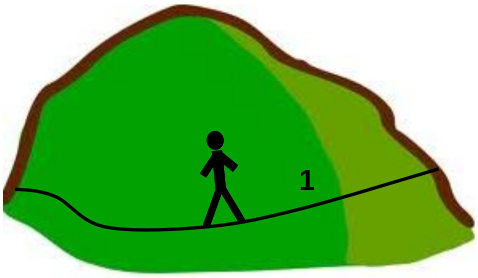
Indira Gandhi National
Open University
School of Sciences

BPHCT-133
ELECTRICITY AND MAGNETISM

Block

1

VECTOR ANALYSIS



How would you determine the path of steepest ascent on a hill?

UNIT 1

SCALAR FIELDS AND THEIR GRADIENT

Structure

STUDY GUIDE

In this unit you will study about scalar fields and their gradient. Before studying this unit, you should revise Units 1 and 2 of the Physics Elective BPHCT-131 entitled Mechanics. You should also be familiar with the basic concepts of differential calculus which you have studied in school. These concepts are also explained in the Mathematics Elective BMTC-131. For calculating the gradient you must know how to obtain the partial derivatives of functions. You can revise this from the Appendix of this unit. Partial derivatives are also explained in the Mathematics course BMTC-132, which you may revise. A brief summary of the basic concepts of vector algebra and derivatives of vector functions is provided in Appendix A1 of this block. You may like to go through it before studying this unit.

> ***"And, believe me, if I were again beginning my studies, I should follow the advice of Plato and start with mathematics."***
>
> ***Galileo Galilei***

1.1 INTRODUCTION

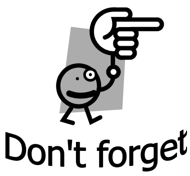
In the first two units of Block 1 of the first course in Physics entitled Mechanics (BPHCT-131), you have studied the basic concepts of vector algebra. You have learnt how to add, subtract and multiply vector quantities using both the geometric and algebraic representation of vectors. You have seen examples of vectors, their sum and difference, as well as scalar and vector products in physics. In Unit 2 you have also studied the preliminary concepts of vector differential calculus. You now know about vector functions or vector valued functions and how to obtain their derivatives with respect to a scalar variable. You have also learnt how to obtain the derivatives of vector and scalar products of vectors. In this unit, we further study vector differential calculus. In Sec. 1.2, we introduce the concept of **scalar fields**. In Sec. 1.3, you will learn about the **gradient** and **directional derivative** of a scalar field.

You may wish to know: Why do you need to learn these concepts? To understand this, consider the example of a bar, plate or a cylinder that is heated non-uniformly. So, its temperature at different points is different. The temperature distribution of the bar/plate/cylinder is represented mathematically by a **scalar field**. If we now wish to know the **rate of change** of the temperature within the object, **in any given direction**, we need to determine the **directional derivative** of this temperature distribution. For this, we must know the **gradient of the scalar field**. In Sec. 1.3.1, you will learn how to determine the gradient of a scalar field. Then you will learn how to determine the directional derivative of the field in Sec. 1.3.2.

Expected Learning Outcomes

After studying this unit, you should be able to:

- explain the concept of scalar fields and give examples in physics;
- determine the gradient of a scalar field; and
- determine the directional derivative of a scalar field.



IN YOUR WRITTEN WORK, ALWAYS USE AN ARROW ABOVE THE LETTER YOU USE TO DENOTE A VECTOR, E.G., $\vec{r}$. USE A CAP ABOVE THE LETTER YOU USE TO DENOTE A UNIT VECTOR, E.G., $\hat{r}$.

1.2 SCALAR FIELDS

In Block 1 of the course BPHCT-131 entitled Mechanics you have learnt about vector functions, which can depend on one or more variable. When we are describing physical quantities, one obvious variable with which many physical quantities change is time. However, you have also studied about many physical quantities which have different values at different points in space. **A function that describes a physical quantity at different points in space is called a field**. As physical quantities are either scalar or vector in nature, we can have both **scalar** and **vector fields**. In this section, we focus on **scalar fields**.

For example, the density of air (in the Earth's atmosphere) is a scalar quantity that changes with the altitude above the sea level. Similarly, the atmospheric pressure is also a scalar having different values at different altitudes. It is also different at different points around the Earth. The temperature of an unevenly heated plate is a function of both space coordinates and time. All these are examples of **scalar** fields.

Let us discuss the concept of a scalar field in more detail.

You have studied in Unit 2 of BPHCT-131 that every point in space can be denoted by the position vector

$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}.$$

So we could also say that corresponding to every point represented by a position vector $\vec{\mathbf{r}}$ we have a unique scalar quantity $f(\vec{\mathbf{r}})$.

1.2.1 Defining a Scalar Field

A scalar field is a function that assigns a unique scalar to every point in a given region. So you can say that it is essentially a scalar function of space coordinates. As you know from your school mathematics, every point in space may be specified by the Cartesian coordinates of the point (x, y, z). So we can write the scalar function or scalar field as $f = f(x, y, z)$. This means that for every point (x, y, z) in space, there exists a unique scalar quantity given by $f(x, y, z)$.

The gravitational potential energy of an object near the surface of the Earth is a simple example of a scalar field. Suppose we take the *xy* plane to lie on the surface of the Earth and the *z*-axis to point upwards. Then the gravitational potential energy of the object is given by

$$\phi(x, y, z) = mgz \qquad (1.1a)$$

where m is the mass of the object. In this case the scalar field depends only on the z coordinate and is, therefore, a one-dimensional scalar field.

Temperature, pressure and density are examples of some other physical quantities which are scalars and can be represented by scalar fields. For example, the temperature on a perfectly flat hot plate can be described by

$$T(x, y) = \frac{250}{x^2 + y^2 + 1} \qquad (1.1b)$$

$T(x, y)$ gives us the temperature at any point (x, y) on the surface of the hot plate (Fig. 1.1). Since the surface of the plate is two-dimensional, the value of T depends only on x and y. This is an example of a two-dimensional scalar field.

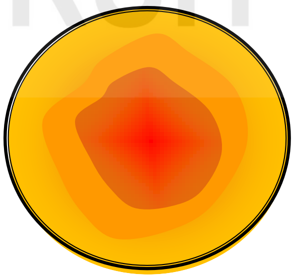


Fig. 1.1: **Temperature on a flat hot plate heated at the centre is a two-dimensional scalar field. At the centre the temperature is very high. The temperature is lower as we move away from the centre of the hot plate.**

Another example of a scalar field is the **electric potential** in free space at a distance r from a point charge q. It is given by

$$V(r) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r} \text{ volt} \qquad (1.1c)$$

where q is measured in coulomb and r in m. Here ε_0 is the permittivity of free space. If we consider the origin of the Cartesian coordinate system to be located at the charge we can also write the electric potential at a distance r as

$$V(x, y, z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{(x^2 + y^2 + z^2)^{1/2}}\right] \tag{1.1d}$$

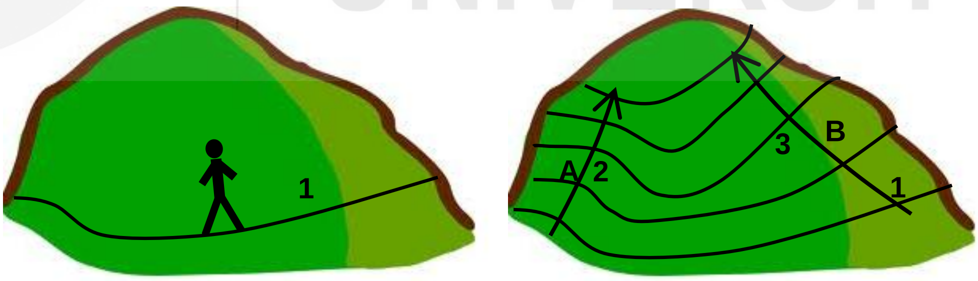
This function describes a scalar field in three-dimensions. In this unit, we are interested in finding out about the rate at which these fields change in space.

In order to understand physically the rate of change of scalar fields, it is a good idea to learn how to represent them pictorially. So our next question is: **How do we represent scalar fields in diagrams**? It is possible to represent scalar fields visually using what are called **contour curves** or **contour surfaces**. Let us see how.

1.2.2 Representations of a Scalar Field

Have you heard about contour lines? Do you remember the maps you studied in your school geography courses? How do we show the heights of places on a map? Recall from school geography courses that these are shown using **contour lines** or **contour curves**. On a map, **contour lines or contour curves connect those points which are at the same height (elevation) above a fixed level**. This is usually the sea level. Remember that **contour lines never cross each other**. If you walk along a contour line you neither gain nor lose elevation. Contour lines are useful because they tell us about the **shape** of the land surface.

Imagine that you are standing at some point on a hill at a certain height (Fig. 1.2a). Study Fig. 1.2b. It shows the contour curve 1 joining all points of that height on the hill. In fact, **each contour curve joins all the points on the hill which are at the same height**. Suppose we write the height at a point (x, y) as z (x, y). Then the contour curves in Fig 1.2b join the points $z(x, y)$ = constant. The contour curves are then a pictorial representation of the scalar function $z(x, y)$.



(a) **(b)**

Fig.1. 2: **a) Imagine you are standing somewhere on a hill; b) contour map of the hill. A plot of several contour lines is called a contour map.**

We have already described the scalar field for the gravitational potential energy in Eq. (1.1a). You know that all points in any given contour curve in Fig. 1. 2b depict points which lie at the same height above the ground. So as per Eq. (1.1a), it is also the curve on which the gravitational potential energy of the object would be constant. Thus, the set of contour curves shown in Fig. 1. 2b can be used to depict the gravitational potential energy field given by Eq. (1.1a).

In general,

A contour curve is defined as a curve in two-dimensions on which the value of the scalar field $f(x, y)$ **is constant**:

$$f(x, y) = C \qquad (1.2a)$$

A plot of several contour curves is called a **contour plot** or a **contour map**.

The contour map for the temperature of the hot plate is shown in Fig. 1.3.

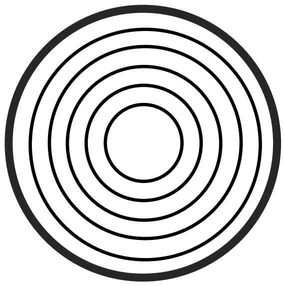


Fig. 1.3: **Contour Map for the temperature of the hot plate. Each contour line corresponds to one value of the temperature defined in Eq. (1.1b).**

We now ask: What else can we determine from the contour map of a scalar field? Study Fig. 1.2b once again. Look at the regions around the points *A* and *B* in it. Note that the contour curves around *A* are close together and those around *B* are far apart. What does this tell us?

Imagine that you are climbing the hill along a steep path (path 2 in Fig. 1.2b). Since the height above the sea level changes rapidly along this path, the contour curves (the curves of equal height above the ground) lie close to each other. On the other hand, if you walk along path 3, where the contour curves are far apart, the height above sea level changes comparatively slowly. So, the spacing of the contour curves on a contour map indicates how rapidly the function is changing: If the contour curves lie close to each other, the scalar field changes rapidly in that region. If these are far apart, the change in the scalar field is slower.

So far we were talking about scalar functions in two-dimensions.

You may now ask: How do we represent scalar fields in three-dimensions? One way of doing this is to define **contour surfaces** as follows:

Contour surfaces are the surfaces on which the value of a three-dimensional scalar field is constant.

So, if a scalar field is defined by the function $\phi(x, y, z)$, the contour surface would be the collection of all points (x, y, z) for which the value of ϕ is constant, say *C*. The contour surface is defined by the equation

$$\phi(x, y, z) = C \qquad (1.2b)$$

We get different contour surfaces for different values of *C*. A collection of such contour surfaces would then be a representation of the scalar field $\phi(x, y, z)$.

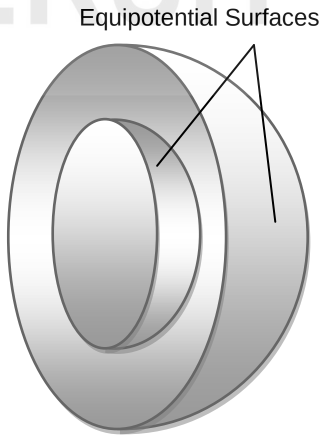


Fig. 1.4: **Contour surfaces of constant electric potential are surfaces of concentric spheres. The radii of the spheres are defined by Eq. (1.3b).**

For example, for the scalar field described by Eq. (1.1d), the contour surfaces are the surfaces of constant electric potential given by $V(x, y, z) = V_0$ and are described by the equation

$$\frac{1}{4\pi\varepsilon_0}\frac{q}{(x^2+y^2+z^2)^{1/2}} = V_0 \Rightarrow x^2+y^2+z^2 = R^2 \qquad (1.3a)$$

where *R* is a constant. You can see that the contour surface given by Eq. (1.3a) is a sphere whose radius is given by the equation:

$$R = \frac{q}{4\pi\varepsilon_0 V_0} \qquad (1.3b)$$

For different values of V_0, R is different. The contour surfaces (for different values of V_0) are surfaces of concentric spheres, as shown in Fig. 1.4.

Any contour curve on which a potential is constant is called an **equipotential curve**. The contour surfaces described by Eq. (1.3a) are called **equipotential surfaces**. Similarly, contour curves or surfaces for the temperature field are called **isothermals**.

Now that we have defined a scalar field and represented it pictorially, we would like to know: **What is the rate of change of a scalar field in a given direction?** For example, if you are standing somewhere on a hill, you may wish to find out the direction of the fastest way down (or the steepest slope) so that you may avoid it! You would of course like to take the direction of least slope to climb down, lest you fall! Recall from school calculus that the slope also gives us the rate of change. We use this fact to arrive at the concept of the gradient operator and the directional derivative. You will learn in the next section that the **gradient of a scalar field gives the maximum rate of change of the function**. So let us define the gradient of a scalar field. Then we shall use it to explain the concept of directional derivative.

1.3 GRADIENT OF A SCALAR FIELD AND THE DIRECTIONAL DERIVATIVE

In your school calculus course, you have learnt about the concept of the slope of a function. Suppose you heat a thin rod at one of its ends. Then the temperature T of the rod would be different at different points along its length. Suppose you wish to know the rate at which T changes along the rod. Let us treat the rod as a one-dimensional object and choose the x-axis to be along the rod. So, $T = T(x)$ is a function of x.

Then the question is: **If you change** x **by a small amount** dx, **by what amount does** $T(x)$ **change?** Let this change in $T(x)$ be denoted by dT. From school calculus, you know that the answer is

$$dT = \left(\frac{dT}{dx}\right)dx \tag{1.4}$$

So if x is changed by an amount dx, T changes by an amount dT given by Eq. (1.4). The derivative $\frac{dT}{dx}$ is the proportionality factor. It gives **the rate of change in** $T(x)$ **along the** x**-direction**.

In this example, the temperature is a function of just one variable x measured along the length of the rod.

Now imagine a closed room with a fire lit in the fireplace (or a room heater) in one corner. The temperature in the room is described by the scalar field $T(x, y, z)$. Standing at a point in the room, say at the centre, we could ask at what rate the temperature changes as we move away from the point in the x-direction, say by a distance Δx. From your school calculus course, you know that the rate of change of a two or three-dimensional function is described by **partial derivatives** of the function. Thus, the partial derivative of $T(x, y, z)$ with respect to x gives its rate of change in the x-direction. Can we then find out the direction in which the rate of change of temperature is maximum? The answer is, yes, we can, in terms of the partial derivatives of the scalar field $T(x, y, z)$. At this stage, you may like to refresh your knowledge of partial derivatives. For this, you should **study the Appendix to this unit before studying further**.

Let us now answer the question: How do we express the **maximum rate of change of any scalar field**? To do so, we define the **gradient** of a scalar field using the concept of partial derivatives.

1.3.1 The Gradient of a Scalar Field

Let us begin by considering a two-dimensional scalar field $f(x, y)$. Then we shall generalize the results to $f(x, y, z)$. Consider two points $P(x, y)$ and $Q(x+\Delta x, y+\Delta y)$ in the region in which the scalar field $f(x, y)$ is defined. Suppose the position vectors of the points are $\vec{\mathbf{r}}$ and $\vec{\mathbf{r}}+\Delta\vec{\mathbf{r}}$, respectively. Then the change in $f(x, y)$ as one goes a small distance from the point P to point Q is given by

Do the steps from Eq. (1.5a) to Eq. (1.7c) yourself on a separate piece of paper as you study them. You will understand this concept better.

$$\Delta f = f(x+\Delta x, y+\Delta y) - f(x, y) \tag{1.5a}$$

To get to the final result, we add and subtract a term $f(x, y+\Delta y)$ and write:

$$\Delta f = \left[f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)\right] + \left[f(x, y+\Delta y) - f(x, y)\right] \tag{1.5b}$$

Then we divide and multiply the two terms on the right hand side of Eq. (1.5b) by Δx and Δy, respectively, and write:

$$\Delta f = \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\Delta x + \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\Delta y \tag{1.5c}$$

Next we consider the terms on the right-hand side of Eq. (1.5c) and take the limits of Δf as $\Delta x \to 0$ and $\Delta y \to 0$:

$$\begin{aligned}\lim_{\Delta x\to 0, \Delta y\to 0} \Delta f &= \lim_{\Delta x\to 0, \Delta y\to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\Delta x \\ &+ \lim_{\Delta x\to 0, \Delta y\to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\Delta y\end{aligned} \tag{1.5d}$$

Using the definition of partial derivative in Eq. (1) of the Appendix to this unit, we can write

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\Delta x = \frac{\partial f(x, y+\Delta y)}{\partial x}\, dx$$

and $$\lim_{\Delta y\to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\Delta y = \frac{\partial f(x, y)}{\partial y}\, dy$$

Using these results, we can write Eq. (1.5d) as

$$\begin{aligned}\lim_{\Delta x\to 0, \Delta y\to 0} \Delta f &= \lim_{\Delta y\to 0} \frac{\partial f(x, y+\Delta y)}{\partial x} dx + \lim_{\Delta x\to 0} \frac{\partial f(x, y)}{\partial y}\, dy \\ &= \frac{\partial f(x, y)}{\partial x} dx + \frac{\partial f(x, y)}{\partial y}\, dy\end{aligned} \tag{1.5e}$$

The left hand side of the Eq. (1.5e) defines a function df called the **total differential** of f. Thus, the total differential of a two-dimensional scalar field $f(x, y)$ or f (to keep the writing of the equation simple) is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \tag{1.6a}$$

We can generalize this result to $f(x, y, z)$ for which the total differential df is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \tag{1.6b}$$

The significance of df is that it is a good approximation to Δf if $dx (= \Delta x)$, $dy = (\Delta y)$ and $dz (= \Delta z)$ represent small changes in x, y and z, respectively. So df becomes a better and better approximation of Δf as dx, dy and dz become smaller.

We are now ready to introduce the concept of the gradient. We can rewrite Eq. (1.6b) in a more convenient form as follows:

$$df = \left(\frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}} + \frac{\partial f}{\partial z} \hat{\mathbf{k}} \right) . (dx\, \hat{\mathbf{i}} + dy\, \hat{\mathbf{j}} + dz\, \hat{\mathbf{k}}) \tag{1.7a}$$

You must verify by simplifying Eq. (1.7a) that it is indeed the same as Eq. (1.6b). You can also identify that

$$dx\, \hat{\mathbf{i}} + dy\, \hat{\mathbf{j}} + dz\, \hat{\mathbf{k}} \equiv d\vec{\mathbf{r}} \tag{1.7b}$$

is the change in the position vector. The vector

$$\frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}} + \frac{\partial f}{\partial z} \hat{\mathbf{k}} = \text{grad } f \equiv \vec{\nabla} f \tag{1.7c}$$

is defined as the **gradient of the scalar field** f.

