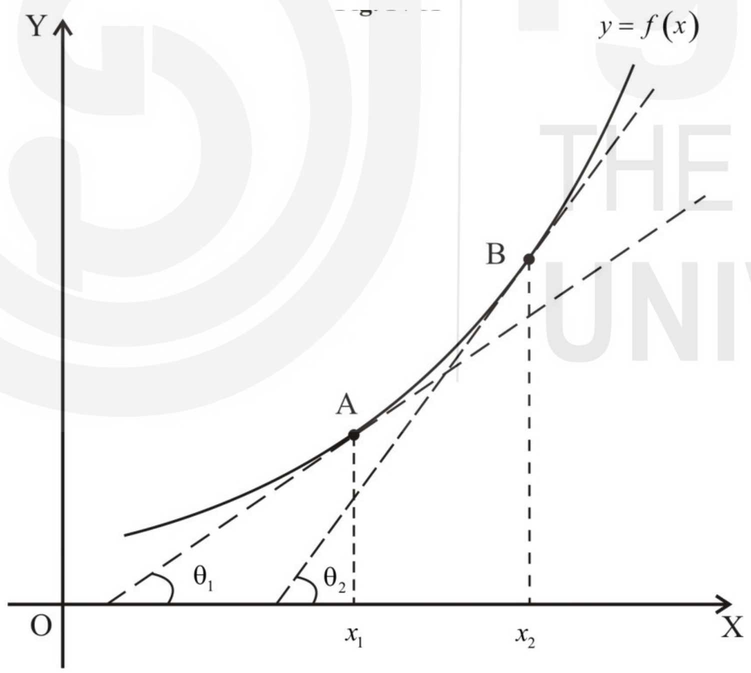

Figure 10.2 A

Case II: $f'(x) > 0$, $f''(x) < 0$ Concave to x-axis

i) $f'(x) > 0$ implies that the function is *increasing*.

ii) $f''(x) < 0$ implies that the function is increasing at a *diminishing rate* $(\theta_2 < \theta_1)$.

Together, it shows that the function is *increasing at a diminishing rate*. (refer Figure 10.2 B)

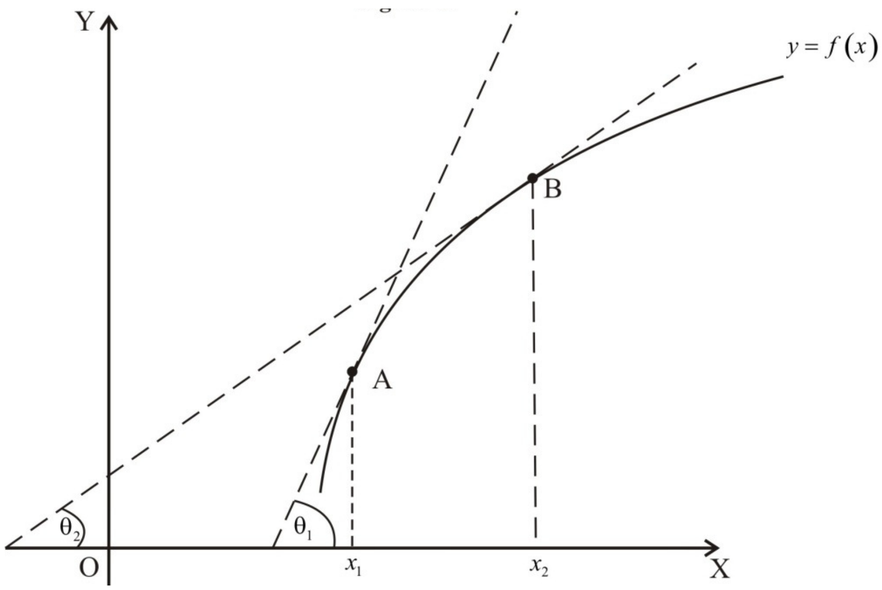

Figure 10.2 B

Case III: $\boxed{\begin{array}{l} f'(x) < 0 \\ f''(x) > 0 \end{array}}$ Convex to origin

i) $f'(x) < 0$ implies that the function is *decreasing*.

ii) $f''(x) > 0$ implies that the function is decreasing at an *increasing rate* $(\theta_2 > \theta_1)$.

