UNIT 7 LIMITS*

Structure

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7.0 OBJECTIVES

After going through this unit, you will be able to:

- explain the idea of limit in a sequence;
- define limit of a function;
- evaluate left- and right-hand limits of a function; and
- apply numerical, algebraic and graphic approaches to finding limits.

7.1 INTRODUCTION

You will see in the following units that the concept of a limit is fundamental to Calculus. Many Calculus concepts like the derivative, the integral, etc. make use of limits. To understand the basic idea behind limit you may have to remember calculation of average and instantaneous speeds. Suppose you are travelling from point A to point B. In order to compute your average speed from A to B, you simply take the ratio of the distance between points A and B and the time it takes to travel this distance. For example, let us take a function $s(t)$ that determines the position of the moving body at time 't'. Assume that at time t_0, the moving body is at point A [= $s(t_0)$], whereas at time t_1 (where $t_1 > t_0$), it is at point B [= $s(t_1)$]. Time spent in travelling between these points will be given by Δt (= $t_1 - t_0$), where Δstands for "change in". Distance between points will be given by $s(t_1) - s(t_0)$. Then, the average speed over the time interval [t_0, t_1] is given by

$$\frac{s(t_1) - s(t_0)}{\Delta t} \qquad \ldots(1)$$

This may also be interpreted as the average rate of change of the position function $s(t)$ over the interval [t_0, t_1].

The instantaneous speed is the speed of the moving body at an instant. As we make the time interval, given by Δt (= $t_1 - t_0$), shorter and shorter, we approach nearer and nearer to the speed of the moving body at the instant in time t, that

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is the instantaneous speed. This is exactly the application of the concept of limit. In the above function (1), notion of instantaneous speed would mean the limiting value of the function (1), as Δt gets closer and closer to 0.

That is, the instantaneous speed at time *t* will be given by,

$\lim_{\Delta t \to 0} \frac{s(t_1) - s(t_0)}{\Delta t}$, provided such a limit exists.

Reformulate such a situation and suppose that between A and B there lies a point C through which you need to pass. To compute instantaneous speed at C, you may try to compute the average speed from C to points close to C. In such a situation, the distance between these points and C is very small as well as the time taken to travel from them to C. Then when you look at the ratio, the value you get will be the instantaneous speed at C. This is the way policeman's radar computes the driver's speed at traffic points in your city.

The concept of a limit involves approaching a point or a value arbitrarily close and still never reaching it. Intuitively, this idea may not seem very appealing. However, understanding this concept is essential to the study of differential calculus, and therefore very important. We will first discuss the concept of the limit of a sequence and then go on to discuss the concept of the limit of a function. Moreover, we will also discuss the basic concepts used for arriving at the limit to a function.

7.2 LIMIT OF A SEQUENCE

The idea of a limit follows from the behaviour of a moving point that comes closer and closer to a fixed point. Roughly speaking, when a moving point approaches a fixed point and the distance between them becomes progressively smaller (but never actually vanishes), we say that the point is tending to a *limit*. Let us try to understand the concept by discussing the *limit of a sequence*. Consider the following two sequences of numbers:

(i) $S_1 = \{1,2,3,4,5\ldots.\}$ (ii) $S_2 = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots\}$

In the first sequence, beginning with the first number, each succeeding number is obtained by adding 1 to the preceding number. In this way, the numbers in this sequence go on increasing without showing any tendency to stabilise at a fixed value. In the second sequence, beginning with the first number, each succeeding number is obtained by adding 1 to both the numerator and the denominator. In this sequence also the numbers seem to go on increasing but with a difference; here the sequence gradually tends to stabilise at a fixed value, which is 1. To fix the idea better, let us write the second sequence of numbers in the decimal form (upto three-decimal point). The sequence then takes the following form:

$$S_2 = 0.888,\ 0.900,\ 0.909,\ 0.916,\ 0.923,\ 0.928,\ 0.933,\ 0.937,\ldots$$

It is clear that each succeeding number of the sequence is closer to 1 than the preceding number. The numerical difference between the first number and 1 is 0.112 and that between the eighth number and 1 is just 0.063. In fact, if we consider the number $\frac{999}{1000}$, which is also a member of the sequence and occurs much later, the numerical difference between it and 1 is only 0.001. It seems that the sequence can continue endlessly and each succeeding number can become progressively as close to 1 as we want it to be. The numbers in the sequence thus appear to steadily approach 1 without ever attaining it. In this case, the sequence is said to '*tend to*' 1 and this 1 is called the *limit* of the sequence. In the above example, the sequence approaches 1 from below, that is, from numbers lower than 1. However, a sequence of numbers may approach a limit in other ways also. For example, it may approach a limit from above, that is, from numbers higher than the limit; or, it may converge to a limit by oscillating around it also. The important point to note is that for the existence of a limit, a sequence of numbers should have a definite tendency to approach a finite value without quite attaining it.

In general, a sequence may

i) increase or decrease indefinitely at a constant or increasing rate. For example,

$S = \{0, -1, -3, -8, -15...\}$

or, $S = \{1, 5, 10, 17, 26, ...\}$

ii) increase or decrease at a decreasing rate. For example,

$S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}...\}$

or, $S = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}...\}$

iii) fluctuate with wider margin. For example,

$S = \{-1, 5, -7, 17, -31...\}$

iv) fluctuate without any tendency for an increase or decrease. For example, $S = \{-1, 1, -1, 1, -1...\}$

v) fluctuate with a decreasing margin. For example,

$S = \{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}...\}$

Sequences given above show characteristics on the basis of which we can categorize them into:

i) heading towards infinity $(\infty, -\infty)$;

ii) heading towards some finite number (positive, negative or zero);

iii) leading nowhere (not settling at any value).

These observations help us explain the limit of a sequence.

Let us consider a sequence of numbers $\{x_n\}_{n\geq 1}$. If the numbers get closer and closer to a number *L*, (*i.e.*, $x_n \approx L$), then we say that the sequence $\{x_n\}$ is convergent and has a limit equal to *L*.

We will write

$$\lim_{n\to\infty} x_n = L$$

or, $$\underset{n\to\infty}{Lt}\ x_n = L$$

or, $$x_n \to L \text{ when } n \to \infty.$$

When we say $n \to \infty$, we imply n is getting larger and larger. Of course we do not have $n = \infty$. But perhaps in the neighbourhood of ∞. If a sequence is not convergent, it is called *divergent*. We will discuss a bit more on the convergent sequence to formally define the limit of a sequence.

We have said above that a sequence $\{x_n\}_{n\geq 1}$ is convergent if there exists a number L such that the numbers x_n get closer and closer to L as n becomes larger. That is,

i) we want to see that $x_n \approx L$;

ii) make sure that $|x_n - L| < \varepsilon$, where ε is a small number greater than zero;

iii) for (ii) to happen, we have $N \geq 1$ such that for every $n \geq N$, we have $|x_n - L| \leq \varepsilon$.