Using Eqs. (1.7b and c), we can write Eq. (1.7a) as

$$df = (\vec{\nabla} f) . d\vec{\mathbf{r}} \tag{1.8}$$

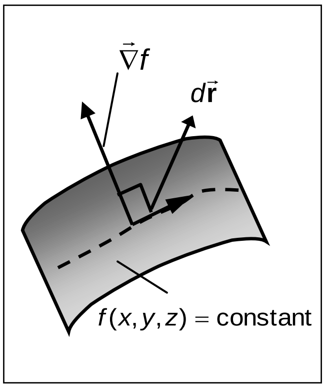


Fig. 1.5: **The gradient of a scalar field is perpendicular to the surface on which the scalar field is constant.**

The symbol ∇ is pronounced as 'del' and $\vec{\nabla} f$ as grad f. Eq. (1.7c) tells us that **the gradient of a scalar field is a vector**. Note that $\vec{\nabla} f$ *is a vector such that the change df in f for an arbitrarily small change* $d\vec{\mathbf{r}}$ *is given by* Eq. (1.8). Let us interpret the physical meaning of this result.

Suppose the angle between $\vec{\nabla} f$ and $d\vec{\mathbf{r}}$ is given by θ. Then we can write Eq. (1.8) as

$$df = \left|\vec{\nabla} f\right| \left|d\vec{\mathbf{r}}\right| \cos\theta = \left|\vec{\nabla} f\right| dr \cos\theta \tag{1.9a}$$

What happens when θ is 90°, that is $\vec{\nabla} f$ is **perpendicular** to $d\vec{\mathbf{r}}$? Eq. (1.9a) tells us that $df = 0$ which means that f is constant. In other words, the **value of the scalar field** f **is constant along the direction perpendicular to its gradient** (Fig. 1.5). We can express this result as follows:

> The vector $\vec{\nabla} f$ is perpendicular (normal) to the curve or surface
>
> $f = \text{constant}$

We now keep *dr* constant and find the change *df* in various *directions* by changing θ. Then we ask: For what value of θ (or in which direction) is the change maximum? From Eq. (1.8), you can immediately say that *df* is maximum when θ is 0°. This means that for a fixed *dr*, the change *df* is **maximum** when you move in the same direction as $\vec{\nabla} f$. In addition, the rate of change of *f* with respect to *r* is given by

$$\frac{df}{dr} = \left|\vec{\nabla} f\right| \cos\theta \qquad (1.9b)$$

So the **maximum rate of increase of the scalar field *f* in space is along the direction of the gradient of the field $\vec{\nabla} f$ and its magnitude is**

$$\left(\frac{df}{dr}\right)_{max} = \left|\vec{\nabla} f\right| \qquad (1.10)$$

> The magnitude of $\vec{\nabla} f$ gives the **maximum rate of change** of the scalar field in space.

In the same way, when $d\vec{\mathbf{r}}$ is in the direction opposite to $\vec{\nabla} f$, then θ is 180° in Eq.(1.8) and $\frac{df}{dr} = -|\vec{\nabla} f|$. This is then the direction in which the **rate of decrease of the field *f* is maximum**.

We have arrived at the **definition of the gradient of a scalar field and learnt its physical meaning**. Let us put these results together.

GRADIENT OF A SCALAR FIELD

The **gradient of a scalar field** *f* is defined as follows:

$$\text{grad } f \equiv \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \qquad (1.11a)$$

For a scalar field $f(x, y)$ in two-dimensions,

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} \qquad (1.11b)$$

The **magnitude** of $\vec{\nabla} f$ gives the maximum rate of change of the scalar field in space.

The **direction** of $\vec{\nabla} f$ is **perpendicular** to the curves or surfaces $f = \text{constant}$.

You should note that: $f(x, y, z)$ is a scalar field, because it assigns a **scalar quantity** to every point (x, y, z) in space. On the other hand, the **gradient** $\vec{\nabla} f$ of the scalar field f **assigns a vector quantity to every point in space**. Therefore, $\vec{\nabla} f$ is a **vector field**.

Let us further understand the concept of gradient with the help of examples from physics and calculate its value.

In physics, many vector quantities can be expressed as the gradient of a scalar field. For example, we can express the gravitational force as $\vec{\mathbf{F}} = -\vec{\nabla} V$, where V is the gravitational potential energy. The electric field $\vec{\mathbf{E}}$ due to a static charge distribution is the gradient of the electric potential: $\vec{\mathbf{E}} = -\vec{\nabla}\phi$. Let us take up an example to calculate the gradient of a potential field.

EXAMPLE 1.1: GRADIENT OF A SCALAR FIELD

The potential that represents an inverse square force is

$V(x, y, z) = \dfrac{k}{(x^2 + y^2 + z^2)^{1/2}}$ where k is a constant. Using the definition $\vec{\mathbf{F}} = -\vec{\nabla} V$, calculate the components of this force.

SOLUTION ■ We use the definition $\vec{\mathbf{F}} = -\vec{\nabla} V$ and calculate the gradient of $V(x, y, z)$ from Eq. (1.11a).

Thus, we have

$$(F_x \hat{\mathbf{i}} + F_y \hat{\mathbf{j}} + F_z \hat{\mathbf{k}}) = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right)$$

or

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}, \quad F_z = -\frac{\partial V}{\partial z}$$

So we have to calculate the above partial derivatives of $V(x, y, z)$.

Now

$$F_x = -\frac{\partial V}{\partial x} = -k\frac{\partial}{\partial x}\left[\frac{1}{(x^2 + y^2 + z^2)^{1/2}}\right]$$

$$= -k\left[-\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right] \quad \text{(read the margin remark)}$$

$$= \frac{kx}{(x^2 + y^2 + z^2)^{3/2}}$$

Similarly,

$$F_y = -\frac{\partial V}{\partial y} = \frac{ky}{(x^2 + y^2 + z^2)^{3/2}}$$

and

$$F_z = -\frac{\partial V}{\partial z} = \frac{kz}{(x^2 + y^2 + z^2)^{3/2}}$$

Let

$$x^2 + y^2 + z^2 = t$$

then

$$\frac{\partial}{\partial x}\left[\frac{1}{(x^2 + y^2 + z^2)^{1/2}}\right] = \left[\frac{\partial}{\partial t}\left(\frac{1}{t^{1/2}}\right)\left(\frac{\partial t}{\partial x}\right)\right]$$

$$= -\frac{1}{2}\left(\frac{1}{t^{3/2}}\right) \times \frac{\partial}{\partial x}[x^2 + y^2 + z^2]$$

$$= -\frac{1}{2t^{3/2}} \cdot 2x$$

$$= -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\therefore \qquad \vec{\mathbf{F}} = -\vec{\nabla} V = \frac{k}{(x^2 + y^2 + z^2)^{3/2}} (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}) \qquad \text{(i)}$$

$$= \frac{k\,\vec{\mathbf{r}}}{r^3}$$

$$= \frac{k}{r^2}\hat{\mathbf{r}} \qquad \text{since } \vec{\mathbf{r}} = r\,\hat{\mathbf{r}}$$

where $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and $r = (x^2 + y^2 + z^2)^{1/2}$.

Do the steps of Example 1.1 yourself on a separate piece of paper.

Example 1.1 illustrates an important application of the gradient. In physics, we use scalar and vector field functions to represent various physical quantities. They are often related in this way: *a vector field is the scalar multiple of the gradient of some scalar field function*. This also suggests that we can construct a vector field from a scalar field by taking its gradient. Notice that in this section, we have used the term 'vector field'. You will learn more about it in Unit 2.

Let us take another example for calculating the gradient of a scalar field.

EXAMPLE 1.2: GRADIENT OF A SCALAR FIELD

Determine the gradient of an arbitrary function $f(r)$ where $r = (x^2 + y^2 + z^2)^{1/2}$. Use this to determine the gradient of the function $\phi(x, y, z) = \dfrac{k}{(x^2 + y^2 + z^2)}$.

SOLUTION ■ We use Eq. (1.11a) with $\phi = f(r)$:

$$\vec{\nabla}\phi = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \qquad \text{(i)}$$

Using the chain rule from calculus we can write

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x}$$

$$= \left(\frac{\partial f}{\partial r}\right)\frac{\partial}{\partial x}\left[(x^2 + y^2 + z^2)^{1/2}\right] \quad [\because\ r = (x^2 + y^2 + z^2)^{1/2}]$$

$$= \frac{\partial f}{\partial r}\cdot\left[\frac{1}{2}\cdot\frac{2x}{(x^2 + y^2 + z^2)^{1/2}}\right]$$

$$= \frac{x}{r}\frac{\partial f}{\partial r} \qquad (\because\ r = (x^2 + y^2 + z^2)^{1/2}) \qquad \text{(ii)}$$

$$\frac{\partial}{\partial x}[(x^2 + y^2 + z^2)^{1/2}]$$

$$= \frac{1}{2}\left[(x^2 + y^2 + z^2)^{\frac{1}{2}-1}\right] \times (2x)$$

Similarly,

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\cdot\frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{1/2}$$

$$= \frac{\partial f}{\partial r} \cdot \left[\frac{1}{2} \cdot \frac{2y}{(x^2 + y^2 + z^2)^{1/2}} \right] = \frac{y}{r} \frac{\partial f}{\partial r} \qquad \text{(iii)}$$

and $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial r} \cdot \frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{1/2}$

$$= \frac{\partial f}{\partial r} \cdot \left[\frac{1}{2} \cdot \frac{2z}{(x^2 + y^2 + z^2)^{1/2}} \right] = \frac{z}{r} \frac{\partial f}{\partial r} \qquad \text{(iv)}$$

Substituting (ii), (iii) and (iv) in (i) we get

$$\vec{\nabla} f(r) = \frac{x}{r} \frac{\partial f}{\partial r} \hat{\mathbf{i}} + \frac{y}{r} \frac{\partial f}{\partial r} \hat{\mathbf{j}} + \frac{z}{r} \frac{\partial f}{\partial r} \hat{\mathbf{k}} = \frac{1}{r} \frac{\partial f}{\partial r} [x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}]$$

or $\vec{\nabla} f(r) = \frac{\partial f}{\partial r} \left[\frac{\vec{\mathbf{r}}}{r} \right] = \frac{\partial f}{\partial r} \hat{\mathbf{r}}$ (v)

where $\hat{\mathbf{r}}$ is the unit vector along $\vec{\mathbf{r}}$.

In physics, we encounter many scalar functions of *r* and hence Eq. (v) is an important result.

To determine the gradient of the scalar field $\phi(x, y, z) = \frac{k}{(x^2 + y^2 + z^2)}$, we write it as $\phi = \frac{k}{r^2} = f(r)$. Using Eq. (v), we can write:

$$\vec{\nabla} \phi = \vec{\nabla} f(r) = \frac{\partial}{\partial r} \left(\frac{k}{r^2} \right) \hat{\mathbf{r}} = -\left(\frac{2k}{r^3} \right) \hat{\mathbf{r}} = -\frac{2k}{r^3} \hat{\mathbf{r}}$$

SAQ 1 - Gradient of a scalar field

a) Calculate the gradient of the following scalar fields:

(i) $\phi(x, y) = \ln(x^2 + y^2)$

(ii) $\phi(x, y, z) = xy + yz + zx$

b) The height of a hill is given by $z = 50 - x^2 y^2$. Calculate the maximum rate of change (also called the steepest ascent) in the height of the hill at the point (1, 2). What is its direction?

The gradient of a scalar field is important in physics, because we use it to express the relationship between a conservative force and a scalar potential. A conservative force field is related to a scalar potential *V* as

$$\vec{\mathbf{F}} = -\vec{\nabla} V \qquad (1.12)$$

You have studied about conservative forces in your first semester course BPHCT-131 entitled “Mechanics”. You know that forces like the electrostatic force, the gravitational force and the spring force are conservative forces. Each of these forces can be related to a corresponding potential function. The negative sign in this equation is important. **It says that the force is in the direction of the negative gradient of the potential**. This tells us, for example, that the gravitational force is directed from the point of higher gravitational potential to lower gravitation potential, like from the top of a building to the ground. In SAQ 2 you will solve problems on some applications of the gradient of a scalar field.

SAQ 2 – **Applications of the gradient**

a) Determine the unit vector normal to the curve $x^2 + 4y^2 = 1$ at the point (1, − 1).

b) Obtain a unit vector normal to the surface $x + 2y - z + 5 = 0$ at any point.

So far we have defined the gradient of a scalar field which tells us the direction of maximum space rate of change of the scalar field. We can determine the rate of change of the scalar field **in any direction** by defining the **directional derivative of the scalar field** *f* in terms of the gradient of the scalar field. This is what we do now.

1.3.2 The Directional Derivative of a Scalar Field

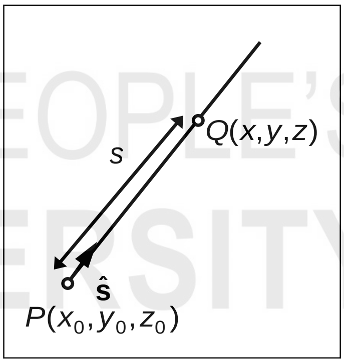


Fig. 1.6: **Directional derivative.**

Let us find the rate of change of *f* (*x*, *y*, *z*) with distance *s*, at a given point $P\ (x_0, y_0, z_0)$ in the field and in a given direction (Fig. 1.6). This is called the **directional derivative** (*df* / *ds*) of the function with distance *s*. Let $\hat{\mathbf{s}}$ be the unit vector in the direction in which we want to find the rate of change of the scalar field. Let the unit vector $\hat{\mathbf{s}}$ be defined by

$$\hat{\mathbf{s}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}} \tag{1.13a}$$

Let us start from the point *P* and go a distance $s\,(\geq 0)$ along the direction of the unit vector $\hat{\mathbf{s}}$ to reach the point *Q* (*x*, *y*, *z*). We can write the displacement vector $\overrightarrow{\mathbf{PQ}} = s\,\hat{\mathbf{s}}$ as,

$$\vec{\mathbf{s}} = (x - x_0)\,\hat{\mathbf{i}} + (y - y_0)\,\hat{\mathbf{j}} + (z - z_0)\,\hat{\mathbf{k}} \tag{1.13b}$$

We can also write $\vec{\mathbf{s}}$ in terms of the unit vector $\hat{\mathbf{s}}$ as

$$\vec{\mathbf{s}} = sa\,\hat{\mathbf{i}} + sb\,\hat{\mathbf{j}} + sc\,\hat{\mathbf{k}} \tag{1.13c}$$

Comparing Eqs. (1.13b and c), we get

$$x - x_0 = sa; \qquad y - y_0 = sb; \qquad z - z_0 = sc$$

or

$$x = x_0 + sa; \qquad y = y_0 + sb; \qquad z = z_0 + sc \tag{1.13d}$$

What do Eqs. (1.13d) tell us? You can see that the variables $x = x(s)$, $y = y(s)$ and $z = z(s)$ are all functions of a single variable *s*. Eqs. (1.13d) are called the **parametric equations** of the line *PQ* which passes through the points (x_0, y_0, z_0) and (x, y, z) with the **parameter** as *s*. Remember that here *s* is the distance between the points *P* and *Q* measured along the line *PQ*. So if we substitute *x*, *y* and z in terms of *s*, the function $f(x, y, z)$ becomes a function of a single variable, the **parameter** *s*.

In Eqs. (1.14a to c), note that since *f* is a function of *x*, *y* and *z*, the derivative of *f* with respect to *x* is a partial derivative. Since *x* is a function of only *s*, the derivative of *x* with respect to *s* is an ordinary derivative.

We can now write the directional derivative (df/ds) using the following chain rule from calculus for the derivative of a function $f(x, y, z)$ where *x*, *y* and *z* are functions of *s*:

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} \tag{1.14a}$$

From Eq. (1.13d), we have

$$\frac{dx}{ds} = a; \qquad \frac{dy}{ds} = b; \qquad \frac{dz}{ds} = c \tag{1.14b}$$

So Eq. (1.14a), which gives the expression for the directional derivative of *f*, now becomes:

$$\frac{df}{ds} = \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b + \frac{\partial f}{\partial z}c \tag{1.14c}$$

We now make use of the expression for the gradient of a scalar field given by Eq. (1.11a):

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

Then we take the scalar or dot product of the vectors $\vec{\nabla} f$ and $\hat{\mathbf{s}}$ and get:

$$\vec{\nabla} f \cdot \hat{\mathbf{s}} = \left(\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}\right) \cdot (a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}})$$

$$= \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b + \frac{\partial f}{\partial z}c \tag{1.14d}$$

which is the same as the right hand side of Eq. (1.14c). So we can write

$$\frac{df}{ds} = \vec{\nabla} f \cdot \hat{\mathbf{s}} \tag{1.15}$$

In other words, **the directional derivative of a scalar field at a point <u>in a given direction</u> is the scalar product of the gradient of the scalar field at that point and the unit vector along the given direction**.

To reconcile Eqs. (1.15) and (1.16), note that $\frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|}$ is the unit vector along $\vec{\mathbf{a}}$.

Note that if the direction is specified by an arbitrary vector $\vec{\mathbf{a}}$ which is not the unit vector in that direction, the expression for the directional derivative becomes

$$\frac{df}{ds} = \vec{\nabla} f \cdot \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} \tag{1.16}$$

Let us now work out an example on calculating the directional derivative of a scalar function using Eq. (1.15 or 1.16).

EXAMPLE 1.3: DIRECTIONAL DERIVATIVE

Determine the directional derivative of the scalar field $\phi = xy^2z^3$ in the direction $(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ at the point (3, 1, – 1).

SOLUTION ■ To determine the directional derivative we must use either Eq. (1.15) or Eq. (1.16). Note that the direction is not given in terms of a unit vector (check for yourself that the vector $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ is not a unit vector). Hence, we will use Eq. (1.16). In the first step we determine the gradient of ϕ.

We can find the $\vec{\nabla}\phi$ using Eq. (1.11a) as follows:

$$\vec{\nabla}\phi = \frac{\partial(xy^2z^3)}{\partial x}\hat{\mathbf{i}} + \frac{\partial(xy^2z^3)}{\partial y}\hat{\mathbf{j}} + \frac{\partial(xy^2z^3)}{\partial z}\hat{\mathbf{k}}$$

$$= y^2z^3\,\hat{\mathbf{i}} + 2xyz^3\,\hat{\mathbf{j}} + 3xy^2z^2\,\hat{\mathbf{k}}$$

Then from Eq. (1.16), the directional derivative in the given direction is

$$\frac{d\phi}{ds} = (y^2z^3\,\hat{\mathbf{i}} + 2xyz^3\,\hat{\mathbf{j}} + 3xy^2z^2\,\hat{\mathbf{k}}).\frac{(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{3}$$

$$= \frac{1}{3}y^2z^3 - \frac{4}{3}xyz^3 + 2xy^2z^2$$

At the point (3, 1, – 1), $x = 3$, $y = 1$, $z = -1$ and hence, $\frac{d\phi}{ds} = \frac{29}{3}$ (read the margin remark).

Substituting $x = 3$, $y = 1$ and $z = -1$ in Eq. (i) we can write,

$$\frac{d\phi}{ds} = \frac{1}{3}.1^2.(-1)^3 - \frac{4}{3}(3).(1).(-1)^3 + 2.(3).(1)^2.(-1)^2$$

$$= -\frac{1}{3} + 4 + 6 = \frac{29}{3}$$

You may now like to work out the following SAQ.