Together it shows that the function is *decreasing at an increasing rate.*(refer Figure 10.2 C)

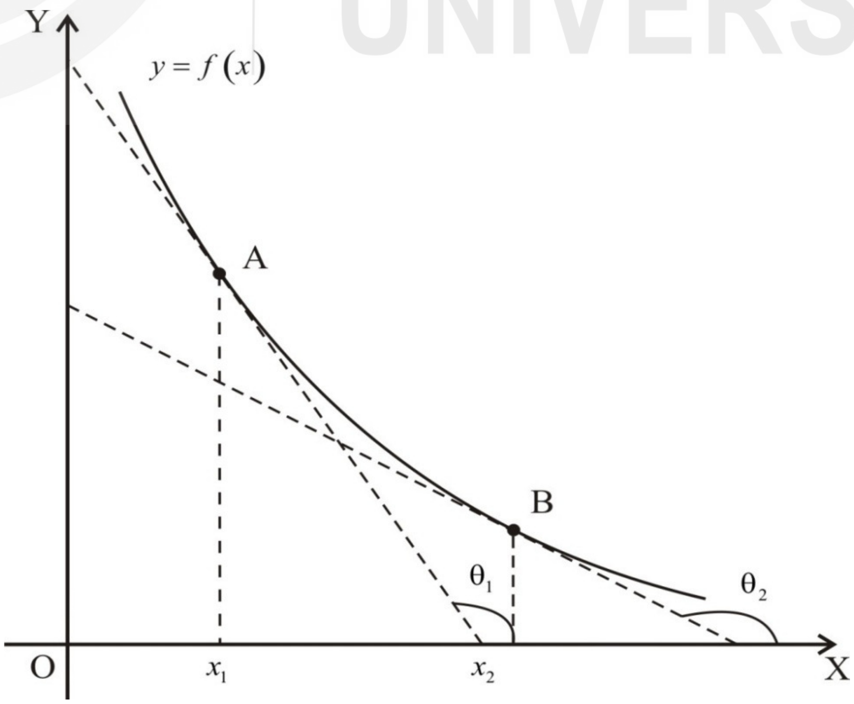

Figure 10.2 C

The curve is convex to origin (O) like an indifference curve which has a negative slope throughout and is convex to origin. An Isoquant, with variable capital coefficients, also slopes downward throughout (*i.e.* has a negative slope) and is convex to the origin.

Case IV: $\boxed{\begin{array}{l} f'(x) < 0 \\ f''(x) < 0 \end{array}}$ Concave to the origin

i) $f'(x) < 0$ implies that the function is *decreasing*.

ii) $f''(x) < 0$ implies that the function is decreasing at *diminishing rate* $(\theta_2 < \theta_1)$.

Together it shows that the slope of the curve is negative and it decreases at diminishing rate. That is angle of inclination of the tangents falls. (refer Figure 10.2 D). The curve is concave downwards from origin or convex from above. Production possibility curve is a fit case for this type.