Note that ε, a Greek symbol called 'Epsilon' represents an arbitrarily small positive quantity. It measures the error between the number x_n and the L. The integer N on the other hand, measures how fast the sequence gets closer to the limit.

Thus, we can say that the sequence $\{x_n\}_{n\geq 1}$ converges to the number L, if and only if, for every $\varepsilon > 0$, there exists $N \geq 1$ such that for every $n \geq N$, $|x_n - L| \leq \varepsilon$.

7.3 LIMIT OF A FUNCTION

We can now extend the concept of the limit of a sequence of numbers to the *limit of a function*.Let us consider a single-variable function $y = f(x)$. We may be interested to know, if x approaches (or tends to) a value say 'a'without ever attaining it (symbolically, $x \to a$), whether y also approaches (or tends to) a finite value say 'b' (*i.e.*$y \to b$). If indeed that is the behaviour of y, we say, *as x tends to a* (note that a need not be a finite value), *the limit of y is b*.

Symbolically, $\lim_{x\to a} f(x) = b$

Thus, a limit L is the value that the function $f(x)$ approaches to (*i.e.,* $f(x) \to L$) as x approaches a ($i.e., x \to a$). It is symbolically written as $\lim_{x\to a} f(x) = L$.

Left and Right Hand Side Limits

There exists two ways by which x may approach a number 'a'. It may approach a either from values smaller than a*i.e.*, from the left hand side or from values greater than a*i.e.*, from the right hand side. When $x \to a$ from right hand side, y approaches a finite value say, L_1, we call L_1 the *right hand limit* of $f(x)$, denoted by $\lim_{x\to a^+} f(x)$;whereas when $x \to a$ from left hand side,y approaches a finite value say, L_2, we call L_2 the *left hand limit* of $f(x)$, denoted by $\lim_{x\to a^-} f(x)$. For example, consider the graph of a function $f(x)$ in Figure 7.1,

where $f(x) = \begin{cases} 2, & x \leq 0 \\ 4, & x > 0 \end{cases}$

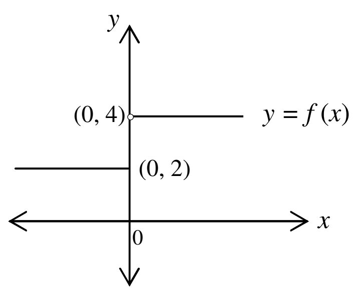

Figure 7.1

Here, it is clear that as $x \to 0$ from the left side, $y \to 2$. Thus, we get the left hand limit as 2, *i.e.*, $\lim_{x\to 0^-} f(x) = 2.$ On the other hand, as $x \to 0$ from the right side, $y \to 4$, and thus we get the right hand limit as 4, *i.e.*, $\lim_{x\to 0^+} f(x) = 4.$

We just observed that, if y has a limit as x tends to a, that limit can be approached either when x tends to a from the left hand side or when it tends to a from the right hand side. Now a pertinent question arises? Does a given function have a limit or not? The answer to this question can be sought in terms of the following observations.

1) A function is said to have a limit if and only if the two limits (LHS and RHS) have a *common finite* value, *i.e.* they exist and are equal to each other.

2) If either of the limits is such that they equal to either $[+\infty]$ or $[-\infty]$, the function has *infinite limit and infinite limit is no limit.*

   Note: If the limit tends to either $+\infty$ or $-\infty$, take only one side limit as it will save time.

Thus, the condition for the existence of the limit of y as x tends to a is,

$$\lim_{x\to a^+} y = \lim_{x\to a^-} y = L \quad \text{(where } L \text{ is some finite number)}$$

With the above formulations we can say that, an increasing sequence with increasing x values leads to a left hand limit, on the other hand a decreasing sequence where x values are decreasing leads to a right hand limit.

Consider the various ways by which x may tend to a in following Figure 7.2 (*a*, *b*, *c*, *d*).

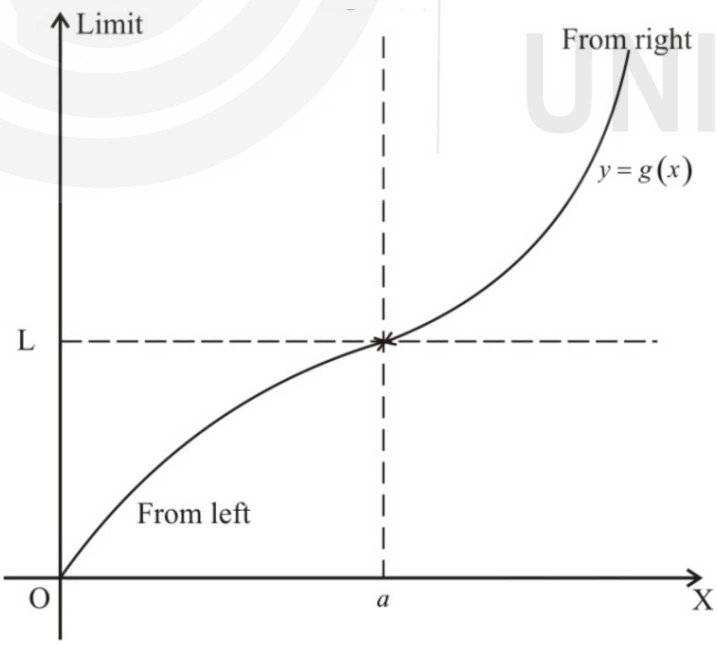

Figure 7.2 (*a*)

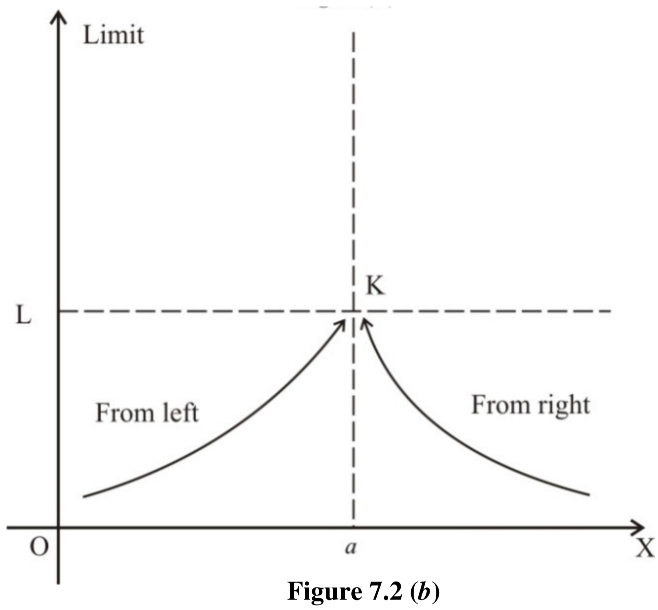

Figure 7.2 (*b*)