SAQ 3 - Directional derivative

Obtain the directional derivative of the scalar field $V = x^2 + \cos y - xz$ at the point $(2, \pi/6, -1)$ in the direction $\frac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$.

Let us now summarise what we have learnt in this unit.

1.4 SUMMARY

Concept	Description
Scalar field	■ A function f which associates a unique scalar with each point in a given region is called a **scalar field function** or a **scalar field**. The curves $f(x, y)$ = constant or the surfaces $f(x, y, z)$ = constant are called the **contour curves** and **contour surfaces**, respectively.

Gradient of a scalar field

- The **gradient** of a scalar field is defined as

$$\vec{\nabla} f = \hat{\mathbf{i}}\frac{\partial f}{\partial x} + \hat{\mathbf{j}}\frac{\partial f}{\partial y} + \hat{\mathbf{k}}\frac{\partial f}{\partial z}$$

It is a vector such that the change df in f for an arbitrarily small change $d\vec{\mathbf{r}}$ is given by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \left(\vec{\nabla} f\right).d\vec{\mathbf{r}}$$

Properties of the gradient

- The gradient of a scalar field is **normal** to the curves of constant value or surfaces of constant value of the scalar field.

The gradient of a scalar field at a point gives the direction of **maximum rate of change of the scalar field**.

Directional derivative of a scalar field.

- The **directional derivative** of a scalar field f in the **direction** specified by the unit vector $\hat{\mathbf{s}}\left(\equiv \frac{\vec{\mathbf{s}}}{s}\right)$ is

$$\frac{\partial f}{\partial s} = \hat{\mathbf{s}}.\vec{\nabla} f = \frac{\vec{\mathbf{s}}}{s}\,.\,\vec{\nabla} f$$

It is the projection of $\vec{\nabla} f$ on $\hat{\mathbf{s}}$.

1.5 TERMINAL QUESTIONS

1. Determine the gradient of the following scalar fields:

   a) $\phi(x, y, z) = e^{xyz}$.

   b) $f = y \sin z - xy$ at the point $(1, 2, \pi/6)$

   In which direction is the scalar field of part (b) decreasing most rapidly?

2. Determine the gradient for the temperature field given by $T(x, y, z) = 2x^2 + xyz + y^2 + 273$ at the point (– 1, 2, 1). What is the direction of heat flow?

3. Determine a unit vector normal to the scalar field $F = e^x \cos y$.

4. If $\vec{\mathbf{A}} = 2yz\hat{\mathbf{i}} - x^2y\hat{\mathbf{j}} + xz^2\hat{\mathbf{k}}$ and $\phi = 2x^2yz^3$, determine $(\vec{\mathbf{A}}\,.\,\vec{\nabla})\phi$.

5. Determine the unit vector normal to the surface $(x-2)^2 + (y+1)^2 + z^2 = 9$ at the point (2, 1, 4).

6. Calculate the gradient of the scalar field $r = (x^2 + y^2 + z^2)^{1/2}$.

7. Given a potential energy function $V(r) = \alpha r^2$, where $r = (x^2 + y^2 + z^2)^{1/2}$ and α a constant, calculate the force field.

8. Determine the force field in three dimensions for a potential energy function given by:

$$V = \frac{e^{-\alpha r}}{r}$$

9. Determine the directional derivative of the scalar field $\phi = x^2 y + xz$ at the point (1, 2, 1) in the direction of the vector $\vec{\mathbf{C}} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$.

10. Obtain the directional derivative of the scalar field $\phi = x^2 + y^2 + z^2$ at the point (3, 0, 1) in the direction $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

1.6 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. a) i) Since $\phi(x, y)$ is two-dimensional scalar field, we use Eq. (1.11b). For $\phi = \ln(x^2 + y^2)$, we have

$$\vec{\nabla}\phi = \frac{\partial}{\partial x}[\ln(x^2 + y^2)]\,\hat{\mathbf{i}} + \frac{\partial}{\partial y}[\ln(x^2 + y^2)]\,\hat{\mathbf{j}}$$

$$= \frac{2x}{x^2 + y^2}\,\hat{\mathbf{i}} + \frac{2y}{x^2 + y^2}\,\hat{\mathbf{j}}$$

ii) Using Eq. (1.11a) with $\phi = xy + yz + zx$, we get,

$$\phi = \frac{\partial}{\partial x}(xy + yz + zx)\,\hat{\mathbf{i}} + \frac{\partial}{\partial y}(xy + yz + zx)\,\hat{\mathbf{j}} + \frac{\partial}{\partial z}(xy + yz + zx)\,\hat{\mathbf{k}}$$

$$= (y + z)\,\hat{\mathbf{i}} + (x + z)\,\hat{\mathbf{j}} + (x + y)\,\hat{\mathbf{k}}$$

b) The magnitude of the maximum rate of change is the magnitude of the gradient, as given by Eq. (1.10). Using Eq. (1.11b),

$$\vec{\nabla}z = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y}\right)(50 - x^2 y^2) = -2xy^2\,\hat{\mathbf{i}} - 2yx^2\,\hat{\mathbf{j}}$$

At the point (1, 2), it is

$$\vec{\nabla}z\Big|_{z=(1,\,2)} = -8\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}$$

$$\therefore \quad |\vec{\nabla}z| = |-8\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}| = 4\sqrt{5}$$

The direction of maximum rate of change of the height, which is the steepest ascent is along the gradient of z at (1, 2) and is given by

$$\vec{\nabla}z\Big|_{z=(1,\,2)} = -8\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}$$

2. a) The gradient of a scalar field $f(x,y)$ is the normal to the curve $f(x, y) = c$, where c is a constant. So we can define the field $f(x, y)$ as

$$f(x, y) = x^2 + 4y^2 - 1$$

We determine $\vec{\nabla}f(x, y)$ which will be normal to the curve $f(x, y) = c$.

Using Eq. (1.11b), we get

$$\vec{\nabla}f = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y}\right)(x^2 + 4y^2 - 1) = (2x\,\hat{\mathbf{i}} + 8y\,\hat{\mathbf{j}})$$

At the point (1, – 1), the value of the gradient is

$$\vec{\nabla}f(1, -1) = 2\,\hat{\mathbf{i}} - 8\,\hat{\mathbf{j}}$$

This vector is normal to the curve $x^2 + 4y^2 = 1$ at (1, – 1). The unit vector normal to the curve at the point (1, –1) is

$$\hat{\mathbf{n}} = \frac{\vec{\nabla} f(1,-1)}{|\vec{\nabla} f(1,-1)|} = \frac{2\,\hat{\mathbf{i}} - 8\,\hat{\mathbf{j}}}{\sqrt{2^2 + 8^2}} = \frac{\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}}{\sqrt{17}}$$

b) For this question, as in SAQ 2(a) we first determine the gradient to the surface $f(x, y, z) = x + 2y - z + 5$ using Eq. (1.11a):

$$\vec{\nabla} f = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(x + 2y - z + 5) = (\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - \hat{\mathbf{k}}) \qquad \text{(i)}$$

Note that $\vec{\nabla} f$ which is normal to the surface $f(x, y, z)$, given by Eq. (i) is a constant vector. The unit vector normal at any point on this surface is then

$$\hat{\mathbf{n}} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{(\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - \hat{\mathbf{k}})}{\sqrt{1^2 + 2^2 + (-1)^2}} = \frac{1}{\sqrt{6}}(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$$

3. We use Eq. (1.15) with the unit vector $\hat{\mathbf{s}} = \frac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$ and

$$f = V(x, y, z) = x^2 + \cos y - xz.$$

We first determine the gradient $\vec{\nabla} V$:

$$\vec{\nabla} V = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(x^2 + \cos y - xz)$$

$$= (2x - z)\,\hat{\mathbf{i}} - \sin y\,\hat{\mathbf{j}} - x\,\hat{\mathbf{k}}$$

The gradient at the point $(2, \pi/6, -1)$ is,

$$\vec{\nabla} V(2, \pi/6, -1) = 5\,\hat{\mathbf{i}} - \sin\pi/6\,\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

$$= 5\,\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

From Eq. (1.15), the directional derivative at that point, along $\frac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$ is:

$$\frac{dV}{ds} = \left(5\,\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}} - 2\hat{\mathbf{k}}\right).\left(\frac{1}{\sqrt{3}}\left(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}\right)\right)$$

$$= \frac{1}{\sqrt{3}}\left(5 - \frac{1}{2} + 2\right) = \frac{13}{2\sqrt{3}}$$

Terminal Questions

1. a) We use Eq. (1.11a) with $\phi = e^{xyz}$ and get

$$\vec{\nabla}\phi = \frac{\partial}{\partial x}e^{xyz}\,\hat{\mathbf{i}} + \frac{\partial}{\partial y}e^{xyz}\,\hat{\mathbf{j}} + \frac{\partial}{\partial z}e^{xyz}\hat{\mathbf{k}}$$

$$= yz\,e^{xyz}\,\hat{\mathbf{i}} + xz\,e^{xyz}\,\hat{\mathbf{j}} + xy\,e^{xyz}\hat{\mathbf{k}}$$

b) We first find the gradient at the point $(1, 2, \pi/6)$:

$$\vec{\nabla} f = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(y\sin z - xy)$$

$$= -y\,\hat{\mathbf{i}} + (\sin z - x)\,\hat{\mathbf{j}} + y\,\cos z\,\hat{\mathbf{k}}$$

$$\vec{\nabla} f\left(1, 2, \frac{\pi}{6}\right) = -2\,\hat{\mathbf{i}} + \left(\sin\frac{\pi}{6} - 1\right)\hat{\mathbf{j}} + 2\cos\frac{\pi}{6}\,\hat{\mathbf{k}}$$

$$= -2\,\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}} + \sqrt{3}\,\hat{\mathbf{k}}$$

The direction in which f decreases most rapidly is **opposite** to the gradient vector at (1, 2, π / 6).

This direction is then along the vector $\vec{\mathbf{a}}$

$$\vec{\mathbf{a}} = -\,\vec{\nabla} f\left(1, 2, \frac{\pi}{6}\right) = 2\,\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}} - \sqrt{3}\,\hat{\mathbf{k}}$$

2. Using Eq. (1.11a) with $f = T$, we get

$$\vec{\nabla} T = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(2x^2 + xyz + y^2 + 273)$$

$$= (4x + yz)\,\hat{\mathbf{i}} + (xz + 2y)\,\hat{\mathbf{j}} + xy\,\hat{\mathbf{k}}$$

$$\vec{\nabla} T\,(-1, 2, 1) = (-4 + 2)\,\hat{\mathbf{i}} + (-1 + 4)\,\hat{\mathbf{j}} - 2\,\hat{\mathbf{k}}$$

$$= -2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} - 2\,\hat{\mathbf{k}}$$

Heat would flow from higher to lower temperature regions. Hence, it would flow along the direction in which the temperature decreases most rapidly at $(-1, 2, 1)$. Therefore, the direction of heat flow is along $-\vec{\nabla} T$. Therefore, it is along $(2\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}})$.

3. The vector normal to the scalar field F is $\vec{\nabla} F$. Using Eq. (1.11b) with $f = F$, we get

$$\vec{\nabla} F = \hat{\mathbf{i}}\,e^x \cos y - \hat{\mathbf{j}}\,e^x \sin y$$

The unit normal vector is then:

$$\hat{\mathbf{n}} = \frac{\vec{\nabla} F}{|\vec{\nabla} F|} = \frac{\hat{\mathbf{i}}\,e^x \cos y - \hat{\mathbf{j}}\,e^x \sin y}{\sqrt{e^{2x}\cos^2 y + e^{2x}\sin^2 y}}$$

$$= \frac{\hat{\mathbf{i}}\,e^x \cos y - \hat{\mathbf{j}}\,e^x \sin y}{e^x} = \hat{\mathbf{i}}\cos y - \hat{\mathbf{j}}\sin y$$

4. We first determine the scalar product of $\vec{\mathbf{A}}$ and the del operator $\vec{\nabla}$:

$$(\vec{\mathbf{A}}.\vec{\nabla}) = (2yz\,\hat{\mathbf{i}} - x^2 y\,\hat{\mathbf{j}} + xz^2\,\hat{\mathbf{k}}) \cdot \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)$$

$$= \left(2yz\frac{\partial}{\partial x} - x^2 y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}\right)$$

Note that this is now a differential operator which can act on the field ϕ:

$$(\vec{\mathbf{A}}\,.\,\vec{\nabla})\phi = \left(2yz\frac{\partial}{\partial x} - x^2 y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}\right)(2x^2 y z^3)$$

$$= 2yz\frac{\partial}{\partial x}(2x^2 y z^3) - x^2 y\frac{\partial}{\partial y}(2x^2 y z^3) + xz^2\frac{\partial}{\partial z}(2x^2 y z^3)$$

$$= 8xy^2 z^4 - 2x^4 y z^3 + 6x^3 y z^4$$

5. Using Eq. (1.11a), we first find the gradient of the scalar function $f(x, y, z) = (x-2)^2 + (y+1)^2 + z^2 - 9$:

$$\vec{\nabla} f = \left[\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right] \left[(x-2)^2 + (y+1)^2 + z^2 - 9 \right]$$

$$= 2(x-2)\,\hat{\mathbf{i}} + 2(y+1)\,\hat{\mathbf{j}} + 2z\,\hat{\mathbf{k}}$$

At the point (2, 1, 4), the gradient vector is $\vec{\nabla} f\,(2, 1, 4) = 4\,\hat{\mathbf{j}} + 8\,\hat{\mathbf{k}}$

The unit vector normal to the surface is,

$$\hat{\mathbf{n}} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{4\,\hat{\mathbf{j}} + 8\,\hat{\mathbf{k}}}{\sqrt{80}} = \frac{1}{\sqrt{5}} (\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

6. Using Eq. (1.11a), we can write

$$\vec{\nabla} r = \frac{\partial r}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial r}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial r}{\partial z}\,\hat{\mathbf{k}} \qquad \text{(i)}$$

Let us determine each component of Eq. (i):

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{1/2} = \frac{x}{(x^2 + y^2 + z^2)^{1/2}} = \frac{x}{r} \qquad \text{(ii)}$$

Similarly, the other two partial derivatives $\frac{\partial r}{\partial y}$ and $\frac{\partial r}{\partial z}$ are:

$$\frac{\partial r}{\partial y} = \frac{y}{(x^2 + y^2 + z^2)^{1/2}} = \frac{y}{r} \qquad \text{(iii)}$$

and $$\frac{\partial r}{\partial z} = \frac{z}{(x^2 + y^2 + z^2)^{1/2}} = \frac{z}{r} \qquad \text{(iv)}$$

Therefore, $$\vec{\nabla} r = \frac{x}{r}\,\hat{\mathbf{i}} + \frac{y}{r}\,\hat{\mathbf{j}} + \frac{z}{r}\,\hat{\mathbf{k}} = \frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{r} = \frac{1}{r}\,\vec{\mathbf{r}} \qquad \text{(v)}$$

where $\vec{\mathbf{r}}$ is the position vector for the point (x, y, z). Can you identify what the right hand side of Eq. (v) represents? You can see that $\vec{\mathbf{r}}/r$ is just the unit vector $\hat{\mathbf{r}}$ in the direction of $\vec{\mathbf{r}}$. This brings us to a result that you will use often in physics:

$$\vec{\nabla} r = \hat{\mathbf{r}} \qquad \text{(vi)}$$

7. We use Eq. (1.12) with $V = \alpha r^2$ to determine the force field $\vec{\mathbf{F}}$:

$$\vec{\mathbf{F}} = -\vec{\nabla}(\alpha r^2)$$

To evaluate $\vec{\nabla} V$, we use the result derived for the gradient of an arbitrary scalar function $f(r)$ in Example 1.2 which is

$$\vec{\nabla} f(r) = \frac{\partial f}{\partial r}\,\hat{\mathbf{r}} \qquad \text{(i)}$$

With $f(r) = \alpha r^2$ we get

$$\vec{\mathbf{F}} = -\frac{1}{r}\,\frac{d}{dr}(\alpha r^2)\,\hat{\mathbf{r}} = -2\alpha r\,\hat{\mathbf{r}}$$

8. The force field $\vec{\mathbf{F}}$ is given by Eq. (1.12).

   Using Eq. (v) of Example 1.2 with $f(r) = V = \dfrac{e^{-\alpha r}}{r}$ we get,

$$\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{\mathbf{r}}$$

   Since $\vec{\mathbf{F}} = -\vec{\nabla} V,$ we can write

$$\vec{\mathbf{F}} = -\left(\frac{\partial V}{\partial r}\right)\hat{\mathbf{r}} = -\frac{\partial}{\partial r}\left(\frac{e^{-\alpha r}}{r}\right)\hat{\mathbf{r}}$$

$$= -\left[-\frac{\alpha e^{-\alpha r}}{r} - \frac{e^{-\alpha r}}{r^2}\right]\hat{\mathbf{r}} = \frac{e^{-\alpha r}}{r}\left[\alpha + \frac{1}{r}\right]\hat{\mathbf{r}}$$

9. We use Eq. (1.16) with $\vec{\mathbf{a}} = \vec{\mathbf{C}}$. The unit vector along $\vec{\mathbf{C}}$ is

$$\hat{\mathbf{C}} = \frac{\vec{\mathbf{C}}}{|\vec{\mathbf{C}}|} = \frac{2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{\sqrt{4+9+16}} = \frac{2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{\sqrt{29}}$$

   Using Eq. (1.11a) with $f = \phi$, we have

$$\vec{\nabla}\phi = \hat{\mathbf{i}}\,(2xy + z) + \hat{\mathbf{j}}\,(x^2) + \hat{\mathbf{k}}\,(x)$$

   At the point (1, 2, 1), the value of the gradient is

$$\vec{\nabla}\phi(1, 2, 1) = 5\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

   Then the directional derivative is

$$\frac{d\phi}{ds} = \hat{\mathbf{C}}.\,(\vec{\nabla}\phi) = \left(\frac{2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{\sqrt{29}}\right).(5\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) = \frac{10}{\sqrt{29}} - \frac{3}{\sqrt{29}} + \frac{4}{\sqrt{29}} = \frac{11}{\sqrt{29}}$$

10. We first determine the gradient of $\phi(x, y, z)$ at the point (3,0,1). Using Eq. (1.11a) with $f = \phi = x^2 + y^2 + z^2,$ we get

$$\vec{\nabla}\phi = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)$$

$$= 2x\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}} + 2z\,\hat{\mathbf{k}}$$

   At (3, 0, 1), the gradient of $\phi(x, y, z)$ is

$$\vec{\nabla}\phi(3, 0, 1) = 6\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{k}}$$

   To find the directional derivative at the point (3, 0, 1) in the direction of $\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}},$ we use Eq. (1.16) with $\vec{\mathbf{a}} = \hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}.$ Thus

$$\frac{d\phi}{ds} = (6\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{k}})\,.\,\frac{(\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}})}{\sqrt{14}} = \frac{10}{\sqrt{14}}$$

APPENDIX PARTIAL DERIVATIVES

By definition, the partial derivative of a function $f(x, y, z)$ with respect to x is

$$\frac{\partial f(x, y, z)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x} \qquad (1)$$

The function $\partial f / \partial x$ is obtained by differentiating the function $f(x, y, z)$ with respect to x as in ordinary calculus, treating ***other variables*** y, z **as constants**. You can similarly determine $\partial f / \partial y$ and $\partial f / \partial z$. The partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ of a function $f(x, y, z)$ give us, respectively, the rate of change of f in the directions of x, y or z-axes. Thus, $\frac{\partial f}{\partial x}$ gives the rate of change of f with respect to x at a given point in space.