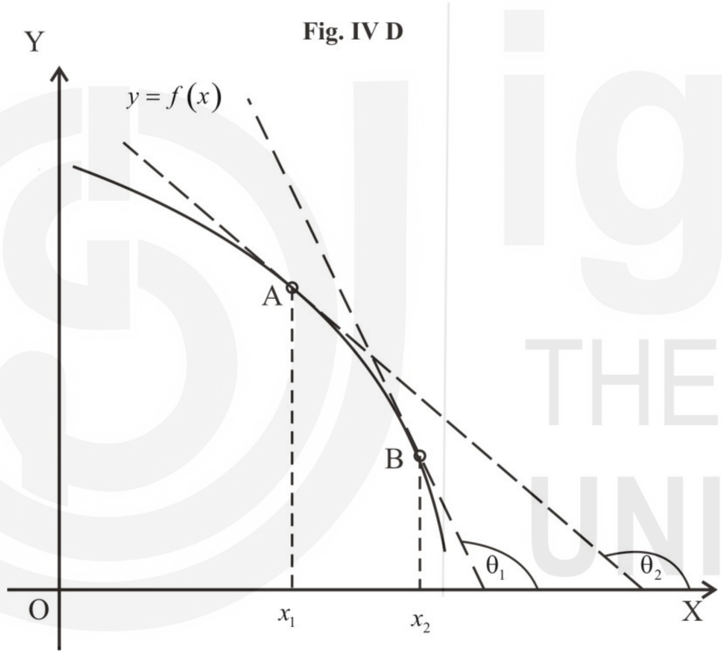

Figure 10.2 D

Case V: $\boxed{\begin{array}{l} f'(x) = 0 \\ f''(x) > 0 \end{array}}$ Minima

i) $f'(x) = 0$ impliesthat the function is neither rising nor falling at a particular point. In Figure 10.2 E, the tangent is horizontal at point A which means that the slope is zero. It is a stationary point, also called a *critical* point. It could be either a minimum point, or a maximum point or a point of inflexion. To identify this, we need second-order derivative.

ii) $f''(x) > 0$ implies that the curve has a minima at point A in a neighbourhood of A. The curve $y = f(x)$ is U-shaped having three phases, namely

a) A *falling phase* upto point A— the curve is falling (*i.e.* has a negative slope) implies that $f'(x) < 0$ on this portion of the curve.

b) A minimum and constant phase at point A, where $f'(x) = 0$.

c) A rising phase beyond point A— the curve is rising (*i.e.* has a positive slope) implies that $f'(x) > 0$ on this portion of the curve. (Refer Figure 10.2 E)

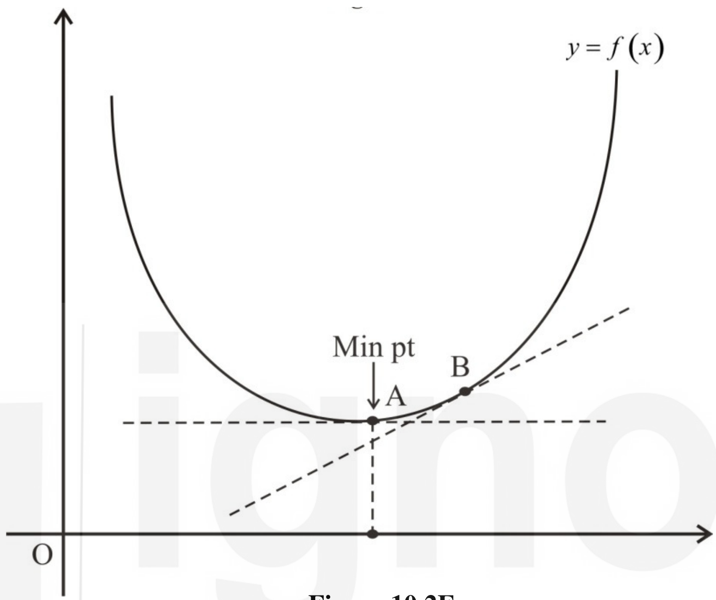

Figure 10.2E

The condition $f'(x) = 0$ is called the *first-order condition* or the *Necessary condition of Minima*. It is same for all extreme values (Minima, Maxima or Inflexion). In order to find which is thecase, we apply second-order condition. For instance, $f''(x) > 0$ in the case of minima. This is called a *Sufficient condition* and is confirmatory in nature.

Let us take one example from Economics. Let $y = f(x)$ be the average cost curve (AC). Let it be represented by a relation, AC $= 5x^2 - 20x + 170$. Let us find its minima.

For minimum AC, the first-order/necessary condition for finding the stationary point is,

$\frac{d}{dx}(\text{AC}) = 0.$

$$\frac{d}{dx}(5x^2 - 20x + 170) = 0$$

$$10x - 20 = 0$$

$$10x = 20$$

$$x = 2$$

Now, the second-order/sufficient condition to find whether the point is minima or maxima, is

$$\frac{d}{dx}\left(\frac{d}{dx}\text{AC}\right) > 0$$

$$\frac{d}{dx}(10x - 20) > 0$$

$$10 > 0$$

Thus, the function has a minima at $x = 2$.