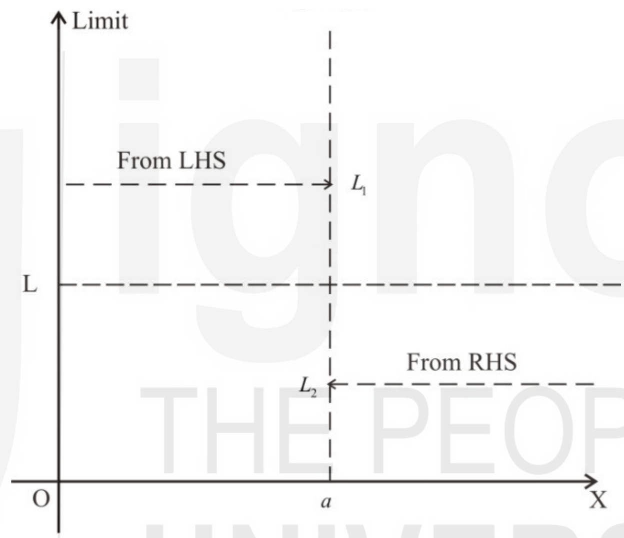

Figure 7.2 (*c*)

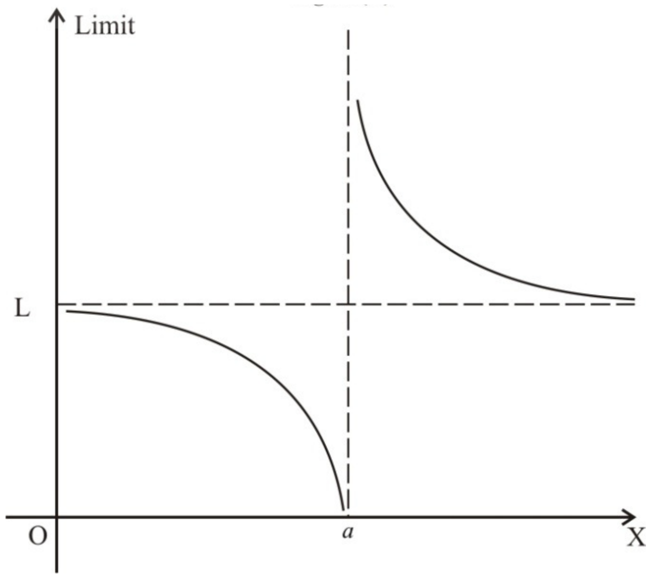

Figure 7.2 (*d*)

Part (*a*) of the diagram presents a smooth curve of the function $y = g(x)$. As variable x approaches the value a from either side on the horizontal axis, variable y approaches L on the vertical axis. Thus, both the left hand limit and the right hand limit exist and they are equal to each other. So in this case, as x tends to a, the limit of y exists and is equal to L.

The curve in Part (*b*) is not smooth. It has a sharp kink at point K directly above the point a on the horizontal axis. However, in this case also y approaches L as x approaches the value a from either side on the horizontal axis. Thus, as x tends to a, the limit of y again exists and is equal to L.

Part (*c*) is the diagram for a step function. Here we can see that as x approaches a from the left hand side, y approaches L_1, *i.e.,* the left hand limit of y is L_1. But, as x approaches a from the right hand side, y approaches L_2, *i.e.,* the right hand limit of y is L_2. In this case, although both the left hand and the right hand limits exist, they are not equal to each other. Hence, the condition for the existence of a limit is violated. Thus, as x tends to a, y does not tend to a limit here.

Finally, a rectangular hyperbola is shown in Part (*d*). Here, as x tends to a from the left hand side, y tends to $-\infty$ (minus infinity). On the other hand, when x tends to a from the right hand side, y tends to $+\infty$ (plus infinity). It is a hyperbolic curve with axis of symmetry being asymptotes. The curve has two branches, both falling and rising indefinitely. Here, both the right hand limit and the left hand limit do not exist. Therefore, in this case also y does not have a limit as x tends to a.

7.4 ALGEBRAIC APPROACH TO COMPUTATION OF LIMITS

We have studied the concept of a limit. Now we shall see the actual procedure of evaluating the limit of a function, with the help of certain rules that can be followed.

7.4.1 Rules for Evaluating a Limit

Limits allow themselves to be subjected to some algebraic manipulations such as addition, subtraction, multiplication and division. Consider the following rules based on two functions $y = f(x)$ and $y = g(x)$ such that $\underset{x\to a}{Lt}\ f(x) \to L_1$ and $\underset{x\to a}{Lt}\ g(x) \to L_2$

1) Limit of a constant (c) is the constant itself $\underset{x\to a}{Lt}\ c = c$

2) Limit of a sum and difference of two function is the sum and difference of their individual limits
$\underset{x\to a}{Lt}\left\{f(x) \pm g(x)\right\} = \underset{x\to a}{Lt}\ f(x) \pm \underset{x\to a}{Lt}\ g(x) = L_1 \pm L_2$

   Hence, limit of sum of a constant and a variable is the sum of their individual limits. $\underset{x\to a}{Lt}\left\{c + f(x)\right\} = \underset{x\to a}{Lt}\ c + \underset{x\to a}{Lt}\ f(x) = c + L_1$

3) Limit of product of two functions is the product of their individual limits.
$\underset{x\to a}{Lt}\left\{f(x)\cdot g(x)\right\} = \underset{x\to a}{Lt}\ f(x) \times \underset{x\to a}{Lt}\ g(x) = L_1 \times L_2$

Hence, limit of a product of a constant and a variable is constant times the limit of the variable $\underset{x\to a}{Lt}\ cf(x) = c\ \underset{x\to a}{Lt}\ f(x) = cL_1$

4) Limit of the quotient of two functions is the quotient of their individual limits, provided the limit of the function in the denominator is not zero.

$$\underset{x\to a}{Lt}\ \frac{f(x)}{g(x)} = \frac{\underset{x\to a}{Lt}\ f(x)}{\underset{x\to a}{Lt}\ g(x)} = \frac{L_1}{L_2}$$, provided $L_2 \neq 0$.

5) Limit of reciprocal of a function is the reciprocal of the limit of the function, provided the limit of the function is not *zero*.

$$\underset{x\to a}{Lt}\ \frac{1}{g(x)} = \frac{1}{\underset{x\to a}{Lt}\ g(x)} = \frac{1}{L_1}$$, provided $L_1 \neq 0$.

6) Limit of a root of a function times n is n times the root of the limit, provided the limit of the function is greater than or equal to *zero*.

$$\underset{x\to a}{Lt}\ n\sqrt{f(x)} = n\cdot \underset{x\to a}{Lt}\sqrt{f(x)} = n\sqrt{L_1}$$, provided $L_1 \geq 0$.