Let us explain how to calculate the partial derivatives of a function $f(x, y, z)$ with respect to x, y and z ***holding other variables to be constant***.

For example, let $f(x, y, z) = 2x^2 y z^3$. Then

$$\frac{\partial f}{\partial x} = \left[\frac{\partial}{\partial x}(x^2)\right](2yz^3) = 4xyz^3$$ since y and z are treated as constants.

Similarly, for the partial derivative with respect to any other variable, we keep the remaining variables as constant. Thus,

$$\frac{\partial f}{\partial y} = \left[\frac{\partial}{\partial y}(y)\right](2x^2 z^3) = 2x^2 z^3 \quad \text{and} \quad \frac{\partial f}{\partial z} = \left[\frac{\partial}{\partial z}(z^3)\right](2x^2 y) = 6x^2 yz^2$$

You may quickly work out a couple of exercises to learn how to calculate partial derivatives of a function.

---

a) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = x^2 y^3 + \exp(x^2 y)$.

b) For the function $u(x, y, z) = 2x + yz - xy$, evaluate $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$.

The solutions are as follows:

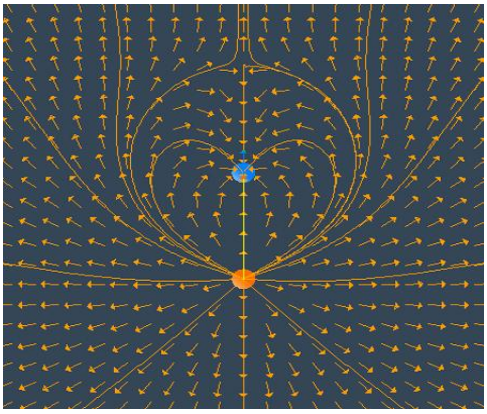
a) $$\frac{\partial f}{\partial x} = \left[\frac{\partial}{\partial x}(x^2 y^3) + \frac{\partial}{\partial x}\exp(x^2 y)\right] = \left[\frac{\partial}{\partial x}(x^2)\right]y^3 + 2x\exp(x^2 y)$$

$$= 2xy^3 + 2x\exp(x^2 y)$$

$$\frac{\partial f}{\partial x} = \left[\frac{\partial}{\partial y}(x^2 y^3) + \frac{\partial}{\partial x}\exp(x^2 y)\right] = 3y^2 x^2 + \exp(x^2 y)$$

b) $$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(2x + yz - xy) = 2 - y$$ since y and z are treated as constants.

Similarly, $$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(2x + yz - xy) = z - x \quad \text{and} \quad \frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(2x + yz - xy) = y$$



Field lines around an electric dipole. How will you find out if the field is spreading out or not?

UNIT 2

VECTOR FIELDS, DIVERGENCE AND CURL

Structure

STUDY GUIDE

In this unit, you will study two new concepts of vector differential calculus, namely, **divergence** and **curl of vector fields**. The concept of vector field may be new for you. So study it carefully. For calculating the divergence and curl of vector fields, you will need to use partial derivatives. These are discussed in the Appendix of Unit 1. You have to be well versed with these. You will also be using scalar and vector products. Therefore, you should revise scalar and vector products in the algebraic notation from Unit 2 of your Physics Elective BPHCT-131 as also in the Appendix A1 of this Block. You will learn how to apply the del operator more than once. So revise Sec. 1.3 of Unit 1. Practice will make you learn the concepts of this unit better. So you must work through all the examples, SAQs and Terminal Questions.

> ***"Mathematics is the tool especially suited for dealing with abstract concepts of any kind and there is no limit to its power in this field."***
>
> ***P.A.M. Dirac***

2.1 INTRODUCTION

In Unit 1, you have studied the concept of a scalar field and learnt how to calculate the **gradient** of a scalar field, which is related to its **directional derivative**. You now know that the directional derivative of a scalar field tells us how the scalar field changes in a particular direction. In this unit, you will learn the concept of a **vector field and operations on vector fields**. Vector fields are quite common in physics. One of the most common examples of a vector field is the velocity field. The gravitational force and the electrostatic force are also familiar examples of vector fields. In the previous unit you have also seen that the gradient of a scalar field is a vector function of position, and is, therefore, a **vector field**.

We begin this unit by giving a formal definition of a **vector field** in Sec. 2.2. In Sec. 1.3, you have learnt about the **del** operator and its operation on a scalar field. In Secs. 2.3 and 2.4, you will learn two ways in which the del operator can operate on a vector field. These give us the **divergence** and **curl** of a vector field. The divergence and curl of vector fields are used extensively in physics. In this course you will learn that Maxwell's equations in electromagnetic theory can be written in a compact form using the del operator. In Sec. 2.5, you will learn about successive applications of the del operator and product rules involving the del operator.

Expected Learning Outcomes

After studying this unit, you should be able to:

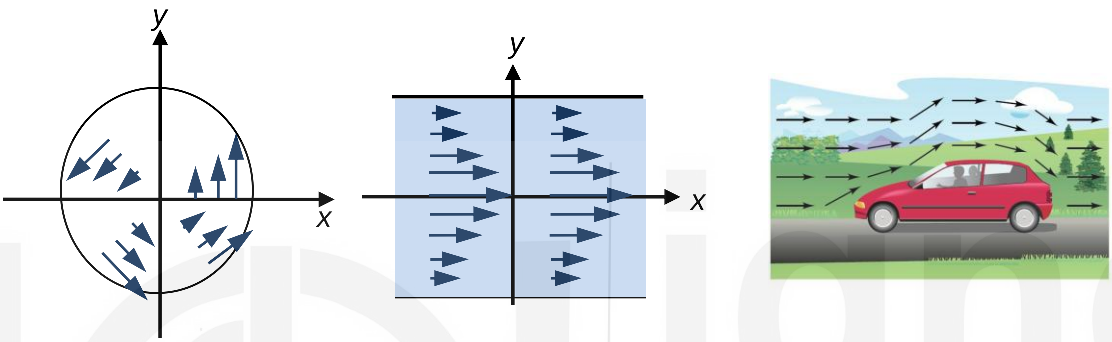
- ❖ explain the concept of a vector field and identify vector fields;
- ❖ calculate the divergence and curl of vector fields; and
- ❖ solve problems based on successive applications of the del operator and product rules involving the del operator.

2.2 VECTOR FIELD

Have you been to a riverside on a calm sunny day and observed the flow of water? Did you observe a leaf floating near the river bank? You may have noticed that it moves very slowly since the water is almost at rest there. Suppose the leaf were dropped in the middle of the river. It would flow faster. Now suppose you want to describe the flow of water in the river. In principle, you could describe the motion of each water particle using Newton's laws. But it would be a cumbersome task since the number of water particles in the river is very large.

Another way to study the flow of water is to specify the velocity at each point in the river. That is, we describe the velocity that a small floating object (e.g., a leaf) would have at each point. Water particles at different points in a flowing river have different velocities. Note that the velocities could change with time. This is an example of a **velocity field**. Since velocity is a vector, the velocity field is a vector field. Velocity fields are used to describe the motion of a system of particles in space.

Refer to Fig. 2.1, where we have shown some more examples of velocity fields. From Fig. 2.1a, you can see the velocity field for a wheel rotating on its axle. The vectors represent the velocity at different points on the wheel and the length of the vector at each point represents its magnitude at that point. As you may recall from your school physics courses or from the course BPHCT-131 entitled Mechanics, the farther away we move from the centre of the wheel, the greater is the velocity. In Fig. 2.1b, we have shown the velocity field for water flowing through a pipe. You may note that the velocity of water is maximum at the centre of the pipe and minimum at its sides. In Fig. 2.1c, you can see the velocity field for air around a moving car. (Car manufacturing companies constantly strive to improve the design of their vehicle so as to increase efficiency through improved aerodynamics.)



(a) (b) (c)

Fig. 2.1: **Velocity field for a) a wheel rotating on its axle; b) water flowing through a pipe; (c) air around a moving car**.

You may recall from Sec. 1.2 of Unit 1 that a scalar field is a scalar function, which associates a scalar to every point in a specified region of space. Similarly, a vector field associates a vector to every point in a specified region, as you have seen in the examples of velocity fields. Let us now define a vector field.

2.2.1 Definition of a Vector Field

We can define the vector field as follows: A **vector field** $\vec{\mathbf{F}}$ over a region **R** in space is a function which assigns a unique vector $\vec{\mathbf{F}}(x, y, z)$ to every point in **R**. Sometimes we refer to vector fields as **vector field functions**.

In a Cartesian coordinate system, we write a vector field in terms of the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ as

$$\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{i}} + F_2(x, y, z)\,\hat{\mathbf{j}} + F_3(x, y, z)\,\hat{\mathbf{k}} \qquad (2.1)$$

where $F_1(x, y, z)$, $F_2(x, y, z)$ and $F_3(x, y, z)$ are the component functions of $\vec{\mathbf{F}}$. Note that **each component function is actually a scalar field** defined over the same region of space as the vector field $\vec{\mathbf{F}}$.

In Fig. 2.1, we have given some examples of a velocity field. Other vector fields that we come across in physics are force fields, and the electric and magnetic fields. Let us consider a few examples.

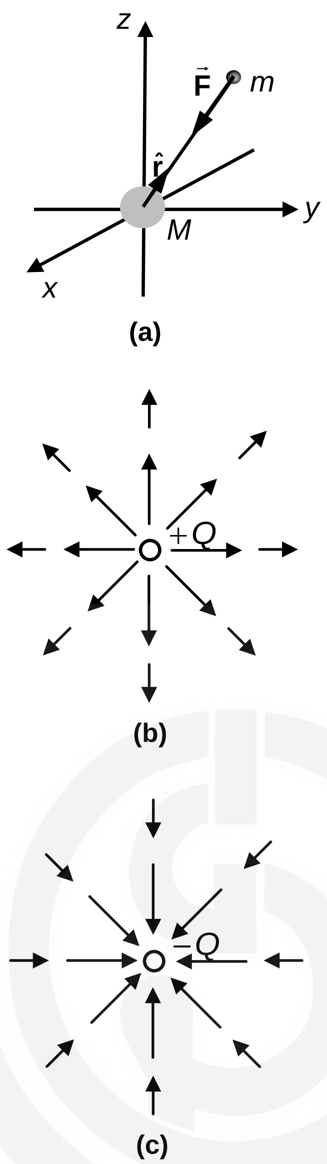


Fig. 2.2: **a) The gravitational force field due a mass *M* located at the origin of the Cartesian coordinate system on a mass *m* located at a distance *r* from it; b) The electric field due to a positive charge located at the origin. The magnitude of the field is greater at a point nearer the charge; c) The electric field due to a negative charge located at the origin.**

EXAMPLE 2.1 : EXAMPLES OF VECTOR FIELDS

a) **Gravitational Force Field**

Consider a particle of mass M located at the origin of the 3D Cartesian coordinate system (Fig. 2.2a). The gravitational force $\vec{\mathbf{F}}$ due to this particle on a particle of mass m located at the point (x, y, z) is directed along the line joining the point (x, y, z) to the origin and is given by

$$\vec{\mathbf{F}}(x, y, z) = -\frac{GMm}{(x^2 + y^2 + z^2)}\,\hat{\mathbf{r}} \qquad \text{(i)}$$

where $\hat{\mathbf{r}}$ is the unit vector along the line joining the origin to the point (x, y, z) pointing away from the origin. Note that $\vec{\mathbf{F}}(x,y,z)$ is a vector field. The negative sign in Eq. (i) means that $\vec{\mathbf{F}}(x,y,z)$ is an attractive force field. Since

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2 + y^2 + z^2)^{1/2}} \qquad \text{(ii)}$$

We can rewrite Eq. (i) as

$$\vec{\mathbf{F}}(x, y, z) = -\frac{GMm}{(x^2 + y^2 + z^2)^{3/2}}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \qquad \text{(iii)}$$

The component functions for the gravitational force field of Eq. (iii) are

$$F_1(x, y, z) = -\frac{GMmx}{(x^2 + y^2 + z^2)^{3/2}} \qquad \text{(iv)}$$

$$F_2\,(x, y, z) = -\frac{GMmy}{(x^2 + y^2 + z^2)^{3/2}} \qquad \text{(v)}$$

$$F_3\,(x, y, z) = -\frac{GMmz}{(x^2 + y^2 + z^2)^{3/2}} \qquad \text{(vi)}$$

b) **Electric Fields**

In your school physics, you have learnt about the **electrostatic force** (also called the Coulomb force) between charged particles **at rest**. Consider a charge Q located at the origin of the Cartesian coordinate system (Fig. 2.2b). The electrostatic force on a charge q located at a point (x, y, z) at a distance $\vec{\mathbf{r}}$ from Q is

$$\vec{\mathbf{F}}(x, y, z) = k\frac{Qq}{r^2}\,\hat{\mathbf{r}} = \frac{kqQ}{(x^2 + y^2 + z^2)^{3/2}}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \qquad \text{(vii)}$$

where $\hat{\mathbf{r}}$ is the unit vector given by Eq. (ii). It points from the origin towards (x, y, z). The electrostatic force on charge q is directed towards the origin if Q and q are unlike charges (since in this case the force is attractive). It points away from the origin if Q and q are like charges (because in this case the force is repulsive). $\vec{\mathbf{F}}$ is a force field. The **electric field** due to the charge Q at the point (x, y, z) is defined as

$$\vec{\mathbf{E}}(x, y, z) = k\frac{Q}{r^2}\,\hat{\mathbf{r}} = \frac{kQ}{(x^2+y^2+z^2)^{3/2}}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \qquad \text{(viii)}$$

The electric field due to Q is also a vector field, which points away from Q if Q is a positive charge and towards Q if Q is a negative charge (Figs. 2.2b and c, respectively).

c) Magnetic Fields

A magnetic field, such as the magnetic field due to a bar magnet or a current carrying wire, is another example of a vector field. You may have traced the lines of force for a bar magnet in your school physics laboratory using a compass needle. The alignment of the compass needle defines the direction of the magnetic field, as shown in Fig. 2.3.

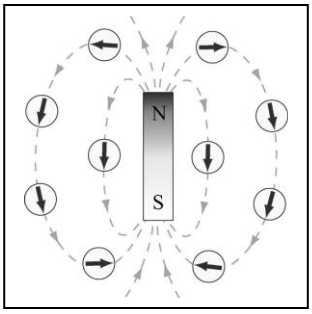


Fig. 2.3: **The magnetic field of a bar magnet as traced by a compass needle.**

Let us now summarize the concept of a vector field.

VECTOR FIELD

Recap

- A vector field is a function that assigns a unique vector to every point of a given region in space.
- A three-dimensional vector field $\vec{\mathbf{F}}$ can be written as

$$\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{i}} + F_2(x, y, z)\,\hat{\mathbf{j}} + F_3(x, y, z)\,\hat{\mathbf{k}} \qquad (2.1)$$

- The components of the vector field $\vec{\mathbf{F}}(x, y, z)$, viz., $F_1(x, y, z)$, $F_2(x, y, z)$ and $F_3(x, y, z)$ are **scalar fields**. These are defined over the same region as the vector field.
- A vector field $\vec{\mathbf{F}}$ in two-dimensions can be written as

$$\vec{\mathbf{F}}(x, y) = F_1(x, y)\,\hat{\mathbf{i}} + F_2(x, y)\,\hat{\mathbf{j}} \qquad (2.2)$$

In the previous unit you have learnt how scalar fields are represented by contour lines and contour maps. We now discuss how to represent a vector field.

2.2.2 Representation of a Vector Field

We know that a scalar field gives us the magnitude of a scalar function at every point in a region of space. However, a vector field gives both the magnitude and the direction of a vector function at every point in a specified region of space. So when you represent the vector field in a diagram, you must show both the magnitude and direction of the vector field at every point in the region. This can be done in two different ways: we can either use the vector field representation or the field lines representation. You will learn it now.

In some textbooks you may observe that the vectors are all of the same length but are colour coded to show relative magnitudes.

In the vector field representation, we draw arrows to represent the vector at each point in the region in which the vector field is defined. We have shown this representation for the velocity fields in Fig. 2.1. The length of the arrow represents the magnitude of the field at a point and the sense of the arrow gives the direction of the field at that point. If we draw the vectors at a sufficient number of points in the region, we can visualize the vector field better.

Now go back to Fig. 2.2a. The gravitational force field is represented by the vectors pointing toward the origin. Note that the force becomes weaker as we move away from the origin. That is why the arrows in the figure are longer at points closer to the origin and become shorter as we move away from it.

Figs. 2.2b and c show the vector field representation for the electric field due to a charge Q located at the origin.

In the example given below, we draw the vector field representation of some simple vector fields.

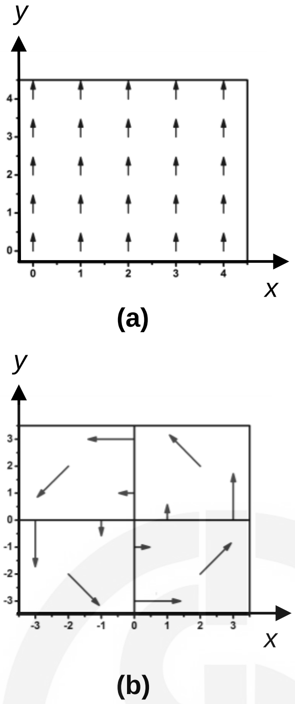


Fig. 2.4: **The vector fields**
a) $\vec{\mathbf{F}} = 2\hat{\mathbf{j}}$;
b) $\vec{\mathbf{F}}(x, y) = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$

EXAMPLE 2.2 : REPRESENTING A VECTOR FIELD

Represent the following vector fields diagramatically:

a) $\vec{\mathbf{F}} = 2\hat{\mathbf{j}}$ b) $\vec{\mathbf{F}} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$

SOLUTION ■ We draw the vectors at different points in space for both fields.

a) Note that $\vec{\mathbf{F}} = 2\hat{\mathbf{j}}$ is a constant vector field. At each point of the vector field, we just have to draw the vector $2\hat{\mathbf{j}}$. This vector field is shown in Fig. 2.4a for the first quadrant of the *xy* plane. You can see that all vectors are of the same length.

b) First let us write down the vectors at some representative points in the *xy* plane corresponding to the vector field $\vec{\mathbf{F}}(x, y) = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$:

x	y	$\vec{\mathbf{F}}(x, y)$	x	y	$\vec{\mathbf{F}}(x, y)$
0	1	$-\hat{\mathbf{i}}$	0	2	$-2\hat{\mathbf{i}}$
1	0	$\hat{\mathbf{j}}$	2	0	$2\hat{\mathbf{j}}$
1	1	$-\hat{\mathbf{i}} + \hat{\mathbf{j}}$	0	–2	$2\hat{\mathbf{i}}$
0	–1	$\hat{\mathbf{i}}$	–2	0	$-2\hat{\mathbf{j}}$
–1	0	$-\hat{\mathbf{j}}$	1	–1	$\hat{\mathbf{i}} + \hat{\mathbf{j}}$
–1	–1	$\hat{\mathbf{i}} - \hat{\mathbf{j}}$	–1	1	$-\hat{\mathbf{i}} - \hat{\mathbf{j}}$

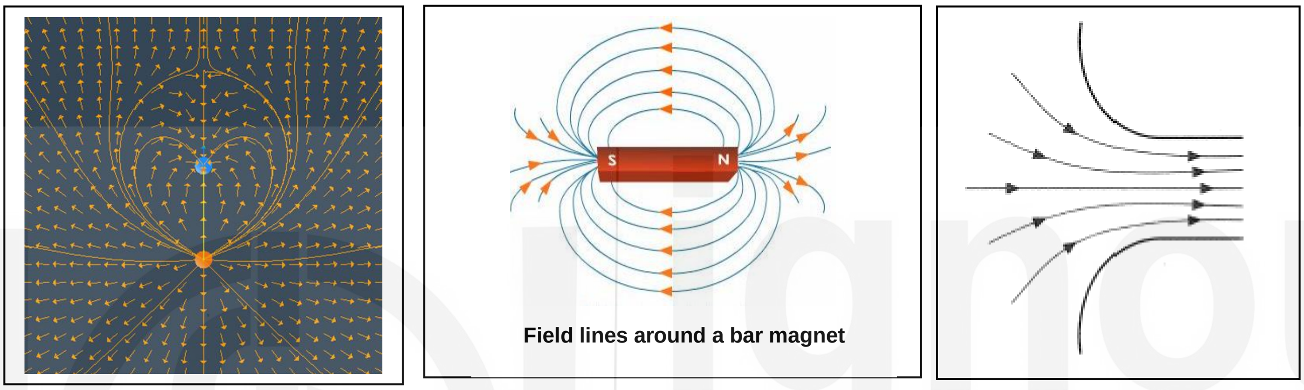
This field is shown in Fig. 2.4b for some values of *x* and *y*.