Let us now find the minimum value of AC. For this we put $x = 2$ in the AC function.

$\therefore$ Minimum AC = $5(2)^2 - 20(2) + 170 = 151$

Case VI: $f'(x) = 0$, $f''(x) < 0$ Maxima

i) As in case V, $f'(x) = 0$ implies that the function is neither rising nor falling at a particular point. In Figure 10.2 F, the tangent is horizontal at point A, meaning that the slope is zero. It is a stationary/critical point which could be either a point of maxima or minima or an inflexion point. As before, to identify this, we need second-order/ sufficient condition.

ii) $f''(x) < 0$ implies that the curve has a maxima at point A in a neighbourhood of A. The curve representing such function is *inverted U-shape* having a maximum point at A.

To illustrate, Let us take an Economic function, say, a profit function (π) as $\pi = 500 + 160x - 2x^2$, where x stands for the advertising expense. We may be interested in finding that level of advertising expense (x) which can maximise our profit.

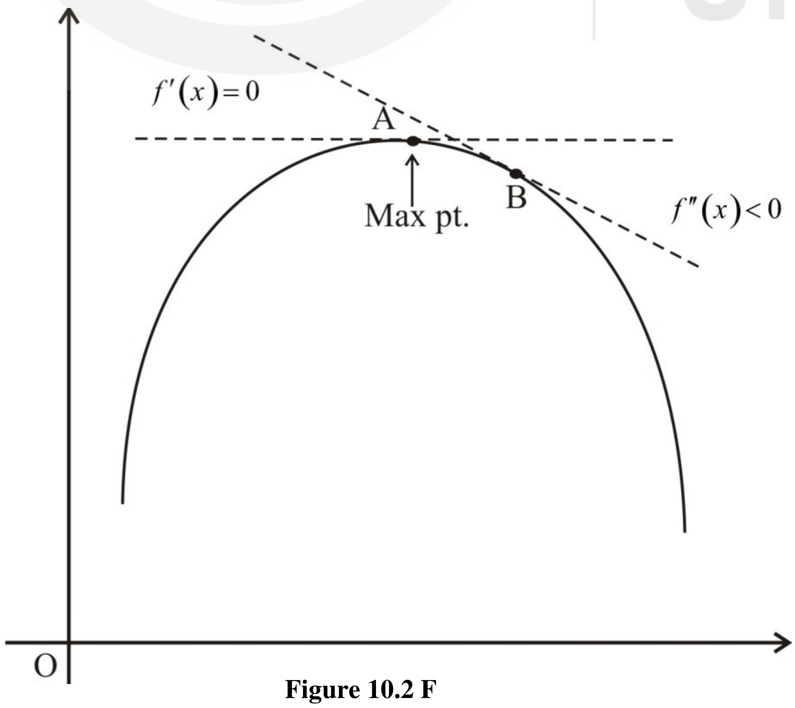

Figure 10.2 F

For maximum profit (π), the first-order/necessary condition for finding the stationary point is,

$\frac{d}{dx}(\pi) = 0.$

$$\frac{d}{dx}(500 + 160x - 2x^2) = 0$$

$$160 - 4x = 0$$

$$x = 40$$

Now, the second-order/sufficient condition to find whether the point is minima or maxima, is

$$\frac{d}{dx}\left(\frac{d}{dx}\pi\right) < 0$$

$$\frac{d}{dx}(160 - 4x) < 0$$

$$-4 < 0$$

∴ Profit is maximum when adverting expenses (x) equals 40 units.

Also, maximum amount of profit is given by,

$$\pi = 500 + 160(40) - 2(40)^2 = 500 + 6400 - 3200 = 3700$$

Thus maximum profit of Rs 3700 can be earned when advertising expenses are 40 units.