7.4.2 Some Standard Limits

You will find these forms useful in your subsequent study of this course, so try to learn these well. There proofs are not required as per the scope of this unit.

1) $\underset{x\to a}{Lt}\ \frac{x^n - a^n}{x-a} = na^{n-1}, a > 0$

2) $\underset{x\to\infty}{Lt}\ 1+\frac{1}{x}^{x} = e = 2.71828$

3) $\underset{x\to 0}{Lt}\ [1+x]^{1/x} = e = 2.71828$

4) $Lt_{x\to 0}\frac{a^x-1}{x} = \log_e a,\ a > 0$ [Note: $\log_e a$ is also denoted as '*ln a*']

5) $Lt_{x\to 0}\frac{e^x-1}{x} = \log_e e = 1$

7.4.3 Finite and Infinite Limits

Limit whose values tends to a finite number are called finite limits. For example, $\underset{x\to 2}{Lt}(5x+7) = 17$ where 17 is a finite number. On the other hand, limits whose values are not finite but tend to an infinite number such as $+\infty$ or $-\infty$ are infinite limits. For example,

$\underset{x\to 0}{Lt}\ \frac{5}{x^2} = \frac{5}{(0)^2} = \frac{5}{0} = \infty$ is an infinite limit. Also, keep in mind that an infinite limit is not considered as a limit.

Example1: Given the function $y = x^2 + 1$, find $\underset{x\to 0}{Lim}\ y$

Solution: To obtain the left hand limit, let us substitute the negative numbers $-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}\ldots$ one by one for x. We find that x^2 (a positive number) becomes smaller and smaller and approaches zero. As a result, $x^2 + 1$ steadily falls and tends to 1. Thus, the left hand limit of the function is 1. Next, we obtain the right hand limit by substituting the positive numbers $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.... We find that x^2 (again a positive number) becomes smaller and smaller and approaches zero. As a result, $x^2 + 1$ steadily falls and tends to 1. Therefore, the right hand limit of the function is also 1. Thus, the two limits are equal. Hence, the limit exists and we write, $\lim_{x \to 0} y = 1$.

Example 2: Find $\lim_{x \to 3} \frac{x^3 - 2x^2 + 2x - 6}{5x^2 - 13x + 3}$

Solution: We shall solve this question by applying various limit theorems.

$$\lim_{x \to 3} \frac{x^3 - 2x^2 + 2x - 6}{5x^2 - 13x + 3}$$

$$= \frac{\lim_{x \to 3} \left(x^3 - 2x^2 + 2x - 6\right)}{\lim_{x \to 3} \left(5x^2 - 13x + 3\right)}$$

$$= \frac{\lim_{x \to 3} \left(x^3\right) - \lim_{x \to 3} \left(2x^2\right) + \lim_{x \to 3} \left(2x\right) - \lim_{x \to 3} \left(6\right)}{\lim_{x \to 3} \left(5x^2\right) - \lim_{x \to 3} \left(13x\right) + \lim_{x \to 3} \left(3\right)}$$

$$= \frac{\lim_{x \to 3} x^3 - 2 \lim_{x \to 3} x^2 + 2 \lim_{x \to 3} x - \lim_{x \to 3} 6}{5 \lim_{x \to 3} x^2 - 13 \lim_{x \to 3} x + \lim_{x \to 3} 3}$$

$$= \frac{27 - 2(9) + 2(3) - 6}{5(9) - 13(3) + 3}$$

$$= 1$$

Example 3: A man invests Rs. 1000 for 2 years at an annual rate of interest of 5 per cent compounded continuously. Determine the amount that the man will get after 2 years. What is the effective rate of interest for the investment?

Solution: To solve this problem, let us discuss the compound interest formula a bit. We know that if an initial amount of *P* is invested for 1 year at a rate of interest of 100 *r*% compounded annually, the final amount *A* that can be obtained at end of 1 year is given by*A*= *P* (1 + *r*). If the same annual rate of interest is compounded *m* times a year, the amount that can be obtained at the end of 1 year is given by $A = P\left(1 + \frac{r}{m}\right)^m$. Here, the expression $\frac{r}{m}$ signifies that for an annual rate of interest of 100 *r*%, a rate of only $\frac{100r}{m}$ % will be applicable for each compounding period. The exponent *m* in the formula denotes that there will be *m* compoundings in one year. Now, with the same annual compounding frequency of *m*, the amount that can be obtained after t years is $A = P\left(1+\frac{r}{m}\right)^{mt}$,

where the exponent mt denotes the number of compoundings in t years. We can also write

$$A = P\left(1+\frac{r}{m}\right)^{mt} = P\left[\left(1+\frac{r}{m}\right)^{m/r}\right]^{rt} = P\left[\left(1+\frac{1}{n}\right)^{n}\right]^{rt} \text{,where } n = \frac{m}{r}$$

. It is clear that when $m \to \infty$, n also $\to \infty$.

In the given question, interest is compounded continuously. As a result, the compounding frequency m approaches infinity. Hence,

the required amount is the limit of $P\left(1+\frac{r}{m}\right)^{mt}$ as $m\to\infty$. This

limit is

$$A = \lim_{m\to\infty} P\left(1+\frac{r}{m}\right)^{mt} = \lim_{m\to\infty} P\left[\left(1+\frac{r}{m}\right)^{m/r}\right]^{rt} = \lim_{n\to\infty} P\left[\left(1+\frac{1}{n}\right)^{n}\right]^{rt}.$$

The limit of

$P\left[\left(1+\frac{1}{n}\right)^{n}\right]^{rt}$ as $n \to \infty$ depends upon the limit of $\left(1+\frac{1}{n}\right)^{n}$ as $n \to \infty$

, that is, $\lim_{n\to\infty} P\left[\left(1+\frac{1}{n}\right)^{n}\right]^{rt} = P\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^{rt}$. From the

standard formulae, we have $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$. Therefore,

$$A = P\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^{rt} = Pe^{rt}$$

We have, P = Rs 1000,

$$r = \frac{\text{percentage rate of interest}}{100} = \frac{5}{100} = 0.05 \text{ and } t = 2 \text{ years}$$

Therefore, A = 1000 ($e^{0.05(2)}$) = 1000 $(2.71828)^{0.1}$ [Because, we have e = 2.71828..]

Taking common log on both the sides

$$\begin{aligned}\log A &= \log 1000 + 0.1\times \log 2.71828\\ &= 3 + 0.1\times 0.4343\\ &= 3.04343\end{aligned}$$

Taking antilog on both the sides

A = antilog 3.04343

= 1105 (approximately)

Thus, the required amount is Rs. 1105.