Faraday observed that iron filings arranged themselves around a bar magnet. He called these curved paths, the lines of force. He visualized a similar pattern of lines around positively and negatively charged bodies. For a bar magnet, the lines appeared to originate on its North pole and terminate on the South pole. So he imagined that the lines of force of an electric field would originate on a positive charge and end on a negative charge.

In physics, we also use vector field lines to depict vector fields, particularly in electricity and magnetism. The concept of field lines was introduced in physics by Michael Faraday, in the context of electromagnetic induction. He called these **lines of force** (read the margin remark).

A **vector field line** is a line such that the tangent drawn to it at any point gives the direction of the vector field at that point. How can you draw such a field line? Choose any point in the region in which the vector field is defined as a starting point. Walk a small distance in the direction of the vector field at that point and draw a line as you walk. From the new point, walk a short distance in the direction of the vector field and draw a line again. As you continue this process, the tangent to the line at any point will give the direction of the vector field at that point. By choosing different starting points, you can generate a set of lines that represents the vector field. In Figs. 2.5a and b, we have shown the field lines for the electric field around a **pair of charges** and the magnetic field around a bar magnet. For electric and magnetic fields, the field lines are also called **the lines of force**. You should remember that these figures give us only a 2-dimensional view of the vector fields that actually exist in 3-dimensional space.

Field lines are also used to represent velocity fields. The field lines for velocity fields are called **streamlines** (Fig. 2.5c). They represent the path followed by a particle whose velocity is given by the velocity field.

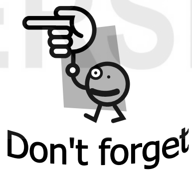


(a) (b) (c)

Fig. 2.5: **a) The field lines around an electric dipole; b) the magnetic field lines around a bar magnet; and c) streamlines for velocity field of water flowing into a pipe.**

1. The tangent drawn to the field line at any point gives the direction of the vector field at that point.

2. The field lines never cross each other. If the field lines were to cross, it would mean that the vector field points in two different directions at the point of intersection. This has no physical meaning.



To be sure that you have understood the concept of a vector field and learnt how to represent it, you should answer the following SAQ.

SAQ 1 - Vector fields

All particles of a fluid flow in one direction with a constant speed. What is the velocity field of the fluid? Draw the vector field lines representing this field.

So far you have learnt the concept of vector fields and how to depict them. We now proceed to learn about another property of vector fields, namely, the presence of sources and sinks. It is useful for understanding the behaviour of several vector fields in physics.

2.2.3 Sources and Sinks of a Vector Field

Refer to Fig. 2.5a again. It shows field lines for the electric field due to an electric dipole. The points *A* and *B* mark the location of the positive and negative charges, respectively. Note that all field lines diverge from point *A* and converge to point *B*. The point *A* is called the **"source"** of the vector field and *B* is called the **"sink".**

In the field line representation of the vector field, some authors prefer to show sources by **full circles** and sinks by **open circles**.

Similarly, in fluid flow, a source in the velocity field is the point at which fluid enters the region, whereas sink is the point where fluid leaves the region. That is, particles flow **out** from a source and hence a source is a point of diverging flow. A sink is a point of converging flow because particles flow **into** it.

In Example 2.1 you have seen that in electric fields, field lines diverge from a positive charge and converge on a negative charge. Hence for electric fields, a positive charge acts as a source and the negative charge as a sink.

In Sec. 1.3 of Unit 1, you have learnt the concept of the gradient of a scalar field. It is the vector differential operator $\left(\vec{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)$ applied to a scalar field *f* and defined as:

$$\vec{\nabla} f = \hat{\mathbf{i}}\frac{\partial f}{\partial x} + \hat{\mathbf{j}}\frac{\partial f}{\partial y} + \hat{\mathbf{k}}\frac{\partial f}{\partial z}$$

You now know that physically, the gradient of a scalar field defines the direction of its maximum rate of change. You may also like to know: How rapidly (at what rate) does a vector field change in a given region? Can we extend the analysis of Sec. 1.3.2 as such? The answer is: We cannot. But why? To answer this question, you may recall from Eq. (2.1) that each component of a vector field is a scalar field. There are two different ways in which the **del** operator can act upon the vector field. Each of these ways defines a type of derivative of the vector field. One of these involves the rate of change of a vector field component in its own direction such as $\partial F_1/\partial x,\ \partial F_2/\partial y,\ \partial F_3/\partial z$ and is called the **divergence** of the vector field. The other type of derivative is called the **curl** of the vector field. It involves the rate of change of the vector field components in directions other than their own, e.g., $\partial F_1/\partial y,\ \partial F_1/\partial z,\ \partial F_2/\partial x$, and so on. We now discuss the divergence of vector fields.

2.3 DIVERGENCE OF A VECTOR FIELD

Consider a three-dimensional vector field function
$\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{i}} + F_2(x, y, z)\,\hat{\mathbf{j}} + F_3(x, y, z)\,\hat{\mathbf{k}}$
where $F_1(x, y, z), F_2(x, y, z)$ and $F_3(x, y, z)$ are the component scalar functions. Its **divergence** is defined as:

In Sec. 1.3 you saw that the gradient of a scalar field is a vector field.

$$\vec{\nabla}\cdot\vec{\mathbf{F}}(x, y, z) = \operatorname{div}\vec{\mathbf{F}}(x, y, z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad (2.3)$$

This expression "$\vec{\nabla}\cdot$" is read as "divergence of" or "del dot". Note that the divergence of a vector field is a scalar field (read also the margin remark). This suggests that we can construct a scalar field from a vector field.

For a two-dimensional vector field $\vec{\mathbf{F}} = F_1(x,y)\,\hat{\mathbf{i}} + F_2(x,y)\,\hat{\mathbf{j}}$, the divergence is

$$\vec{\nabla}.\vec{\mathbf{F}}(x,y) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \qquad (2.4)$$

The meaning of the divergence of a vector field is contained in its name itself. Divergence of the vector field $\vec{\mathbf{F}}$ at a point is a measure of its spread from the point. To appreciate this, go through the following example carefully.

EXAMPLE 2.3 : DIVERGENCE OF A VECTOR FIELD

Calculate the divergence of the following vector fields:

(i) $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ (ii) $\vec{\mathbf{F}} = -x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$ (iii) $\vec{\mathbf{F}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$

SOLUTION ■ We use Eq. (2.4) to calculate the divergence of these two-dimensional vector fields.

(i) For $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ from Eq. (2.4), we get

$$\vec{\nabla}.\vec{\mathbf{F}} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2$$

(ii) For $\vec{\mathbf{F}} = -x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$ from Eq. (2.4), we get

$$\vec{\nabla}.\vec{\mathbf{F}}(x,\,y) = \frac{\partial(-x)}{\partial x} + \frac{\partial(-y)}{\partial y} = -1 - 1 = -2$$

(iii) For $\vec{\mathbf{F}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\quad \vec{\nabla}.\vec{\mathbf{F}}(x,\,y) = \dfrac{\partial(1)}{\partial x} + \dfrac{\partial(1)}{\partial y} = 0$

While going through the solution of Example 2.3, you must have noted that the divergence of the vector field $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ is positive, the divergence of $\vec{\mathbf{F}} = -x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$ is negative whereas $\vec{\mathbf{F}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ has zero divergence. Plots of the two-dimensional vector fields of Example 2.3 are shown in Fig. 2.6. These plots suggest that the vector field $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ has a source at the origin (Fig. 2.6a), $\vec{\mathbf{F}} = -x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$ has a sink at the origin (Fig. 2.6b) and the field $\vec{\mathbf{F}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ has neither a source nor a sink (Fig. 2.6c). **In general, a point of positive divergence is a source and point of negative divergence is a sink.**

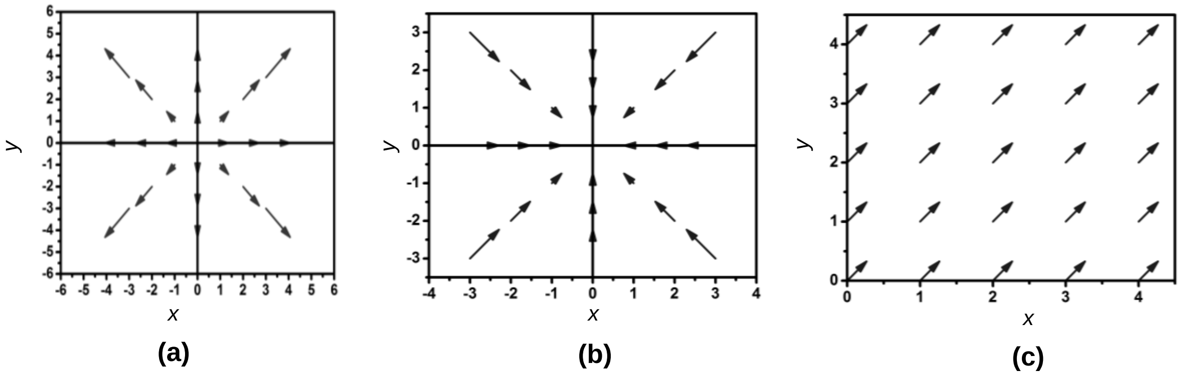


Fig. 2.6: **Plots of the vector fields a)** $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ **; b)** $\vec{\mathbf{F}} = -x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$ **; c)** $\vec{\mathbf{F}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$**.**

A vector field $\vec{\mathbf{F}}$ is called **divergence-free** or **solenoidal** in a given region, if for all points in that region

$$\vec{\nabla} \cdot \vec{\mathbf{F}} = 0 \quad (2.5)$$

Solenoidal comes from a greek word meaning a tube.

The vector field $\vec{\mathbf{F}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ shown in Fig. 2.6c is solenoidal at all points in the *xy* plane. The magnetic field is an example of a solenoidal vector field:

$$\vec{\nabla} \cdot \vec{\mathbf{B}} = 0$$

Let us now summarize what you have learnt about the concept of divergence.

Recap

DIVERGENCE OF A VECTOR FIELD

- The **divergence** of a three-dimensional vector field $\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\hat{\mathbf{i}} + F_2(x, y, z)\hat{\mathbf{j}} + F_3(x, y, z)\hat{\mathbf{k}}$ is defined as

$$\text{div}\,\vec{\mathbf{F}}(x, y, z) = \vec{\nabla} \cdot \vec{\mathbf{F}}(x, y, z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad (2.3)$$

- The divergence of a two-dimensional vector field $\vec{\mathbf{F}}(x, y) = F_1(x, y)\hat{\mathbf{i}} + F_2(x, y)\hat{\mathbf{j}}$ is defined as

$$\text{div}\,\vec{\mathbf{F}}(x, y) = \vec{\nabla} \cdot \vec{\mathbf{F}}(x, y) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \quad (2.4)$$

- A non-zero value of the divergence at any point in a vector field signifies the presence of a source or a sink at that point: $\vec{\nabla}.\vec{\mathbf{F}} > 0$ for a **source** and $\vec{\nabla}.\vec{\mathbf{F}} < 0$ for a **sink**.

- A vector field is called **"divergence-free"** or **"solenoidal"** if its divergence is zero: $\vec{\nabla}.\vec{\mathbf{F}} = 0$ (2.5)

Let us now calculate the divergence of a 3D vector field.

EXAMPLE 2.4 : DIVERGENCE OF A 3D VECTOR FIELD

Calculate the divergence of the vector field $\vec{\mathbf{E}} = \dfrac{\vec{\mathbf{r}}}{r^3}$.

SOLUTION ■ We define the given vector field $\vec{\mathbf{E}} = \dfrac{\vec{\mathbf{r}}}{r^3} = \dfrac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2 + y^2 + z^2)^{3/2}}$.

Then, using Eq. (2.3) with $\vec{\mathbf{F}} = \vec{\mathbf{E}}$, we can write:

$$\vec{\nabla}.\vec{\mathbf{E}} = \vec{\nabla}.\left\{\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2 + y^2 + z^2)^{3/2}}\right\}$$

$$= \frac{\partial}{\partial x}\left[\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2 + y^2 + z^2)^{3/2}}\right]$$

$$+ \frac{\partial}{\partial z}\left[\frac{z}{(x^2 + y^2 + z^2)^{3/2}}\right] \quad \text{(i)}$$

We evaluate each derivative separately:

$$\frac{\partial}{\partial x}\left\{\frac{x}{(x^2+y^2+z^2)^{3/2}}\right\} = \frac{\partial}{\partial x}(x).\left[\frac{1}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$+x.\frac{\partial}{\partial x}\left[\frac{1}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$=\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} = \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}} \quad \text{(ii)}$$

Similarly

$$\frac{\partial}{\partial y}\left\{\frac{y}{(x^2+y^2+z^2)^{3/2}}\right\} = \frac{x^2+z^2-2y^2}{(x^2+y^2+z^2)^{5/2}} \quad \text{(iii)}$$

and

$$\frac{\partial}{\partial z}\left\{\frac{z}{(x^2+y^2+z^2)^{3/2}}\right\} = \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}} \quad \text{(iv)}$$

Substituting from Eqs. (ii), (iii) and (iv) in Eq. (i) we get

$$\vec{\nabla}.\vec{\mathbf{E}} = \frac{2(x^2+y^2+z^2)-2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$$

Before we discuss the physical meaning of the divergence of a vector field, you may like to work out a few problems.

SAQ 2 - Divergence of a vector field

a) Determine the divergence of the following vector fields:

(i) $(x^2-y^2)\hat{\mathbf{i}}+(y^2-z^2)\hat{\mathbf{j}}+(z^2-x^2)\hat{\mathbf{k}}$

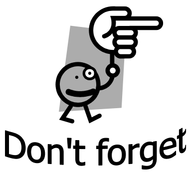
(ii) $y^2z\,\hat{\mathbf{i}}+xy^3\,\hat{\mathbf{j}}-z^2\,\hat{\mathbf{k}}$

b) Calculate the value of the constant *a* such that the vector field $\vec{\mathbf{u}}=(x+3y)\,\hat{\mathbf{i}}+(y+2z)\,\hat{\mathbf{j}}+(x+az)\,\hat{\mathbf{k}}$ is solenoidal.

You may now like to know: **What is the physical significance of the divergence of a vector field?**

The divergence of a vector field can be related to **the flux of that vector field.** You will study about the flux of a vector field in Unit 4.

It is important that we understand another aspect of the divergence at this point. We have written down the expression for the divergence in the rectangular Cartesian coordinate system. However, since the divergence of a vector field can be interpreted as the flux of the vector field per unit volume, it means that:



The value of the divergence of the vector field at any point is independent of the coordinate system.

In your physics courses you will often encounter quantities which involve the sum and products of vector fields or products of scalar fields and vector fields. You will be required to calculate the divergence of these quantities and for this you need to know the rules governing these operations.

IDENTITIES INVOLVING THE DIVERGENCE OF A VECTOR FIELD

The proof of these identities is beyond the scope of this syllabus.

We can write down the following rules for the divergence of the sum and product of vector fields $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$, and the product of a scalar field $f = f(x, y, z)$ with $\vec{\mathbf{F}}$:

$$\vec{\nabla}\,.(\vec{\mathbf{F}} \pm \vec{\mathbf{G}}) = \vec{\nabla}\,.\,\vec{\mathbf{F}} \pm \vec{\nabla}\,.\,\vec{\mathbf{G}} \qquad (2.6a)$$

$$\vec{\nabla}\,.(k\vec{\mathbf{F}}) = k\,\vec{\nabla}\,.\,\vec{\mathbf{F}} \quad \text{where } k \text{ is a constant} \qquad (2.6b)$$

$$\vec{\nabla}\,.(f\,\vec{\mathbf{F}}) = f\,(\vec{\nabla}\,.\,\vec{\mathbf{F}}) + \vec{\mathbf{F}}\,.\,(\vec{\nabla} f) \qquad (2.6c)$$

You may now like to work out an SAQ.

SAQ 3 **– Identities on the divergence of a vector field**

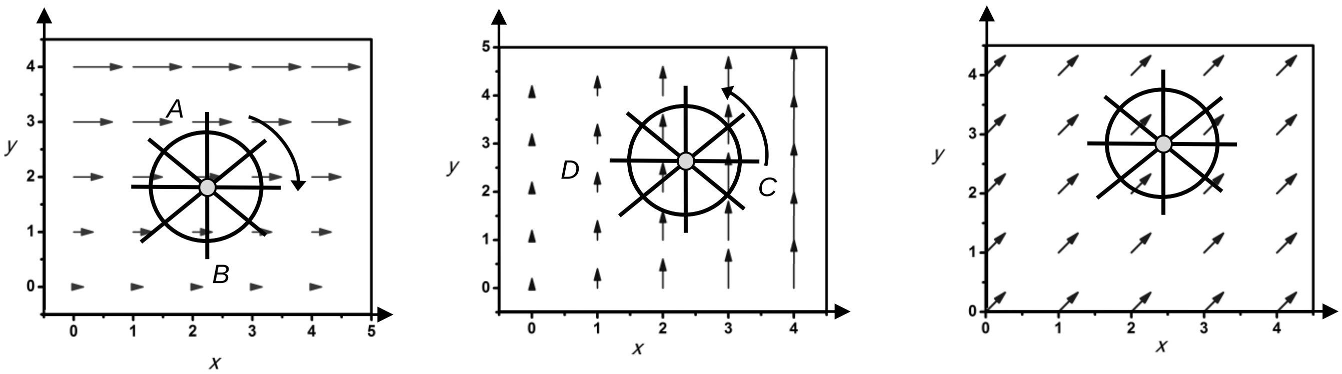
For a scalar field *f*, show that $\vec{\nabla}\,.\,(f\,\vec{\mathbf{r}}) = 3f + \vec{\mathbf{r}}\,.\,(\vec{\nabla} f)$, where $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$.

Let us now study the concept of the curl of a vector field.

2.4 CURL OF VECTOR FIELD

Suppose that you are standing near a pond. Float a small cork with toothpicks (or needles) sticking out radially from it or a paddle wheel in the pond. You may observe that sometimes the cork or paddle wheel rotates. Does this tell us anything about the velocity vector field of the water on the surface of the pond? We can describe this observation in terms of the curl of the velocity vector field. If the cork rotates, it means that the velocity vector field has a non-zero curl at that point. Let us first try to understand, intuitively: **What is the curl of a vector field?**

Let us consider the velocity vector field of water particles on the pond's surface. To keep things simple, we consider a two-dimensional flow in the *xy* plane. Imagine that we put a paddle wheel in the pond with its axis in the z-direction (Fig. 2.7).