Case VII: $\boxed{\begin{array}{l} f'(x) = 0 \\ f''(x) = 0 \end{array}}$ Inflexion point

A single valued function $y = f(x)$ is defined to have an inflexional value at a point where the corresponding curve crosses from one side of its tangent to the other. The point so described is called *inflexion point*. A point of inflexion marks a change in *curvature*.The curvatureof the curve in such case changes from convex to concave or from concave to convex from below as we pass from left to right through the point. All panels of Figure 10.3 show this.

Two classes of Points of Inflexion

Class I: Change of curvature from *Convex to Concave,* irrespective of the slope of the tangent at the point of inflexion. [See Figure 10.3 (A) and (B)]

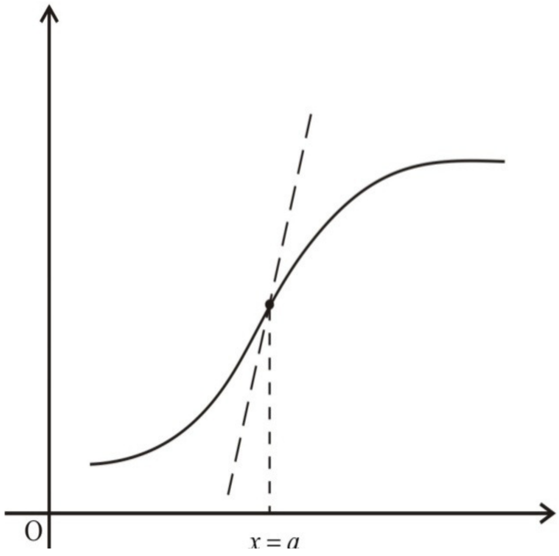

Figure 10.3 A

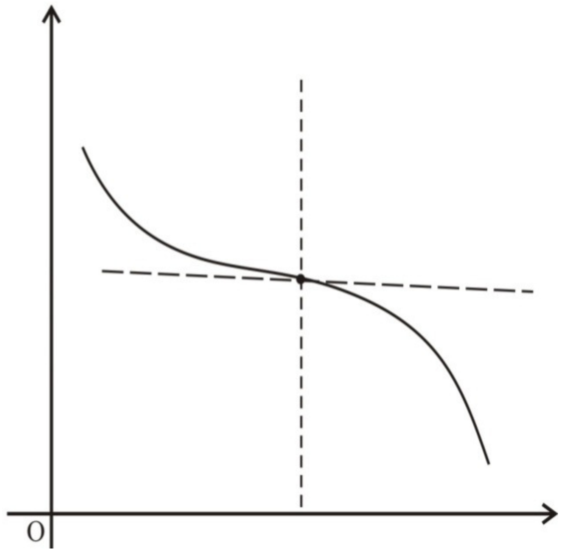

Figure 10.3 B

Class II: Change of curvature from *Concave to Convex*, irrespective of the slope of the tangent at the point of inflexion. [See Figure 10.3 (C) and (D)]