The quoted annual rate of interest that is offered on an investment is called the *nominal rate*. However, the *annual rate* at which a given sum of money *actually* grows depends upon the frequency of compounding in a year. This rate is called the *effective rate of interest*. Thus, the effective rate is the equivalent annual compounding interest rate of a quoted rate that is compounded a given number of times (say, m) in a year. Now, if an amount A is due on a sum P after t years at a rate of interest 100 r% compounded continuously, we can write

$$A = Pe^{rt} \quad (1)$$

If 100 i% is the effective rate of interest, we can also write

$$A = P(1+i)^t \quad (2)$$

Equating (1) and (2)

$$(1+i)^t = e^{rt}$$

$$\text{or } 1+i = e^r \quad (3)$$

Equation (3) can be solved for the effective rate of interest 100 i%.

We have $r = 0.05$, putting this value in (3)

$$1+i = e^{0.05}$$

Taking common log on both the sides

$$\log(1+i) = 0.05 \times \log 2.71828$$
$$= 0.021715$$

Taking antilog on both the sides

$$1+i = \text{antilog}(0.021715)$$
$$\text{or } 1+i = 1.052$$
$$\text{or } i = 0.052$$

Thus, the effective rate of interest is 5.2%.

Check Your Progress 1

1) Evaluate the following limits

a) $\underset{x\to -2}{Lt}\left(x^2+5x\right)$

b) $\underset{x\to 0}{Lt}\ \dfrac{3x+1}{5x-1}$

c) $\underset{x\to 4}{Lt}\ \dfrac{2x^{3/2}-\sqrt{x}}{x^2-15}$

d) $\underset{x\to a}{Lt}\ Ax^n$

e) $\underset{x\to 1}{Lim}\ \dfrac{4x^2-3}{2x^2+1}$

f) $lim_{x \to -1} \frac{x^2+x-6}{x+2}$

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2) The total cost of production for a firm is given by $C = 4q + 2$. Does average cost approach a finite value when the output tends to infinity? If yes, what is that value?

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3) A trader has borrowed a certain sum of money from a moneylender. He has promised to pay Rs. 5000 at the end of 4 years. How much has the trader borrowed if interest is compounded continuously at 10 per cent per annum? What is the effective rate of interest that will be paid by him?

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In the above discussion we encountered L (the limit of a function) as a finite number. However, it could be infinite as well. In such a situation, we need to reconsider the left and right limits formulations given above.

If we have a situation of $\lim_{x \to N} f(x) = \infty$ (or $-\infty$), then the function does not have a limit in real sense of the term. As you may observe $f(x) \to \infty$ implies the function is ever increasing in terms of its values, hence without a limit point. In case of ∞ values, we also have to forget about deriving left and right hand limits seen above and work with one side limit. For example, if you are evaluating the limit of $f(x)$ as $x \to \infty^-$, only left hand limit of $f(x)$ is relevant. If you are having $x \to \infty^+$, from which higher point of ∞ is seen, you start to evaluate the right hand limit. Remember that in case of infinity, we do not have the benefit of usual arithmetic operations. For example, we have $(\infty + 1) = \infty$ or $(\infty - 1) = \infty$. A similar logic applies to $-\infty$ and consequently we do not have a lower value than $-\infty$ to evaluate a left hand limit.

Before looking into a few examples on limit of a function, it will be useful to note that while estimating a limit, we are interested only with the values of the function $f(x)$ as x takes values closer and closer to say, N, but not when $x = N$. If you are evaluating a function, $f(x) = \frac{x^2 - 1}{x - 1}$ for example, you cannot define it for $x = 1$, as the denominator becomes zero. However, the function can be defined for all other values of $x \in R$. The question in this problem before us, therefore, could be, what is its limit as $x \to 1$. See that we write x tends to 1 but not $x = 1$. We will find the limit value of the function in such a case as well. Please note that direct substitution of the limiting number into a function does not help getting the appropriate solution. So we have to be careful while evaluating such functions.

Now let us specify cases where the limits are either zero or infinity or become indeterminate and no general conclusion is possible.

1) When $Lt\, f(x) = A, Lt\, g(x) = \infty$, then $Lt\, f(x) + Lt\, g(x) = A + \infty = \infty$ (an indeterminate form)

2) When $Lt\, f(x) = A, Lt\, g(x) = +\infty$, then $Lt\, f(x) - Lt\, g(x) = A - \infty = -\infty$ (an indeterminate form)

3) When $Lt\, f(x) = \infty, Lt\, g(x) = \infty$, then $Lt\, f(x) + Lt\, g(x) = \infty + \infty = \infty$

4) When $Lt\, f(x) = \infty, Lt\, g(x) = +\infty$, then $Lt\, f(x) - Lt\, g(x) = \infty - \infty$. It is an *indeterminate case* and no general conclusion is possible.

5) When $Lt\, f(x) = A, Lt\, g(x) = 0$, then $\frac{Lt\, f(x)}{Lt\, g(x)} = \frac{A}{0} = \infty$ (Infinite Lt)

6) When $Lt\, f(x) = 0, Lt\, g(x) = 0$, then $\frac{Lt\, f(x)}{Lt\, g(x)} = \frac{0}{0}$. It is an *indeterminate case* and no general conclusion is possible.

7) When $Lt\, f(x) = A,\ Lt\, g(x) = \infty$, then $\frac{Lt\, f(x)}{Lt\, g(x)} = \frac{\infty}{\infty}$

8) When $Lt\, f(x) = \infty,\ Lt\, g(x) = \infty$, then $\frac{Lt\, f(x)}{Lt\, g(x)} = \frac{\infty}{\infty}$. It is an *indeterminate case* and no general conclusion is possible.

9) If $Lt\ f(x) = A, Lt\ g(x) = \infty$, then $Lt\ f(x) \times Lt\ g(x) = A \times \infty = \infty$

10) If $Lt\ f(x) = \infty, Lt\ g(x) = \infty$, then $Lt\ f(x) \times Lt\ g(x) = \infty \times \infty = \infty$

Some Rules for Limits to treat Indeterminate forms

Algebraically, the technique for evaluating limits involves certain operations such as

i) Factorisation

ii) Rationalisation

iii) Simplification

iv) Other types of manipulations

The aim of these operations is to avoid indeterminate cases such as

i) $0 \times \infty$

ii) $\frac{0}{0}$

iii) $\frac{\infty}{\infty}$

iv) $\infty - \infty$

v) $|0| + |\infty|$

Let us take some examples requiring above mentioned operations.

Example 4: Find $\lim_{x\to 1} f(x)$, where $f\ (x) = \frac{x^2-1}{x-1}$the limit of the function .

Solution: We know that for $x = 1$, we cannot define the function. So simplify the function first. We write

$$f(x) = \frac{(x+1)(x-1)}{(x-1)} = (x+1)$$. Now, we have to find $\lim_{x\to 1}(x+1)$. Inserting $x = 1$ in the function $(x + 1)$, we get the limit as $(1+1) = 2$.