(a) (b) (c)

Fig. 2.7: A paddle wheel in the vector field a) $\vec{\mathbf{v}} = (y+1)\,\hat{\mathbf{i}}$; **b)** $\vec{\mathbf{v}} = (x+1)\,\hat{\mathbf{j}}$; **c)** $\vec{\mathbf{v}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$.

Let us first consider the velocity vector field given by $\vec{\mathbf{v}} = (y+1)\,\hat{\mathbf{i}}$ shown in Fig. 2.7a. Note that this velocity vector field is directed along the x-axis and its magnitude increases with y. Since the velocity of water is greater at the top end of the paddle wheel (A) than at the bottom *(B)*, the wheel will have a tendency to rotate in the ***clockwise*** direction, as shown in Fig. 2.7a.

The vector field in Fig. 2.7a is a **particular example** of a vector field which is along the x-direction but the magnitude of the field increases with y.

Next, let us consider the velocity vector field $\vec{\mathbf{v}} = (x+1)\,\hat{\mathbf{j}}$. Here the field is directed along the y-axis and its magnitude increases with x. Since the velocity of water is greater at the right end of the paddle wheel *(C)* than at the left *(D)*, the wheel will have a tendency to rotate ***counterclockwise*** as shown in Fig. 2.7b. For a constant velocity field $\vec{\mathbf{v}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, shown in Fig. 2.7c, the paddle wheel ***will not rotate*** at all.

We use the concept of curl of a vector field to describe these three observations mathematically.

The curl of a vector field is defined as follows:

CURL OF A VECTOR FIELD

The curl of the vector field

$\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{i}} + F_2(x, y, z)\,\hat{\mathbf{j}} + F_3(x, y, z)\,\hat{\mathbf{k}}$ is given by

$$\vec{\nabla} \times \vec{\mathbf{F}} = \text{curl } \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \quad (2.7a)$$

$$\vec{\nabla} \times \vec{\mathbf{F}} = \text{curl } \vec{\mathbf{F}} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}} \quad (2.7b)$$

For three-dimensional vector fields the curl gives the net rotation of the field which would be about some axis. The axis may not be so easy to visualize.

The expression "$\text{curl}\,\vec{\mathbf{F}}$" is pronounced as $\text{curl}\,\vec{\mathbf{F}}$ and "$\vec{\nabla} \times \vec{\mathbf{F}}$" as "del cross F".

The curl of a **two-dimensional vector field** $\vec{\mathbf{F}}(x, y) = F_1(x, y)\,\hat{\mathbf{i}} + F_2(x, y)\,\hat{\mathbf{j}}$ is:

$$\vec{\nabla}\times\vec{\mathbf{F}} = \text{curl } \vec{\mathbf{F}} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}} \qquad (2.8)$$

Note that the curl of a two-dimensional field is normal to the field. If the field is in the *xy* plane, the curl of the field is along the *z*-direction. You can use Eq. (2.8) to obtain the curl of the vector fields depicted in Fig. 2.7.

For $\vec{\mathbf{v}} = (y+1)\,\hat{\mathbf{i}}$, $\vec{\nabla}\times\vec{\mathbf{v}} = -\hat{\mathbf{k}}$,

for $\vec{\mathbf{v}} = (x+1)\,\hat{\mathbf{j}}$, $\vec{\nabla}\times\vec{\mathbf{v}} = \hat{\mathbf{k}}$ and

for $\vec{\mathbf{v}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\vec{\nabla}\times\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

So the velocity vector fields in Figs. 2.7a and b have a finite non-zero curl and the vector field in Fig. 2.7c has a zero curl. Note that the paddle wheel turns anticlockwise if the curl is positive and clockwise if the curl is negative. So in the two-dimensional *xy* plane the curl of a vector field $\vec{\mathbf{F}}$ is a measure of the tendency of the vector field to produce a rotation about the *z*-axis. You will see later in Example 2.6 that the angular velocity of rotation of a rigid body is proportional to the curl of the velocity vector field.

A vector field with zero curl at every point is said to be an **irrotational vector field.** The gravitational force field and electric fields (Example 2.1, Fig. 2.2) are examples of irrotational fields.

Although the expression for the vector product and the curl of a vector look similar, there are some important differences:

1. $\vec{\nabla}\times\vec{\mathbf{F}}$ is not necessarily orthogonal to $\vec{\mathbf{F}}$. In general, it can lie at any angle to $\vec{\mathbf{F}}$ or even be parallel to $\vec{\mathbf{F}}$. **For any two-dimensional vector field, however, the curl of the vector field is always normal to the vector field.**
2. $\vec{\nabla}$ is a vector differential operator. It means that $\vec{\nabla}\times\vec{\mathbf{F}}$ is not the same as $\vec{\mathbf{F}}\times\vec{\nabla}$ and $\vec{\mathbf{F}}\times\vec{\nabla} \neq -\vec{\nabla}\times\vec{\mathbf{F}}$.

In the following example we calculate the curl of three vector fields.

EXAMPLE 2.5 : CURL OF A VECTOR FIELD

Calculate the curl of the following vector fields:

(i) $\vec{\mathbf{F}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$

(ii) $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$

(iii) $\vec{\mathbf{F}} = xe^{y}\,\hat{\mathbf{i}} + ye^{z}\,\hat{\mathbf{j}} + ze^{x}\,\hat{\mathbf{k}}$

SOLUTION ■ (i) Substituting $F_1 = -y$, and $F_2 = x$ in Eq. (2.8), we get:

$$\vec{\nabla}\times\vec{\mathbf{F}} = \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}\right)\hat{\mathbf{k}} = 2\hat{\mathbf{k}}$$

The curl of this two-dimensional vector field is a constant vector in the *z*-direction.

ii) Substituting $F_1 = x$ and $F_2 = y$ in Eq. (2.8) we get:

$$\vec{\nabla} \times \vec{\mathbf{F}} = \left[\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right]\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

The curl of this two dimensional vector field is a null vector so the vector field is **irrotational**.

ii) When we use Eq. (2.7a) with $F_1 = xe^y; F_2 = ye^z$ and $F_3 = ze^x$, we get

$$\vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^y & ye^z & ze^x \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left[\frac{\partial}{\partial y}(ze^x) - \frac{\partial}{\partial z}(ye^z)\right] + \hat{\mathbf{j}}\left[\frac{\partial}{\partial z}(xe^y) - \frac{\partial}{\partial x}(ze^x)\right] + \hat{\mathbf{k}}\left[\frac{\partial}{\partial x}(ye^z) - \frac{\partial}{\partial y}(xe^y)\right]$$

$$= \hat{\mathbf{i}}[-ye^z] + \hat{\mathbf{j}}[-ze^x] + \hat{\mathbf{k}}[-xe^y] \quad = -ye^z\,\hat{\mathbf{i}} - ze^x\,\hat{\mathbf{j}} - xe^y\,\hat{\mathbf{k}}$$

The vector field of Example 2.5(i) is shown in Fig. 2.4b. It has a **positive curl.** As you can see in the figure, the field has the appearance of a *whirlpool* rotating anticlockwise. What is the divergence of this vector field? Let us calculate it using Eq. (2.4):

$$\vec{\nabla}.\vec{\mathbf{F}} = \vec{\nabla}.(-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{k}}) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$$

So $\vec{\mathbf{F}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{k}}$ is an example of a vector field which has **zero divergence and a finite (positive) curl**. This is an example of a **circulating field**, which has no sources or sinks.

On the other hand, the vector field of Example 2.5(ii) which is shown in Fig. 2.6a has zero curl and is an **irrotational** vector field. From the figure, it appears that this vector field has a source.

What is the divergence of this vector field? Using Eq. (2.4) we can write

$$\vec{\nabla}.\vec{\mathbf{F}} = \vec{\nabla}.(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{k}}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2$$

So $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ is an example of a vector field which has **finite (positive) divergence and zero curl.** This is an example of a **diverging field** that has no rotation. We also say that it has no **circulation**.

Let us now summarise what you have learnt about the curl of a vector field.

Recap

CURL OF A VECTOR FIELD

- The curl of a three-dimensional vector field $\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{i}} + F_2(x, y, z)\,\hat{\mathbf{j}} + F_3(x, y, z)\,\hat{\mathbf{k}}$ is defined as

$$\text{curl}\,\vec{\mathbf{F}} = \vec{\nabla}\times\vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \qquad (2.7a)$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}} \qquad (2.7b)$$

- The curl of a two-dimensional vector field $\vec{\mathbf{F}}(x, y) = F_1(x, y)\,\hat{\mathbf{i}} + F_2(x, y)\,\hat{\mathbf{j}}$ is defined as

$$\text{curl}\,\vec{\mathbf{F}} = \vec{\nabla}\times\vec{\mathbf{F}} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}} \qquad (2.8)$$

- If the curl of the vector field is zero, i.e.,

$$\vec{\nabla}\times\vec{\mathbf{F}} = \vec{\mathbf{0}}$$

the vector field is said to be "**irrotational**".

In the next section, we establish some identities for the curl of vector fields. But before studying further, you may like to work out the following SAQ.

SAQ 4 – Curl of a vector field

Determine the curl of the following vector fields:

a) $\vec{\mathbf{F}} = (2x - y)\,\hat{\mathbf{i}} - 2yz^2\,\hat{\mathbf{j}} + -2zy^2\,\hat{\mathbf{k}}$

b) $\vec{\mathbf{F}} = z\cos x\,\hat{\mathbf{i}} + (y + \sin x)\,\hat{\mathbf{j}} + xyz\,\hat{\mathbf{k}}$

We now write down important identities for the curl of the sum and product of vector fields and the product of a scalar field and a vector field.

IDENTITIES INVOLVING THE CURL OF A VECTOR FIELD

For the vector fields $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$, and the scalar field $f = f(x, y, z)$:

$$\vec{\nabla}\times(\vec{\mathbf{F}} \pm \vec{\mathbf{G}}) = \vec{\nabla}\times\vec{\mathbf{F}} \pm \vec{\nabla}\times\vec{\mathbf{G}} \qquad (2.9a)$$

$$\vec{\nabla}\times(k\vec{\mathbf{F}}) = k\vec{\nabla}\times\vec{\mathbf{F}} \qquad \text{where } k \text{ is a constant} \qquad (2.9b)$$

$$\vec{\nabla}\times(f\,\vec{\mathbf{F}}) = f\,(\vec{\nabla}\times\vec{\mathbf{F}}) - \vec{\mathbf{F}}\times(\vec{\nabla}f) \qquad (2.9c)$$

$$\vec{\nabla}\times(\vec{\mathbf{F}}\times\vec{\mathbf{G}}) = \vec{\mathbf{F}}(\vec{\nabla}.\vec{\mathbf{G}}) - \vec{\mathbf{G}}(\vec{\nabla}.\vec{\mathbf{F}}) - (\vec{\mathbf{F}}.\vec{\nabla})\vec{\mathbf{G}} + (\vec{\mathbf{G}}.\vec{\nabla})\vec{\mathbf{F}} \qquad (2.9d)$$

The proof of these identities is beyond the scope of this course.

We also add the following rule for the divergence of the vector product of the two vector fields because it also involves the curl of a vector field:

$$\vec{\nabla}\,.(\vec{\mathbf{F}}\times\vec{\mathbf{G}}) = \vec{\mathbf{G}}\,.\,(\vec{\nabla}\times\vec{\mathbf{F}}) - \vec{\mathbf{F}}\,.\,(\vec{\nabla}\times\vec{\mathbf{G}}) \qquad (2.9e)$$

Let us now study some simple applications of these product rules.

EXAMPLE 2.6 : ROTATION OF A RIGID BODY

Consider a rigid body rotating about a fixed axis with a constant angular velocity $\vec{\omega}$, directed along the axis of rotation (Fig. 2.8). The velocity $\vec{\mathbf{v}}$ of a particle on the rigid body is $\vec{\omega}\times\vec{\mathbf{r}}$. Here $\vec{\mathbf{r}}$ is the position vector of the particle relative to the origin of the coordinate system located at some point on the axis of rotation. Calculate $\vec{\nabla}\times\vec{\mathbf{v}}$.

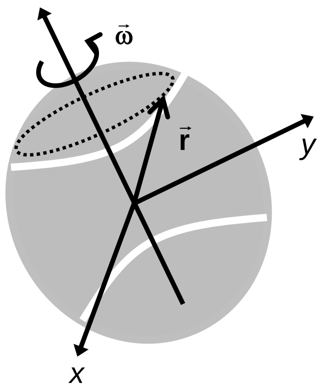


Fig. 2.8: **A rigid body rotating about an axis.**

SOLUTION ■ Since velocity $\vec{\mathbf{v}}$ of a particle on the rigid body is given by $\vec{\omega}\times\vec{\mathbf{r}}$, we can write $\vec{\nabla}\times\vec{\mathbf{v}} = \vec{\nabla}\times(\vec{\omega}\times\vec{\mathbf{r}})$. To obtain the desired expression, we use Eq. (2.9d) for the curl of the cross product of two vector fields. Substituting $\vec{\mathbf{F}}$ by $\vec{\omega}$ and $\vec{\mathbf{G}}$ by $\vec{\mathbf{r}}$ in Eq. (2.9d) we get

$$\vec{\nabla}\times(\vec{\omega}\times\vec{\mathbf{r}}) = \vec{\omega}(\vec{\nabla}.\vec{\mathbf{r}}) - \vec{\mathbf{r}}(\vec{\nabla}.\vec{\omega}) - (\vec{\omega}.\vec{\nabla})\vec{\mathbf{r}} + (\vec{\mathbf{r}}.\vec{\nabla})\vec{\omega} \qquad \text{(i)}$$

The position vector is $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and the angular velocity is $\vec{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$. As the angular velocity ω is a constant, all terms in Eq. (i) which involve the derivatives of ω_x, ω_y and ω_z are zero. For example:

$$(\vec{\mathbf{r}}\;.\;\vec{\nabla})\vec{\omega} = \left[(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})\;.\;\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\right]\vec{\omega}$$

$$= \left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right]\vec{\omega} = \vec{\mathbf{0}} \qquad \text{(ii)}$$

Also $\vec{\mathbf{r}}(\vec{\nabla}.\vec{\omega}) = \vec{\mathbf{0}}$ (iii)

On combining Eqs. (i), (ii) and (iii), we get

$$\vec{\nabla}\times(\vec{\omega}\times\vec{\mathbf{r}}) = \vec{\omega}(\vec{\nabla}\;.\;\vec{\mathbf{r}}) - (\vec{\omega}\;.\;\vec{\nabla})\vec{\mathbf{r}} \qquad \text{(iv)}$$

Now $$\vec{\omega}(\vec{\nabla}\;.\;\vec{\mathbf{r}}) = \vec{\omega}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 3\vec{\omega} \qquad \text{(v)}$$

and $$(\vec{\omega}\;.\;\vec{\nabla})\vec{\mathbf{r}} = \left(\omega_x\frac{\partial}{\partial x} + \omega_y\frac{\partial}{\partial y} + \omega_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})$$

$$= \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}} = \vec{\omega} \qquad \text{(vi)}$$

The final expression for $\vec{\nabla}\times\vec{\mathbf{v}}$ is obtained by substituting Eqs. (v) and (vi) into Eq. (iv):

$$\vec{\nabla}\times\vec{\mathbf{v}} = 3\vec{\omega} - \vec{\omega} = 2\vec{\omega} \qquad \text{(vii)}$$

So we can now write the angular velocity of the rigid body as the curl of the velocity as:

$$\vec{\omega} = \frac{1}{2}(\vec{\nabla}\times\vec{\mathbf{v}}) \qquad \text{(viii)}$$

Therefore, $\vec{\nabla} \times \vec{\mathbf{v}}$ describes the rate at which the body rotates about the axis of rotation.

If $\vec{\mathbf{v}}$ is a two dimensional velocity field describing fluid flow, instead of the velocity of the particles on a rigid body, we can say that $\vec{\nabla} \times \vec{\mathbf{v}}$ at any point (x, y) in the field, is **twice the angular velocity of an infinitesimal paddle wheel placed at the point** (x, y).

Let us study another important example in physics.

EXAMPLE 2.7 : CURL OF A CENTRAL FORCE FIELD

A central force field is a force field of the form $\vec{\mathbf{F}} = f(r)\vec{\mathbf{r}}$. Determine $\vec{\nabla} \times \vec{\mathbf{F}}$.

SOLUTION ■ In Eq. (2.9c), we replace *f* by $f(r)$ and $\vec{\mathbf{F}}$ by $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ to write:

$$\vec{\nabla} \times (f(r)\vec{\mathbf{r}}) = f(r)\,(\vec{\nabla} \times \vec{\mathbf{r}}) - \vec{\mathbf{r}} \times (\vec{\nabla} f(r)) \qquad \text{(i)}$$

From Example 1.2 of Unit 1, you may recall that

$$\vec{\nabla} f(r) = \frac{df}{dr}\hat{\mathbf{r}} \quad \text{and} \quad \vec{\mathbf{r}} = r\,\hat{\mathbf{r}} \qquad \text{(ii)}$$

So we can write $\vec{\mathbf{r}} \times (\vec{\nabla} f(r)) = (r\,\hat{\mathbf{r}}) \times \left(\frac{df}{dr}\hat{\mathbf{r}}\right) = \vec{\mathbf{0}}$ (iii)

since $\hat{\mathbf{r}} \times \hat{\mathbf{r}} = \vec{\mathbf{0}}$. Further, you can show that

$$\vec{\nabla} \times \vec{\mathbf{r}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{\mathbf{0}} \qquad \text{(iv)}$$

Using Eqs. (iii) and (iv) in Eq. (i), we get

$$\vec{\nabla} \times (f(r)\vec{\mathbf{r}}) = \vec{\mathbf{0}} \qquad \text{(v)}$$

Thus, a central force field $\vec{\mathbf{F}}$ of the form $\vec{\mathbf{F}} = f(r)\vec{\mathbf{r}}$ is **irrotational**.

Before you study the next section you may like to work out an SAQ.

SAQ 5 – Identities of the Curl Operator

a) If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are both irrotational, show that $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is solenoidal.

b) A vector function $\vec{\mathbf{f}}(x, y, z)$ is not irrotational but its product with a scalar function *g* (*x*, *y*, *z*) is irrotational. Show that $\vec{\mathbf{f}} \cdot (\vec{\nabla} \times \vec{\mathbf{f}}) = 0$.

2.5 SUCCESSIVE APPLICATIONS OF THE DEL OPERATOR

We now write down five basic identities involving repeated applications of the del operator. These are commonly used in physics: In Poisson's equation and Laplace's equation in electrostatics, electromagnetic wave equation and to describe conservative force fields.

We have presented the repeated applications of del operator in the light of their applications.