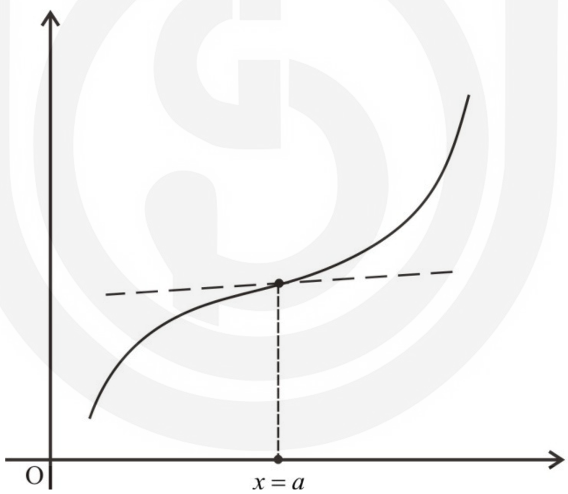

Figure 10.3 C

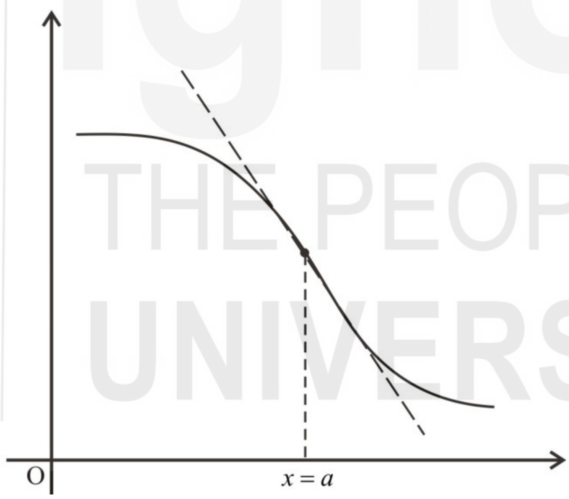

Figure 10.3 D

Equation of the Tangent

One very important use of differential coefficient $\frac{dy}{dx}$ of a function $y = f(x)$ is the equation of a tangent that can be drawn to it at any given point (x_1, y_1) (refer Figure 10.4).

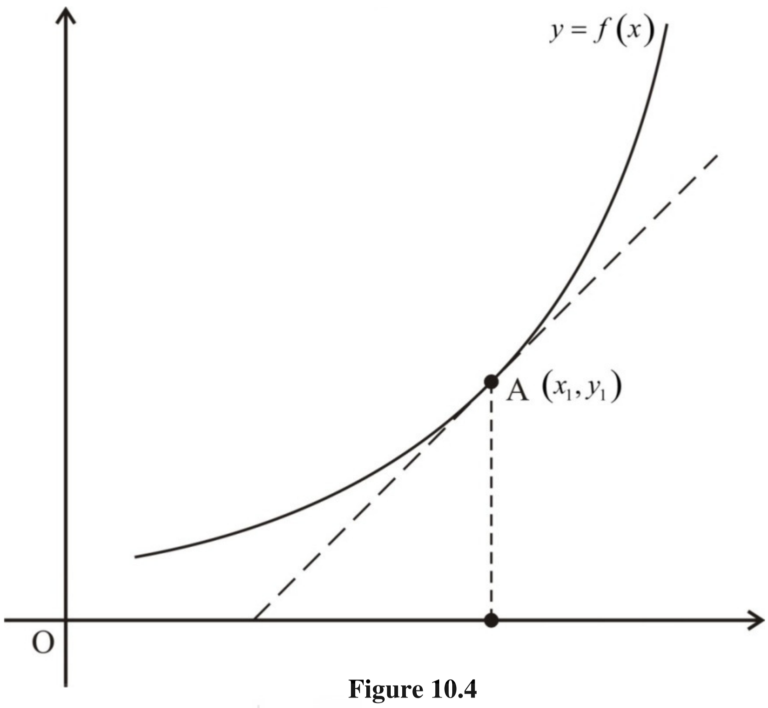

Figure 10.4

The equation takes the form of a point slope equation as discussed in coordinate geometry. It is given by:

$$y - y_1 = m(x - x_1)$$

where, m is the slope and $A(x_1, y_1)$ is the point of tangency with the curve of the given function $y = f(x)$. Slope of the tangent to the curve is the first derivative $\frac{dy}{dx}$ of the given function.

Now, the equation of the tangent can be written as:

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

If the point of tangency is $A(a, b)$, the equation of the tangent can be written as:

$$y - b = \frac{dy}{dx}(x - x_1)$$

Let us find the equation of the tangent to a parabola $y = x^2 + 3x - 2$ at point $A(2,3)$. We know slope of the tangent will be given by $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 2x + 3$$

Therefore, the required equation is:

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$y - y_1 = (2x + 3)(x - x_1)$$

$$y = y_1 + (2x+3)(x - x_1)$$

Substituting the values of point A (2, 3) in the above equations, we get

Slope $\frac{dy}{dx} = 2 \times 2 + 3 = 7$

Tangent $y = 3 + 7(x-2)$

or $y = 3 + 7x - 14$ or $y = 7x - 11$

Check Your Progress 3

1) What are conditions for a function to have,

a) A critical point

b) A minimum value

c) A maxima

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2) What can be said about the rate of change of a function on the basis of concavity or convexity of the function?