Example 5: Find $\lim_{x\to\infty} f(x)$, where $f\ (x) = \frac{x}{1+x}$.

Solution: If you equate x with ∞, then $f(x)$ becomes $\frac{\infty}{\infty}$, which you cannot evaluate. So try to simplify the function. If you divide both numerator and denominator by x, then you get $\frac{1}{(1/x)+1}$. Now substitute the value of x as ∞. You will find that

$$f(x) = \frac{1}{(1/\infty)+1} = \frac{1}{(0+1)} = \frac{1}{1} = 1.$$

Hence, $\lim_{x\to\infty} \frac{x}{1+x} = 1$

Example 6: Find $\lim_{x\to\infty} f(x)$, where $f\ (x) = \frac{1+2x}{1+3x}$.

Solution: Direct substitution of $x = \infty$, makes the function indeterminate. So, simplify it first. Apply the trick we have adopted in the proceeding example and divide the numerator and denominator by x. Then substitute the value of x as ∞. You will find the limit to be 2/3.

Example 7: Find $\underset{x\to 0}{lim} f(x)$, where $f\ (x) = \frac{[(x+1)^2-1]}{x}$.

Solution: Here again direct substitution will not yield the limit. So you need to simplify the function. Just write,

$$\frac{\left[(x+1)^2 - 1\right]}{x} \text{ as } \frac{1+x^2+2x-1}{x} = \frac{x(x+2)}{x} = (x+2) .$$

Then we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} (x+2)$. Direct substitution of $x=0$ will result in $\lim_{x\to 0} f(x) = 2$.

Example 8: Examine the behaviour of the following functions when (a) $x \to \infty$; (b) When $x \to -\infty$

(i) $\dfrac{5x^2+x+1}{x^2+2}$ (ii) $\dfrac{1-x^5}{x^4+x+1}$

Solution:

(i) $\underset{x\to\infty}{Lt} \dfrac{5x^2+x+1}{x^2+2} = \dfrac{5(\infty)^2+\infty+1}{(\infty)^2+2} = \dfrac{\infty}{\infty}$ (Indeterminate)

Therefore this limit requires the algebraic operation wherein both numerator as well as denominator will be divided by the highest power of x, that is x^2 in this case.

$$\therefore \underset{x\to\infty}{Lt} \frac{5x^2+x+1}{x^2+2} = \underset{x\to\infty}{Lt} \frac{\frac{5x^2}{x^2}+\frac{x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{2}{x^2}} = \underset{x\to\infty}{Lt} \frac{5+\frac{1}{x}+\frac{1}{x^2}}{1+\frac{2}{x^2}}$$

Now put $x=\infty$, to get $\dfrac{5+\frac{1}{\infty}+\frac{1}{\infty^2}}{1+\frac{2}{\infty^2}} = \dfrac{5+0+0}{1+0} = \dfrac{5}{1} = 5$

Similarly, $\underset{x\to-\infty}{Lt} \dfrac{5x^2+x+1}{x^2+2} = \dfrac{5+\frac{1}{-\infty}+\frac{1}{(-\infty)^2}}{1+\frac{2}{(-\infty)^2}} = \dfrac{5-0+0}{1+0} = \dfrac{5}{1}$

(ii) $\underset{x\to\infty}{Lt} \dfrac{1-x^5}{x^4+x+1} = \underset{x\to\infty}{Lt} \dfrac{\frac{1}{x^5}-1}{\frac{1}{x}+\frac{1}{x^4}+\frac{1}{x^5}} = \dfrac{\frac{1}{(\infty)^2}-1}{\frac{1}{\infty}+\frac{1}{(\infty)^4}+\frac{1}{(\infty)^5}}$

$$= \frac{0-1}{0+0+0} = \frac{-1}{0} = -\infty$$

For $x \to -\infty$, we will get the same answer. The students may try this themselves.

Example 9: Evaluate $\underset{x\to 0}{Lt} \dfrac{1/x^2}{1/x^4}$.

Solution: When we put $x=0$ in the given expression, we get the value $\dfrac{1/0}{1/0} = \dfrac{\infty}{\infty}$, which is an indeterminate case. It requires some algebraic operation to simplify the expression and then put $x=0$:

$$\underset{x\to 0}{Lt}\ \frac{1/x^2}{1/x^4} = \underset{x\to 0}{Lt}\ \frac{1}{x^2}\times\frac{x^4}{1}\quad \underset{x\to 0}{Lt}\ \left(x^2\right)=(0)^2=0$$

Example 10: If $f(x)=3+\dfrac{1}{x}$ and $g(x)=5-\dfrac{1}{x}$, examine the limits as $x\to 0$ of $f(x)+g(x)$

Solution: $\underset{x\to 0}{Lt}\ f(x)\quad g(x)\qquad \underset{x\to 0}{Lt}\ 3\ \ \frac{1}{x}\qquad 5\ \ \frac{1}{x}$

$= \underset{x\to 0}{Lt}\ 3+\dfrac{1}{x}\ +\underset{x\to 0}{Lt}\ 5-\dfrac{1}{x}\ =\infty+\infty$ case. But a little careful observation will reveal that we can avoid this situation by just opening the brackets.

That is, $Lt_{x\to 0}\left[3+\frac{1}{x}+5-\frac{1}{x}\right]=\underset{x\to 0}{Lt}\ 3+\cancel{\frac{1}{x}}+5-\cancel{\frac{1}{x}}\ =\underset{x\to 0}{Lt}\ 8=8$.

Hence the limit is 8.

Example 11: Evaluate the following limits

a) $\underset{x\to 1}{Lt}\ \dfrac{x-1}{2x^2-7x+5}$ b) $\underset{x\to 1}{Lt}\ \dfrac{x^4-3x^3+2}{x^3-5x^2+3x+1}$

Solution: a) If we put $x=1$ in the given expression, then

$\underset{x\to 1}{Lt}\ \dfrac{x-1}{2x^2-7x+5}$ becomes $\dfrac{1-1}{2(1)^2-7(1)+5}=\dfrac{0}{0}$

(Indeterminate)

Therefore, it requires some algebraic operation. Let us try factorisation method where $2x^2-7x+5=(x-1)(2x-5)$

$\therefore$ we get, $\underset{x\to 1}{Lt}\ \dfrac{(x-1)}{(x-1)(2x-5)}=\underset{x\to 1}{Lt}\ \dfrac{1}{(2x-5)}=\dfrac{1}{2(1)-5}=\dfrac{-1}{3}$

b) If we put $x=1$ in the expression $\dfrac{x^4-3x^3+2}{x^3-5x^2+3x+1}$

We get $\dfrac{(1)^4-3(1)^3+2}{(1)^3-5(1)^2+3(1)+1}=\dfrac{1-3+2}{1-5+3+1}=\dfrac{0}{0}$ (Indeterminate)