Since $\vec{\nabla} f$ is a vector field, we can take its divergence and curl:

i) Divergence of $\vec{\nabla} f : \vec{\nabla} \,.\, (\vec{\nabla} f)$

ii) Curl of $\vec{\nabla} f : \vec{\nabla} \times (\vec{\nabla} f)$

Since $\vec{\nabla} . \vec{\mathbf{F}}$ is a scalar field, we obtain its gradient as

iii) Gradient of $\vec{\nabla}.\vec{\mathbf{F}}$: $\vec{\nabla}(\vec{\nabla}.\vec{\mathbf{F}})$

Since $\vec{\nabla} \times \vec{\mathbf{F}}$ is vector field we can take its divergence and curl:

iv) Divergence of $\vec{\nabla} \times \vec{\mathbf{F}} : \vec{\nabla} . (\vec{\nabla} \times \vec{\mathbf{F}})$

v) Curl of $\vec{\nabla} \times \vec{\mathbf{F}} : \vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{F}})$

Thus, we can construct five different second order derivatives of scalar and vector fields. Let us consider them one at a time with examples.

i) Divergence of $\vec{\nabla} f$

Using Eqs. (1.11a) and (2.3) we can write

$$\vec{\nabla} . (\vec{\nabla} f) = \left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z}\right) . \left(\hat{\mathbf{i}} \frac{\partial f}{\partial x} + \hat{\mathbf{j}} \frac{\partial f}{\partial y} + \hat{\mathbf{k}} \frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

or

$$\vec{\nabla} . (\vec{\nabla} f) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f \equiv \nabla^2 f \qquad (2.10a)$$

The operator $\nabla^2 (\equiv \vec{\nabla} . \vec{\nabla})$ is called the **Laplace operator** and $\nabla^2 f$ is called the **Laplacian** of *f*. Notice that $\nabla^2 f$ is a scalar field. The Laplace operator plays an extremely important role in determining the charge density ρ (*x, y, z*) of a charge distribution which gives rise to an electrostatic potential ϕ. This is done by solving the following equation, known as **Poisson's equation**:

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$$

Conversely, given ρ (*x, y, z*), we can obtain ϕ from this equation but the method of solving for ϕ is beyond the scope of this course. You will learn to do so in a course on partial differential equations.

To obtain ϕ in a charge-free region we solve **Laplace's equation**:

$$\nabla^2 \phi = 0$$

In electromagnetic theory, you will come across the Laplacian of a vector field: $\nabla^2 \vec{\mathbf{F}}$. This means that $\nabla^2 \vec{\mathbf{F}}$ is a vector quantity whose *x, y* and z components are the Laplacians $\nabla^2 F_x$, $\nabla^2 F_y$ and $\nabla^2 F_z$, respectively, i.e.

$$\nabla^2 \vec{\mathbf{F}} = (\nabla^2 F_x)\,\hat{\mathbf{i}} + (\nabla^2 F_y)\,\hat{\mathbf{j}} + (\nabla^2 F_z)\,\hat{\mathbf{k}} \tag{2.10b}$$

ii) Curl of $\vec{\nabla} f$

We can show that **curl** $\vec{\nabla} f$ is always zero:

$$\vec{\nabla} \times (\vec{\nabla} f) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

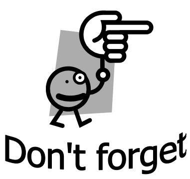
$$= \hat{\mathbf{i}}\left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\right) + \hat{\mathbf{j}}\left(\frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z}\right) + \hat{\mathbf{k}}\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right)$$

From the theory of partial derivatives we know that $\dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial y}$ $\dfrac{\partial}{\partial z}\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial z}$, and so on. Therefore, it follows that

$$\vec{\nabla} \times (\vec{\nabla} f) = 0 \tag{2.10c}$$

You may think that $\vec{\nabla} \times (\vec{\nabla} f) = 0$ is an obvious result: Isn't it just $(\vec{\nabla} \times \vec{\nabla}) f$, and the cross product of a vector with itself is zero. This reasoning is **not correct**. This is because $\vec{\nabla}$ is an operator and does not multiply in the usual way. The proof of Eq. (2.10c), in fact, depends on the relation $\dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial y}$, etc.

There are some vector fields like the inverse square force field $\vec{\mathbf{F}}$ (e.g., the gravitational force field or the electrostatic force field) which can be expressed as the gradient of the scalar field $\phi = k(x^2 + y^2 + z^2)^{-1/2}$. For such fields, using the identity (2.10c) you can see that $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$. **Such vector fields with zero curl can be expressed as gradients of scalar fields and are called conservative fields.**



For a conservative vector field $\vec{\mathbf{F}}$, $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$ everywhere.

iii) Gradient of $\vec{\nabla}.\vec{\mathbf{F}}$

Using the definitions of $\vec{\nabla}$ and the divergence, we can write

$$\vec{\nabla}(\vec{\nabla}.\,\vec{\mathbf{F}}) = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)$$

$$= \hat{\mathbf{i}}\left(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial}{\partial x}\frac{\partial F_y}{\partial y} + \frac{\partial}{\partial x}\frac{\partial F_z}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial}{\partial y}\frac{\partial F_x}{\partial x} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial}{\partial y}\frac{\partial F_z}{\partial z}\right)$$

$$+ \hat{\mathbf{k}}\left(\frac{\partial}{\partial z}\frac{\partial F_x}{\partial x} + \frac{\partial}{\partial z}\frac{\partial F_y}{\partial y} + \frac{\partial^2 F_z}{\partial z^2}\right) \tag{2.10d}$$

This operator has no name of its own and is called the gradient of the divergence. It appears in the wave equation of an electromagnetic wave $\vec{\mathbf{E}}$:

$$\nabla^2\vec{\mathbf{E}} - \vec{\nabla}\,(\vec{\nabla}.\,\vec{\mathbf{E}}) = -\,\mu_0\,\varepsilon_0\,\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2} - \mu_0\,\frac{\partial\vec{\mathbf{J}}}{\partial t}$$

where μ_0 and ε_0 are the permeability and permittivity of free space.

Remember that $\vec{\nabla}\,(\vec{\nabla}\,.\,\vec{\mathbf{F}})$ is not the same as $\nabla^2 f$:

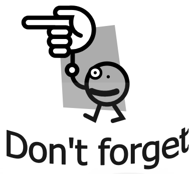
$$\nabla^2 f = \vec{\nabla}.\,(\vec{\nabla} f) \neq \vec{\nabla}(\vec{\nabla}\,.\,\vec{\mathbf{F}}) \qquad (2.10e)$$

iv) Divergence of $(\vec{\nabla}\times\vec{\mathbf{F}})$

We can show that $\vec{\nabla}.(\vec{\nabla}\times\vec{\mathbf{F}})$ is zero. (2.10f)

You can try this out for yourself in SAQ 6.

Do not equate $\vec{\nabla}.(\vec{\nabla}\times\vec{\mathbf{F}}) = 0$ with the property $\vec{\mathbf{A}}\,.\,(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) = 0$ for vectors, as $\vec{\nabla}$ is a differential operator.



v) Curl of $(\vec{\nabla}\times\vec{\mathbf{F}})$

This can be expressed as the following:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathbf{F}}) = \vec{\nabla}(\vec{\nabla}\,.\,\vec{\mathbf{F}}) - \nabla^2\vec{\mathbf{F}} \qquad (2.10g)$$

Of course, $\nabla^2\vec{\mathbf{F}}$ has the meaning as explained in this section before Eq. (2.10b).

Using Eq. (2.10g) you can express the electromagnetic wave equation as

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathbf{E}}) = -\mu_0\varepsilon_0\,\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2} - \mu_0\,\frac{\partial\vec{\mathbf{J}}}{\partial t}$$

We can apply the del operator once more to get a few more identities but this is beyond the scope of this course.

Let us understand some physical implications of what you have learnt in this section.

You have seen in Eq. (2.10c) that the curl of the gradient of a scalar field is zero. This means that if **a vector field is irrotational or has a zero curl, you may write it as the gradient of a scalar field**. In other words, an irrotational vector field may be generated from **a scalar field alone**.

Similarly, if a vector field is solenoidal, its divergence is zero. It can then be written as the curl of a vector field as you can see from Eq. (2.10f). Therefore, a solenoidal vector field can be generated from a **vector field** alone.

The most general vector field, which has both a non-zero divergence and a non-zero curl, can therefore be written as the sum of a solenoidal field and an irrotational field (see margin remark).

This is also called the Helmholtz Theorem.

The magnetic field $\vec{\mathbf{B}}$ is an example of a solenoidal vector field. Since $\nabla.\vec{\mathbf{B}} = 0$ we can write, using Eq. (2.10f)

$$\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}} \qquad (2.10h)$$

The vector field $\vec{\mathbf{A}}$ associated with the magnetic field is also called the vector potential.

You may now like to work out an SAQ on the repeated applications of the del operator.

***SAQ 6* - Successive applications of the Del Operator**

a) Verify Eq. (2.10f).

b) For a function $\phi = xyz^2$, show that $\vec{\nabla}.(\vec{\nabla}\phi) = \nabla^2\phi.$

We now summarise what you have learnt in this unit.

2.6 SUMMARY

Concept	Description
Vector field	■ A vector field is a function that assigns a vector to every point of a given region in space. A three-dimensional vector field $\vec{\mathbf{F}}$ can be written as follows: $\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\hat{\mathbf{i}} + F_2(x, y, z)\hat{\mathbf{j}} + F_3(x, y, z)\hat{\mathbf{k}}$ The components of the vector field $\vec{\mathbf{F}}(x, y, z)$ namely $F_1(x, y, z)$, $F_2(x, y, z)$ and $F_3(x, y, z)$ are **scalar fields** defined over the same region as the vector field. A vector field $\vec{\mathbf{F}}$ in two-dimensions can be written as follows: $\vec{\mathbf{F}}(x, y) = F_1(x, y)\hat{\mathbf{i}} + F_2(x, y)\hat{\mathbf{j}}$
Divergence of a vector field	■ The divergence of a two-dimensional vector field $\vec{\mathbf{F}}(x, y) = F_1(x, y)\hat{\mathbf{i}} + F_2(x, y)\hat{\mathbf{j}}$ is defined as $\text{div } \vec{\mathbf{F}}(x, y) = \vec{\nabla} . \vec{\mathbf{F}}(x, y) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$ The divergence of a three-dimensional vector field $\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\hat{\mathbf{i}} + F_2(x, y, z)\hat{\mathbf{j}} + F_3(x, y, z)\hat{\mathbf{k}}$ is defined as $\text{div } \vec{\mathbf{F}}(x, y) = \vec{\nabla}.\vec{\mathbf{F}}(x, y) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$ The divergence of a vector field gives the extent to which the vector field flow behaves like source or a sink at a given point. A non-zero value of the divergence at any point in a vector field signifies the presence of a source or a sink: $\vec{\nabla}.\vec{\mathbf{F}} > 0$ for a source and $\vec{\nabla}.\vec{\mathbf{F}} < 0$ for a sink. If the divergence of the vector field is zero, the vector field is called “divergence-free” or “solenoidal”.

Curl of a vector field

- The curl of a two-dimensional vector field $\vec{\mathbf{F}}(x, y) = F_1(x, y)\,\hat{\mathbf{i}} + F_2(x, y)\,\hat{\mathbf{j}}$ is defined as

$$\vec{\nabla}\times\vec{\mathbf{F}} = \text{curl } \vec{\mathbf{F}} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}}$$

The curl of a three-dimensional vector field
$\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{i}} + F_2(x, y, z)\,\hat{\mathbf{j}} + F_3(x, y, z)\,\hat{\mathbf{k}}$ is defined as

$$\text{curl } \vec{\mathbf{F}} = \vec{\nabla}\times\vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}}$$

If the curl of the vector field is zero, the vector field is called **irrotational**.

Identities involving the divergence and curl of a vector field

- For arbitrary vector fields $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$, and a scalar field $f = f(x,y,z)$:

$$\vec{\nabla}\,.(\vec{\mathbf{F}} \pm \vec{\mathbf{G}}) = \vec{\nabla}\,.\,\vec{\mathbf{F}} \pm \vec{\nabla}\,.\,\vec{\mathbf{G}}$$

$$\vec{\nabla}\,.(k\vec{\mathbf{F}}) = k\vec{\nabla}\,.\,\vec{\mathbf{F}} \quad \text{where } k \text{ is a constant}$$

$$\vec{\nabla}\,.(f\vec{\mathbf{F}}) = f(\vec{\nabla}\,.\,\vec{\mathbf{F}}) + \vec{\mathbf{F}}\,.\,(\vec{\nabla}f)$$

$$\vec{\nabla}\times(\vec{\mathbf{F}} \pm \vec{\mathbf{G}}) = \vec{\nabla}\times\vec{\mathbf{F}} \pm \vec{\nabla}\times\vec{\mathbf{G}}$$

$$\vec{\nabla}\times(k\vec{\mathbf{F}}) = k\vec{\nabla}\times\vec{\mathbf{F}} \quad \text{where } k \text{ is a constant}$$

$$\vec{\nabla}\times(f\vec{\mathbf{F}}) = f(\vec{\nabla}\times\vec{\mathbf{F}}) - \vec{\mathbf{F}}\times(\vec{\nabla}f)$$

$$\vec{\nabla}\times(\vec{\mathbf{F}}\times\vec{\mathbf{G}}) = (\vec{\mathbf{G}}\,.\,\vec{\nabla})\vec{\mathbf{F}}) - (\vec{\mathbf{F}}\,.\,\vec{\nabla})\vec{\mathbf{G}} + \vec{\mathbf{F}}(\vec{\nabla}\,.\,\vec{\mathbf{G}}) - \vec{\mathbf{G}}(\vec{\nabla}\,.\,\vec{\mathbf{F}})$$

Successive application of the Del operator

- For an arbitrary vector field $\vec{\mathbf{F}}$ and a scalar field f

$$\vec{\nabla}.(\vec{\nabla}f) = \nabla^2 f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)f$$

$$\vec{\nabla}\times(\vec{\nabla}f) = 0$$

$$\vec{\nabla}.(\vec{\nabla}\times\vec{\mathbf{F}}) = 0$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathbf{F}}) = \vec{\nabla}(\vec{\nabla}\,.\,\vec{\mathbf{F}}) - \nabla^2\vec{\mathbf{F}}$$

2.7 TERMINAL QUESTIONS

1. Determine the divergence and curl of the following vector fields

   (i) $\vec{\mathbf{u}} = x^2y^2\hat{\mathbf{i}} - x^2y\hat{\mathbf{j}}$

   (ii) $\vec{\mathbf{u}} = \ln(x)\,\hat{\mathbf{i}} + \ln(xy)\hat{\mathbf{j}} + \ln(xyz)\hat{\mathbf{k}}$

2. Calculate $\vec{\nabla}.\dfrac{\vec{\mathbf{r}}}{r}$, given $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, $r = \sqrt{x^2 + y^2 + z^2}$.

3. Determine whether the vector field $\vec{\mathbf{F}} = \dfrac{\vec{\mathbf{r}}}{r^2}$ is: (i) irrotational, (ii) solenoidal.

4. Given $\vec{\mathbf{u}} = 2y\hat{\mathbf{i}} + 4z\hat{\mathbf{j}} + x^2z^2\,\hat{\mathbf{k}}$, calculate $\vec{\nabla}.\vec{\mathbf{u}}$ and $\vec{\nabla}\times\vec{\mathbf{u}}$ at the point (0, 1, 2).

5. If $\vec{\mathbf{a}}$ is a constant vector show that $\vec{\nabla}\times(\vec{\mathbf{a}}\times\vec{\mathbf{r}}) = 2\vec{\mathbf{a}}$

6, Determine the value of the constant *k* for which curl of the vector field

$$\vec{\mathbf{F}} = -\frac{y}{(x^2+y^2)^{k/2}}\hat{\mathbf{i}} + \frac{x}{(x^2+y^2)^{k/2}}\hat{\mathbf{j}}$$

is (i) positive (ii) negative and (iii) a null vector.

7. If $\vec{\mathbf{A}} = 2yz\hat{\mathbf{i}} - x^2y\hat{\mathbf{j}} + xz^2\hat{\mathbf{k}}$ and $f = xyz$ show that $(\vec{\mathbf{A}}\,.\,\vec{\nabla})f = \vec{\mathbf{A}}\,.\big(\vec{\nabla}f\big)$.

8. Determine the values of *a*, *b* and *c* such that the vector field $\vec{\mathbf{A}} = (3x - y + az)\,\hat{\mathbf{i}} + (bx + 2y + z)\,\hat{\mathbf{j}} + (x + cy - 2z)\,\hat{\mathbf{k}}$ is irrotational.

9. Prove that $\vec{\nabla}\,.\,(\vec{\nabla}f\times\vec{\nabla}g) = 0$.

10. Determine $\nabla^2\phi$ for (i) $\phi = \ln(x^2+y^2)$ and (ii) $\phi = xyz(x^2 - y^2 + z^2)$

2.8 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. We choose a constant velocity in the *y*-direction. Then the velocity vector at every point in the field is the same and given by $\vec{\mathbf{v}} = a\,\hat{\mathbf{j}}$. We sketch the field in Fig. 2.9.

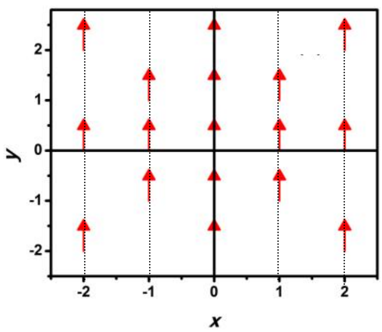


Fig. 2.9: **The vector field** $\vec{\mathbf{v}} = a\,\hat{\mathbf{j}}$

2. a) We use Eq. (2.3) to evaluate the divergence of a three dimensional field.

i) $\vec{\nabla}.\,[(x^2 - y^2)\,\hat{\mathbf{i}} + (y^2 - z^2)\,\hat{\mathbf{j}} + (z^2 - x^2)\hat{\mathbf{k}}]$

$$= \frac{\partial}{\partial x}(x^2 - y^2) + \frac{\partial}{\partial y}(y^2 - z^2) + \frac{\partial}{\partial z}(z^2 - x^2)$$

$$= 2x + 2y + 2z$$

ii) $\vec{\nabla}.(y^2 z\hat{\mathbf{i}} + xy^3\hat{\mathbf{j}} - z^2\hat{\mathbf{k}})$

$$= \frac{\partial}{\partial x}(y^2 z) + \frac{\partial}{\partial y}(xy^3) - \frac{\partial}{\partial z}(z^2)$$

$$= 3y^2 x - 2z$$

b) For a vector field to be solenoidal, its divergence has to be zero. Imposing this condition on the given vector field we can write

$$\vec{\nabla}.[(x + 3y)\hat{\mathbf{i}} + (y + 2z)\hat{\mathbf{j}} + (x + az)\hat{\mathbf{k}}] = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y + 2z) + \frac{\partial}{\partial z}(x + az) = 0$$

$$\Rightarrow 1 + 1 + a = 0 \quad \text{or} \quad a = -2.$$

For the value of $a = -2$ the divergence of the vector field is zero and the field is solenoidal.

3. We use Eq. (2.6c) with $\vec{\mathbf{F}} = \vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ to obtain the result.

$$\vec{\nabla}.(f\,\vec{\mathbf{r}}) = f\,(\vec{\nabla}.\vec{\mathbf{r}}) + \vec{\mathbf{r}}.(\vec{\nabla} f) \qquad \text{(i)}$$

$$\vec{\nabla}.\vec{\mathbf{r}} = \vec{\nabla}.(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3 \qquad \text{(ii)}$$

Substituting Eq. (ii) in Eq. (i) we get

$$\vec{\nabla}.(f\vec{\mathbf{r}}) = 3f + \vec{\mathbf{r}}.\vec{\nabla} f$$

4. a) Using Eq. (2.7a) with $F_1 = 2x - y$, $F_2 = -2yz^2$ and $F_3 = -2zy^2$, we get :

$$\vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -2yz^2 & -2zy^2 \end{vmatrix}$$

$$= \hat{\mathbf{i}}\,(-4yz + 4yz) + \hat{\mathbf{j}}\,(0 - 0) + \hat{\mathbf{k}}\,(1) = \hat{\mathbf{k}}$$

b) We use Eq. (2.7a) with $F_1 = z\cos x, F_2 = y + \sin x$ and $F_3 = xyz$.

$$\therefore \qquad \vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\cos x & y + \sin x & xyz \end{vmatrix}$$

$$= \hat{\mathbf{i}}[xz - 0] + \hat{\mathbf{j}}[\cos x - yz] + \hat{\mathbf{k}}[\cos x - 0]$$

$$= xz\hat{\mathbf{i}} + (\cos x - yz)\hat{\mathbf{j}} + \cos x\,\hat{\mathbf{k}}$$

5. a) To show that the vector field $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is solenoidal we must prove that the divergence of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is zero or $\vec{\nabla}.(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = 0$.