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10.4 TAYLOR SERIES FORMULA AND MEAN VALUE THEOREM

We know that the differential $dy = f'(x)dx$ can be used to give us an approximation to the change in the y variable, $dy \cong \Delta y$, for a given change in the x variable, $dx \equiv \Delta x$. The percentage error from usingdy as an approximation to the actual change, Δy can be made arbitrarily small if we are willing to consider changes in x that are made arbitrarily smaller. However, we may not be satisfied with the restriction that Δx be infinitesimally small, and for non-infinitesimally small changes in the x variable, this approximation may not be very accurate.

The Taylor series expansion formula gives us to go deeper into this issue. The basic idea behind the Taylor series formula is to use information about the value of a function $y = f(x)$ at a specific point, $x = a$, together with information about the value of the derivative function of $f(x)$ at the point $x = a$, in order to obtain the value of the function at a different value of x,

$x = x_0$, within some neighbourhood of the point $x = a$. Let f be the function, differentiable $(n + 1)$ times, on an open interval containing points a and x. Then as per the Taylor theorem:

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f^{(3)}(a)(x-a)^3}{3!} + \frac{f^{(4)}(a)(x-a)^4}{4!} + \ldots + \frac{f^{n}(a)(x-a)^n}{(n)!} + R_n(x)$$

where, $R_n(x) = \dfrac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$ and c lies between a and x.The above formula is sometimes called the remainder form of the Taylor series expansion formula, where $R_n(x)$ is the remainder term.$f^{(n)}(a)$ is the n^{th} derivative of f at the point a, and is assumed to exist. We give the same expression by making use of the summation notation:

$f(x) = f(a) + \sum_{k=1}^{n-1}\left[\dfrac{f^{k}(a)(x-a)^k}{k!}\right] + R_n(x)$. This function is presumed to possess derivatives to the $(n+1)^{th}$ order. Since we have got the formula or equation, we can see that if we know the value of a function at some point $x = a$, and then use the formula to find the value of the function at some other point $x = x_0$, given by $f(x_0) = f(a) + \sum_{k=1}^{n-1}\left[\dfrac{f^{k}(a)(x_0-a)^k}{k!}\right] + R_n(x_0)$, it is the same as finding how the function f changes as a result of changing x by the amount $\Delta x = x_0 - a$. This we can see by shifting the term $f(a)$ of the expression above to the left side to get $f(x_0) - f(a) \equiv \Delta y$.

Restatement of the Mean Value Theorem from the Taylor's Series formula

We can illustrate the Mean Value theorem for the derivative by taking only one term in the Taylor series formula above with $n = 0$: $f(x) = f(a) + f'(c)(x-a)$ for some $c$between x and a. This brings us to the Mean Value theorem:

If the function $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there must be some $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

On rearranging terms, we get

$f(b) = f(a) + f'(c)(b - a)$, which is what the Taylor theorem claims for n=0.

Check Your Progress 3

1) Explain Taylor series formula. What is its significance?

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2) What is the connection between the mean-value theorem and the Taylor series formula?

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10.5 LET US SUM UP

This unit carried on from the previous unit the discussion on derivatives. However, the present unit focused on higher-order derivatives. We investigated what happens when we take derivatives of derivatives. The unit looked at how to compute derivatives of derivatives and also discussed some Economic applications of higher order derivatives. After this the unit moved on to the discussion of some very important geometric properties of functions, namely, concavity and convexity. We saw that the sign of the second-order being negative indicated concavity of the function, while it being positive indicated that the function is convex. The unit also discussed the implications of the function being concave or convex. Finally, the unit discussed in detail two important results that allow us to talk about approximations to a function around a point. The two important results were the Taylor Series and Mean-Value theorem.

10.6 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) $180x^2$

2) $9e^{3x+2}$

Check Your Progress 2

1) a) $f'(x)=0$

   b) $f'(x)=0$ and $f''(x)>0$

   c) $f'(x)=0$ and $f''(x)<0$

2) Refer section 10.3 and answer.

Check Your Progress 3

1) Refer section 10.4 and answer.

2) Refer section 10.4 and answer.