Therefore, it requires some algebraic operation. Here, we factorise both numerator as well as denominator, to get

$$\underset{x\to 1}{Lt}\ \frac{x^4-3x^3+2}{x^3-5x^2+3x+1}=\underset{x\to 1}{Lt}\ \frac{(x-1)\left(x^3-2x^2-2x-2\right)}{(x-1)\left(x^2-4x-1\right)}$$

(**Hint:** Given a function $f(x)$ and $x = a$ as one of its roots, that is, $f(a) = 0$, then $(x - a)$ will be one of the factors of $f(x)$. On dividing $f(x)$ with $(x - a)$ by way of long division, other factors can be ascertained.)

$$= \underset{x \to 1}{Lt} \frac{x^3 - 2x^2 - 2x - 2}{x^2 - 4x - 1}$$

$$= \frac{(1)^3 - 2(1)^2 - 2(1) - 2}{(1)^2 - 4(1) - 1} = \frac{1-2-2-2}{1-4-1} = \frac{-5}{-4} = \frac{5}{4}$$

Example 12: Find the limit of the following

(a) $\underset{x \to 0}{Lt} \frac{\sqrt{2-x} - \sqrt{2+x}}{x}$ and (b) $\underset{x \to \infty}{Lim}\left(\sqrt{x^2+5x+4} - \sqrt{x^2-3x+4}\right)$

Solution: (a) This question involves irrational expression and, therefore, the algebraic operation here is rationalisation.

$$\therefore \underset{x \to 0}{Lt} \frac{\sqrt{2-x} - \sqrt{2+x}}{x} = \frac{\sqrt{2} - \sqrt{2}}{0} = \frac{0}{0}$$ (Indeterminate case)

On rationalization expression becomes

$$= \frac{\sqrt{2-x} - \sqrt{2+x} \quad \sqrt{2-x} + \sqrt{2+x}}{x\left(\sqrt{2-x} + \sqrt{2+x}\right)}$$

$$= \frac{2-x-2-x}{x\left(\sqrt{2-x} + \sqrt{2+x}\right)} = \frac{-2x}{x\left(\sqrt{2-x} + \sqrt{2+x}\right)} = \frac{-2}{\sqrt{2-x} + \sqrt{2+x}}$$

Now take the limit to get

$$\underset{x \to 0}{Lt} \frac{-2}{\sqrt{2-x} + \sqrt{2+x}} = \frac{-2}{\sqrt{2-0} + \sqrt{2+0}} = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

(b) Before rationalization we get

$$\underset{x \to \infty}{Lt} \sqrt{x^2+5x+4} - \sqrt{x^2-3x+4} = \infty - \infty$$ (Indeterminate case)

Therefore we rationalise and make use of the formulae $(a+b)(a-b) = a^2 - b^2$

$$= \underset{x \to \infty}{Lt} \frac{\left[\sqrt{(x^2+5x+4)} - \sqrt{(x^2-3x+4)}\right]\left[\sqrt{(x^2+5x+4)} + \sqrt{(x^2-3x+4)}\right]}{\left[\sqrt{(x^2+5x+4)} + \sqrt{(x^2-3x+4)}\right]}$$

$$= \underset{x \to \infty}{Lt} \frac{x^2 + 5x + 4 - x^2 + 3x - 4}{\left[\sqrt{(x^2+5x+4)} + \sqrt{(x^2-3x+4)}\right]}$$

$$= \underset{x\to\infty}{Lt} \frac{8x}{\left[\sqrt{(x^2+5x+4)}+\sqrt{(x^2-3x+4)}\right]}$$

(Dividing numerator and denominator by x, and under the root x becomes x^2)

$$= \underset{x\to\infty}{Lt} \frac{8}{\left[\sqrt{\left(\frac{x^2}{x^2}+\frac{5x}{x^2}+\frac{4}{x^2}\right)}+\sqrt{\left(\frac{x^2}{x^2}-\frac{3x}{x^2}+\frac{4}{x^2}\right)}\right]}$$

$$= \underset{x\to\infty}{Lt} \frac{8}{\left[\sqrt{\left(1+\frac{5}{x}+\frac{4}{x^2}\right)}+\sqrt{\left(1-\frac{3}{x}+\frac{4}{x^2}\right)}\right]}$$

$$= \frac{8}{\sqrt{1+0+0}+\sqrt{1-0-0}} = \frac{8}{1+1} = 4$$

Example 13: Evaluate a) $\underset{x\to\infty}{Lt} \frac{x+1}{3x-1}+\frac{2x+1}{x-1}$ and b) $Lt_{x\to\infty}\left(\frac{x+1}{3x-1}\right)\left(\frac{2x+1}{x-1}\right)$

(a) $\underset{x\to\infty}{Lt} \frac{x+1}{3x-1}+\frac{2x+1}{x-1} = \underset{x\to\infty}{Lt} \frac{x+1}{3x-1} + \underset{x\to\infty}{Lt} \frac{2x+1}{x-1}$

$$\underset{x\to\infty}{Lt} \frac{1+\frac{1}{x}}{3-\frac{1}{x}} + \underset{x\to\infty}{Lt} \frac{2+\frac{1}{x}}{1-\frac{1}{x}} = \frac{1+\frac{1}{\infty}}{3-\frac{1}{\infty}}+\frac{2+\frac{1}{\infty}}{1-\frac{1}{\infty}} = \frac{1+0}{3-0}+\frac{2+0}{1-0}$$

$$= \frac{1}{3}+2 = \frac{7}{3}$$

(b) $\underset{x\to\infty}{Lt} \frac{x+1}{3x-1}\ \frac{2x+1}{x-1} = \underset{x\to\infty}{Lt} \frac{x+1}{3x-1} \cdot \underset{x\to\infty}{Lt} \frac{2x+1}{x-1}$

$= \frac{1}{3}\times 2 = \frac{2}{3}$ [see part (a)]

Example 14: Find the limit of $\frac{x^m-a^m}{x^n-a^n}$ when $x \to a$.

Solution: When we put $x=a$ in the given expression, we get $\frac{0}{0}$ an indeterminate case. Also we have a standard form as $\underset{x\to a}{Lt} \frac{x^n-a^n}{x-a} = n\,a^{n-1}$. Now express an expression in this form. That is,

$$= \underset{x \to a}{Lt} \frac{x^m - a^m}{x^n - a^n} = \underset{x \to a}{Lt} \left[\frac{x^m - a^m}{x - a} \times \frac{x - a}{x^n - a^n} \right]$$

$$= \underset{x \to a}{Lt} \left[\frac{x^m - a^m}{x - a} \times \frac{1}{\frac{x^n - a^n}{x - a}} \right]$$

$$= \underset{x \to a}{Lt} \left[\frac{x^m - a^m}{x - a} \div \frac{x^n - a^n}{x - a} \right] = ma^{m-1} \div na^{n-1} = \frac{m}{n} a^{m-n}$$

Check Your Progress 2

1) Given the function: $y = \dfrac{x^2 + x - 56}{x - 7}$, $(x \neq 7)$, find the left hand limit and the right hand limit of y as x approaches 7. Can we conclude from these answers that y has a limit as x approaches 7?

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2) Given the function, $y = \dfrac{x^3 - 6x^2 + 12x - 8}{x^2 - 4x + 4}$, $(x \neq 2)$, find the limit of y as x approaches 2.

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3) Find(a) $\underset{x \to 3}{Lim} \dfrac{x^2 - 7x + 12}{x - 3}$ (b) $\underset{x \to 0}{Lim} \dfrac{\sqrt{2 + 3x} - \sqrt{2 - 5x}}{4x}$

(c) $\underset{x \to \infty}{Lim} \dfrac{2x^3 + 3x^2 + 5}{5x^3 + 8x - 17}$

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4) Consider a demand function $p = \dfrac{5}{q + 2}$. What is the total revenue when the quantity demanded approaches infinity?

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L' Hospital's Rule

When two differentiable functions are such that their quotient $\frac{f(x)}{g(x)}$ approaches indeterminate cases ($\frac{0}{0}$ or $\frac{\infty}{\infty}$) as x tends to some value, say, a, then the L' Hospital's Rule implies that:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)} \text{ and so on.}$$

This means:

i) If $\underset{x \to a}{Lt}$ of $\frac{f(x)}{g(x)}$ does not exist, then we can take the limit of quotient of their derivatives.

ii) If $\underset{x \to a}{Lt} \frac{f'(x)}{g'(x)}$ is also indeterminate, then use can find limit of quotient of their second order derivative and so on; the process can be continued.

Note: i) There are several types of indeterminate forms, a few of which are:

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, 0^0, \infty^\infty$, and 1^∞

But, L' Hospital's Rule only applies directly to indeterminate forms of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The other types of indeterminate forms must first be reduced to one of these two types if we wish to apply L' Hospital's Rule.

ii) L' Hospital's Rule also applies to one-sided limits and limits where $x \to \pm\infty$.

Example 15

Evaluate the following limits using L' Hospital's Rule.

a) $\underset{x \to a}{Lt} \frac{x^4 - a^4}{x - a}$ b) $\underset{x \to 0^+}{Lt} \frac{a^x - b^x}{x}$ c) $Lt_{x \to 0^+} x^x$

Solution

a) $\underset{x \to a}{Lt} \frac{x^4 - a^4}{x - a}$ is of standard form $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

Here, $n = 4$, therefore, the limit is $4a^{4-1} = 4a^3$

b) $\underset{x \to 0^+}{Lt} \frac{a^x - b^x}{x} = \frac{0}{0}$.

Therefore, in order to apply L' Hospital's rule we first find

$\frac{d}{dx} \frac{a^x - b^x}{x}$ which is a standard form and equals

$\left(a^x \log a - b^x \log b\right)$.

Now using L' Hospital's Rule, we get

$$\underset{x\to 0^+}{Lt}\ \frac{a^x - b^x}{x} = \underset{x\to 0^+}{Lt}\ \frac{d}{dx}\ \frac{a^x - b^x}{x} = \underset{x\to 0^+}{Lt}\ \left(a^x \log a - b^x \log b\right)$$

$$= a^0 \log a - b^0 \log b = \log a - \log b = \log\ \frac{a}{b}\ .$$

c) Notice that $\text{Lt}_{x\to 0^+} x^x$ is an indeterminate form of type 0^0. This can be simplified by writing

$$x^x = e^{lnx^x} = e^{xlnx} = xlnx$$

Now, $$Lt_{x\to 0^+} x^x = Lt_{x\to 0^+} xlnx$$

$$\underset{x\to 0^+}{Lt} xlnx = \underset{x\to 0^+}{Lt} \frac{lnx}{\frac{1}{x}} = \frac{\infty}{\infty}\text{type}$$

Now, we can apply the L'Hospital's rule:

$$\underset{x\to 0^+}{Lt} \frac{lnx}{\frac{1}{x}} = \underset{x\to 0^+}{Lt} \frac{\frac{1}{x}}{\frac{1}{x^2}} = \underset{x\to 0^+}{Lt}(-x) = 0$$

7.5 LET US SUM UP

This unit discussed the very important concept of limits of a function. We saw that the concept of a limit has to do with "approaching a value" eventually as some other quantity or variable 'tends to' a value. The unit discussed first the basic notion of a limit intuitively and subsequently provided a rigorous definition. Following this the unit took up for discussion the limit of a sequence, and that of a function.

The unit also made a detailed discussion of left-hand and right-hand limits. Following this, the unit discussed some important properties of limits, the rules that are followed to arrive at the limit of a function, including the very useful L'Hopital's rule. You were also familiarised with some standard limits. On the whole, with the help of various examples, the learner has been introduced to various techniques of approaching the limit of a function. Not only to the finite extent, but also the infinite extent of approaching to the limit of a function has been touched.

7.6 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1)

a) $$\underset{x\to -2}{Lt}\left(x^2 + 5x\right) = \underset{x\to -2}{Lt} x^2 + 5\ \underset{x\to -2}{Lt} x = (-2)^2 + 5(-2) = 4 - 10 = -6$$

b) $$\underset{x\to 0}{Lt}\ \frac{3x+1}{5x-1} = \frac{\underset{x\to 0}{Lt}(3x+1)}{\underset{x\to 0}{Lt}(5x-1)} = \frac{3(0)+1}{5(0)-1} = \frac{1}{-1} = -1$$

c) $\underset{x\to 4}{Lt}\ \dfrac{2x^{3/2}-\sqrt{x}}{x^2-15} = \dfrac{\underset{x\to 4}{Lt}\left(2x^{3/2}-\sqrt{x}\right)}{\underset{x\to 4}{Lt}\left(x^2-15\right)} = \dfrac{2(4)^{3/2}-\sqrt{4}}{(4)^2-15}$

$= \dfrac{2\times 4\times 2-2}{1} = 14$

d) $\underset{x\to a}{Lt}\ Ax^n = \underset{x\to a}{Lt}\ A\cdot \underset{x\to a}{Lt}\ x^n = A\cdot(a)^n$

e) $\frac{1}{3}$

f) -6

2) Average Cost (AC) = $4+\frac{2}{q}$. Now, $\lim_{q\to\infty} 4+\frac{2}{q} = 4$

3) Principal = Rs 3357 (Approx.); Effective rate of interest = 10.5%

Check Your Progress 2

1) 15

2) 0

3) (a) -1 ; (b) $\frac{1}{\sqrt{2}}$; (c) $\frac{2}{5}$

4) 5