   It is given that the vector fields $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are irrotational, so $\vec{\nabla} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$ and $\vec{\nabla} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$. Using Eq. (2.9e) with $\vec{\mathbf{F}} = \vec{\mathbf{u}}$ and $\vec{\mathbf{G}} = \vec{\mathbf{v}}$ we get:

$$\vec{\nabla}.(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = \vec{\mathbf{v}}.(\vec{\nabla} \times \vec{\mathbf{u}}) - \vec{\mathbf{u}}.(\vec{\nabla} \times \vec{\mathbf{v}}) = 0 \text{ since } \vec{\nabla} \times \vec{\mathbf{u}} = \vec{\mathbf{0}} \text{ and } \vec{\nabla} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$$

   b) Since $g\vec{\mathbf{f}}$ is irrotational $\vec{\nabla} \times (g\vec{\mathbf{f}}) = \vec{\mathbf{0}}$. So, using Eq. (2.9c)

$$\vec{\nabla} \times (g\vec{\mathbf{f}}) = g(\vec{\nabla} \times \vec{\mathbf{f}}) - \vec{\mathbf{f}} \times (\vec{\nabla} g) = \vec{\mathbf{0}}$$

$$\Rightarrow \quad g(\vec{\nabla} \times \vec{\mathbf{f}}) = \vec{\mathbf{f}} \times (\vec{\nabla} g) \qquad \text{(i)}$$

   Taking the scalar product of Eq. (i) with $\vec{\mathbf{f}}$, we have

$$g\vec{\mathbf{f}}.(\vec{\nabla} \times \vec{\mathbf{f}}) = \vec{\mathbf{f}}.[\vec{\mathbf{f}} \times \vec{\nabla} g] \qquad \text{(ii)}$$

   Since a scalar triple product of the kind $\vec{\mathbf{a}}.(\vec{\mathbf{a}} \times \vec{\mathbf{b}})$ is always zero, the RHS of Eq. (ii) is zero. Hence $\vec{\mathbf{f}}.(\vec{\nabla} \times \vec{\mathbf{f}}) = 0$

6. a) With $\vec{\mathbf{F}} = F_1\hat{\mathbf{i}} + F_2\hat{\mathbf{j}} + F_3\hat{\mathbf{k}}$, we can write using Eq. (2.7b)

$$\vec{\nabla} \times \vec{\mathrm{F}} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}}$$

$$\therefore \vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathrm{F}}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0$$

$$\because \frac{\partial^2 F_3}{\partial x \partial y} = \frac{\partial^2 F_3}{\partial y \partial x}, \; \frac{\partial^2 F_2}{\partial x \partial z} = \frac{\partial^2 F_2}{\partial z \partial x} \text{ and } \frac{\partial^2 F_1}{\partial y \partial z} = \frac{\partial^2 F_1}{\partial z \partial y}$$

   b) We first determine $\vec{\nabla}\phi = \vec{\mathbf{F}}$. Using Eq. (1.11a) for the gradient of a scalar function with $f = \phi = xyz^2$ we can write:

$$\vec{\mathbf{F}} = \vec{\nabla}\phi = \frac{\partial}{\partial x}(xyz^2)\hat{\mathbf{i}} + \frac{\partial}{\partial y}(xyz^2)\hat{\mathbf{j}} + \frac{\partial}{\partial z}(xyz^2)\hat{\mathbf{k}}$$

$$= yz^2\hat{\mathbf{i}} + xz^2\hat{\mathbf{j}} + 2xyz\hat{\mathbf{k}}$$

   Next we find $\vec{\nabla}.\vec{\mathbf{F}}$ using Eq. (2.3):

$$\vec{\nabla}.\vec{\mathbf{F}} = \vec{\nabla}.(\vec{\nabla}\phi) = \frac{\partial}{\partial x}(yz^2) + \frac{\partial}{\partial y}(xz^2) + \frac{\partial}{\partial z}(2xyz) = 2xy \qquad \text{(i)}$$

We next determine $\nabla^2\phi$:

$$\nabla^2\phi = \frac{\partial}{\partial x^2}(xyz^2) + \frac{\partial^2}{\partial y^2}(xyz^2) + \frac{\partial^2}{\partial z^2}(xyz^2) \qquad \text{(ii)}$$

We now calculate the following partial derivatives:

$$\frac{\partial}{\partial x}(xyz^2) = yz^2; \ \frac{\partial^2}{\partial x^2}(xyz^2) = \frac{\partial}{\partial x}(yz^2) = 0 \qquad \text{(iii)}$$

$$\frac{\partial}{\partial y}(xyz^2) = xz^2; \ \frac{\partial^2}{\partial y^2}(xyz^2) = \frac{\partial}{\partial y}(xz^2) = 0. \qquad \text{(iv)}$$

and $$\frac{\partial}{\partial z}(xyz^2) = 2zxy; \frac{\partial^2}{\partial z^2}(xyz^2) = \frac{\partial}{\partial z}(2xyz) = 2xy \qquad \text{(v)}$$

Substituting from Eqs. (iii), (iv) and (v) in Eq. (ii) we get

$$\nabla^2\phi = 0 + 0 + 2xy = 2xy \qquad \text{(vi)}$$

Comparing Eqs. (i) and (vi), we can see that

$$\vec{\nabla}.(\vec{\nabla}\phi) = \nabla^2\phi = 2xy$$

Terminal Questions

ln (x y) = ln x + ln y

ln (xyz) = ln x + ln y + ln z

$$\frac{\partial}{\partial x}(\ln x) = \frac{1}{x}$$

1. i) We use Eq. (2.4) with $F_1 = x^2y^2,\ F_2 = -x^2y$

$$\vec{\nabla}.\,\vec{\mathbf{F}} = \frac{\partial}{\partial x}(x^2y^2) + \frac{\partial}{\partial y}(-x^2y) = 2xy^2 - x^2$$

Using Eq. (2.8) for the curl we get:

$$\vec{\nabla}\times\vec{\mathbf{F}} = \hat{\mathbf{k}}\left[\frac{\partial}{\partial x}(-x^2y) - \frac{\partial}{\partial y}(x^2y^2)\right] = (-2xy - 2x^2y)\,\hat{\mathbf{k}}$$

ii) We use Eq. (2.3) with $F_1 = \ln x, F_2 = \ln xy$ and $F_3 = \ln xyz$ to get

$$\vec{\nabla}.\,\vec{\mathbf{F}} = \frac{\partial}{\partial x}(\ln x) + \frac{\partial}{\partial y}(\ln x + \ln y) + \frac{\partial}{\partial z}(\ln x + \ln y + \ln z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

Using Eq. (2.7a) for the curl we get:

$$\vec{\nabla}\times\vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \ln x & \ln x + \ln y & \ln x + \ln y + \ln z \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left(\frac{1}{y}\right) + \hat{\mathbf{j}}\left(-\frac{1}{x}\right) + \hat{\mathbf{k}}\left(\frac{1}{x}\right)$$

2. Here the given field is $\vec{\mathbf{F}} = \dfrac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{r}$ where $r = (x^2 + y^2 + z^2)^{1/2}$

$$\therefore \vec{\nabla}.\vec{\mathbf{F}} = \vec{\nabla}.\left[\frac{x}{(x^2+y^2+z^2)^{1/2}}\hat{\mathbf{i}} + \frac{y}{(x^2+y^2+z^2)^{1/2}}\hat{\mathbf{j}} + \frac{z}{(x^2+y^2+z^2)^{1/2}}\hat{\mathbf{k}}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{1/2}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{1/2}}\right] + \frac{\partial}{\partial z}\left[\frac{z}{x^2+y^2+z^2)^{1/2}}\right] \quad \text{(i)}$$

Differentiate this as the product of two functions, as you have learnt to do for ordinary derivatives in your school calculus course. If you are still not clear about partial derivatives, please read the Appendix of Unit 1.

Let us evaluate each of the three partial derivatives separately.

$$\frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{1/2}}\right] = \frac{1}{(x^2+y^2+z^2)^{1/2}} - \frac{x^2}{(x^2+y^2+z^2)^{3/2}} \quad \text{(See MR)}$$

$$= \frac{(y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} \quad \text{(ii)}$$

Similarly,

$$\frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{1/2}}\right] = \frac{(x^2+z^2)}{(x^2+y^2+z^2)^{3/2}} \quad \text{(iii)}$$

and

$$\frac{\partial}{\partial z}\left[\frac{z}{(z^2+y^2+z^2)^{1/2}}\right] = \left[\frac{(x^2+y^2)}{(x^2+y^2+z^2)^{3/2}}\right] \quad \text{(iv)}$$

Substituting from Eqs. (ii), (iii) and (iv) in Eq. (i) we get

$$\vec{\nabla}.\vec{\mathbf{F}} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{(x^2+y^2+{}^2)^{1/2}} = \frac{2}{r}$$

3. The field $\vec{\mathbf{F}}$ is written in Cartesian coordinates as

$$\vec{\mathbf{F}} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2+y^2+z^2)}$$

i) For an irrotational field $\vec{\nabla}\times\vec{\mathbf{F}} = \vec{\mathbf{0}}$. Using Eq. (2.7a) we evaluate $\vec{\nabla}\times\vec{\mathbf{F}}$ as:

$$\vec{\nabla}\times\vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{(x^2+y^2+z^2)} & \dfrac{y}{(x^2+y^2+z^2)} & \dfrac{z}{(x^2+y^2+z^2)} \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left[\frac{\partial}{\partial y}\left(\frac{z}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial z}\left(\frac{y}{x^2+y^2+z^2}\right)\right]$$

$$+\hat{\mathbf{j}}\left[\frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial x}\left(\frac{z}{x^2+y^2+z^2}\right)\right]$$

$$+\hat{\mathbf{k}}\left[\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2+z^2}\right)\right]$$

$$= \hat{\mathbf{i}}\left[\frac{-2yz}{(x^2+y^2+z^2)^2}+\frac{2yz}{(x^2+y^2+z^2)^2}\right]$$

$$+\hat{\mathbf{j}}\left[\frac{-2xz}{(x^2+y^2+z^2)^2}+\frac{2xz}{(x^2+y^2+z^2)^2}\right]$$

$$+\hat{\mathbf{k}}\left[\frac{-2xy}{(x^2+y^2+z^2)^2}+\frac{2xy}{(x^2+y^2+z^2)^2}\right]$$

$$=\vec{\mathbf{0}}$$

So, the vector filed $\vec{\mathbf{F}}$ is irrotational at all points (because $\vec{\nabla}\times\vec{\mathbf{F}}$ is always zero), **except at the origin**. The field is **not defined** at the origin.

ii) To find whether the field is solenoidal, we must calculate $\vec{\nabla}.\vec{\mathbf{F}}$. Using Eq. (2.3) we get:

$$\vec{\nabla}.\vec{\mathbf{F}} = \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2+z^2}\right]+\frac{\partial}{\partial y}\left[\frac{y}{x^2+y^2+z^2}\right]+\frac{\partial}{\partial z}\left[\frac{z}{x^2+y^2+z^2}\right]$$

$$=\left[\frac{1}{(x^2+y^2+z^2)}-\frac{2x^2}{(x^2+y^2+z^2)^2}\right]$$

$$+\left[\frac{1}{(x^2+y^2+z^2)}-\frac{2y^2}{(x^2+y^2+z^2)^2}\right]$$

$$+\left[\frac{1}{(x^2+y^2+z^2)}-\frac{2z^2}{(x^2+y^2+z^2)^2}\right]$$

$$=\frac{3}{(x^2+y^2+z^2)}-\frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}$$

$$=\frac{3}{\left(x^2+y^2+z^2\right)}-\frac{2}{\left(x^2+y^2+z^2\right)}$$

$$=\frac{1}{(x^2+y^2+z^2)}$$

Since the value of $\vec{\nabla}.\vec{\mathbf{F}}$ is not zero, the field is not solenoidal.

4. To evaluate the divergence of the vector field $\vec{\mathbf{u}}=2y\,\hat{\mathbf{i}}+4z\,\hat{\mathbf{j}}+x^2z^2\,\hat{\mathbf{k}}$ we use Eq. (2.3)

$$\vec{\nabla}.\,\vec{\mathbf{u}}=\frac{\partial}{\partial x}(2y)+\frac{\partial}{\partial y}(4z)+\frac{\partial}{\partial z}\left(x^2z^2\right)$$

$$=0+0+2x^2z=2x^2z$$

At point (0, 1, 2), $\vec{\nabla}.\,\vec{\mathbf{u}}=0$.

To evaluate $\vec{\nabla}\times\vec{\mathbf{u}}$ we use Eq. (2.7a) as follows:

$$\vec{\nabla}\times\vec{\mathrm{u}}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2y & 4z & x^2z^2\end{vmatrix}=\hat{\mathbf{i}}(-4)+\hat{\mathbf{j}}\left(-2xz^2\right)+\hat{\mathbf{k}}(-2)$$

$$=-4\hat{\mathbf{i}}-2xz^2\hat{\mathbf{j}}-2\hat{\mathbf{k}}$$

At point (0, 1, 2)

$$\vec{\nabla}\times\vec{\mathbf{u}} = -4\hat{\mathbf{i}} - 2\hat{\mathbf{k}}$$

5. We can write the vector $\vec{\mathbf{a}}$ as $\vec{\mathbf{a}} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and the position vector $\vec{\mathbf{r}}$ as $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. We first calculate the cross product $\vec{\mathbf{a}}\times\vec{\mathbf{r}}$. Using Eq. (2.21b) from Unit 2, BPHCT-131, we get,

$$\vec{\mathbf{a}}\times\vec{\mathbf{r}} = \hat{\mathbf{i}}(a_2 z - a_3 y) + \hat{\mathbf{j}}(a_3 x - a_1 z) + \hat{\mathbf{k}}(a_1 y - a_2 x)$$

Using Eq. (2.7a) for the curl we get:

$$\vec{\nabla}\times(\vec{\mathbf{a}}\times\vec{\mathbf{r}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= 2a_1\hat{\mathbf{i}} + 2a_2\hat{\mathbf{j}} + 2a_3\hat{\mathbf{k}} = 2\vec{\mathbf{a}}$$

6. We use Eq. (2.8) with:

$$F_1 = \frac{-y}{(x^2+y^2)^{k/2}} \quad ; \; F_2 = \frac{x}{(x^2+y^2)^{k/2}}$$

$$\therefore \; \vec{\nabla}\times\vec{\mathbf{F}} = \hat{\mathbf{k}}\left[\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2)^{k/2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2)^{k/2}}\right)\right]$$

$$= \vec{\mathbf{k}}\left[\frac{1}{(x^2+y^2)^{k/2}} - \frac{kx^2}{(x^2+y^2)^{(k/2)+1}} + \frac{1}{(x^2+y^2)^{k/2}} - \frac{ky^2}{(x^2+y^2)^{(k/2)+1}}\right]$$

$$= \hat{\mathbf{k}}\left[\frac{2}{(x^2+y^2)^{k/2}} - \frac{k(x^2+y^2)}{(x^2+y^2)^{(k/2)+1}}\right] = \frac{(2-k)}{(x^2+y^2)^{k/2}}\hat{\mathbf{k}}$$

Now, let us find the values of k for which

(i) Curl $\hat{\mathbf{F}}$ is positive: $\vec{\nabla}\times\vec{\mathbf{F}} > 0 \Rightarrow 2-k > 0 \Rightarrow k < 2$

(ii) Curl $\vec{\mathbf{F}}$ is negative: $\vec{\nabla}\times\vec{\mathbf{F}} < 0 \Rightarrow 2-k < 0 \Rightarrow k > 2$

(iii) Curl $\vec{\mathbf{F}}$ is zero: $\nabla\times\vec{\mathbf{F}} = 0 \Rightarrow 2-k = 0 \Rightarrow k = 2$

7. Let us write an expression for $(\vec{\mathbf{A}}.\vec{\nabla})$, using the rules of the scalar product.

$$\vec{\mathbf{A}}.\vec{\nabla} = \left[2yz\hat{\mathbf{i}} - x^2 y\hat{\mathbf{j}} + xz^2\hat{\mathrm{k}}\right]\cdot\left[\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right]$$

$$= \left[2yz\frac{\partial}{\partial x} - x^2 y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}\right]$$

This is now an operator and can act of on the scalar field f.

$$(\vec{\mathbf{A}}.\vec{\nabla})f = \left[2yz\frac{\partial}{\partial x} - x^2 y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}\right]\;[xyz]$$

$$= \; 2yz\frac{\partial}{\partial x}(xyz) - x^2 y\frac{\partial}{\partial y}(xyz) + xz^2\frac{\partial}{\partial z}(xyz)$$

$$= 2yz(yz) - x^2y\,(xz) + xz^2(xy)$$

$$= 2y^2z^2 - x^3yz + x^2yz^2 \qquad \text{(i)}$$

Next we evaluate $\vec{\mathbf{A}}.\left(\vec{\nabla}f\right)$:

We first determine $\vec{\nabla}f$ using Eq. (1.11a).

$$\vec{\nabla}f = \hat{\mathbf{i}}\frac{\partial}{\partial x}(xyz) + \hat{\mathbf{j}}\frac{\partial}{\partial y}(xyz) + \hat{\mathbf{k}}\frac{\partial}{\partial z}(xyz)$$

$$= yz\,\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}.$$

$\vec{\nabla}f$ is a vector. So $\vec{\mathbf{A}}.(\vec{\nabla}f)$ is evaluated as a scalar product as follows:

$$\vec{\mathbf{A}}.(\vec{\nabla}f) = (2yz\hat{\mathbf{i}} - x^2y\hat{\mathbf{j}} + xz^2\hat{\mathbf{k}}).(yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}) \qquad \text{(ii)}$$

$$= 2y^2z^2 - x^3yz + x^2yz^2.$$

Comparing Eqs. (i) and (ii) we can see that :

$$(\vec{\mathbf{A}}.\vec{\nabla})f = \vec{\mathbf{A}}.(\vec{\nabla}f)$$

8. Let us first find $\vec{\nabla}\times\vec{\mathbf{A}}$ using Eq. (2.7a):

$$\vec{\nabla}\times\vec{\mathbf{A}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x - y + az & bx + 2y + z & x + cy - 2z \end{vmatrix}$$

$$= (c-1)\,\hat{\mathbf{i}} + (a-1)\,\hat{\mathbf{j}} + (b+1)\hat{\mathbf{k}} \qquad \text{(i)}$$

For $\vec{\mathbf{A}}$ to be irrotational, $\vec{\nabla}\times\vec{\mathbf{A}} = \vec{\mathbf{0}}$. In other words, each component of $\vec{\nabla}\times\vec{\mathbf{A}}$ is zero. So from Eq. (i) we can write:

$$(c-1) = 0,\ (a-1) = 0 \text{ and } (b+1) = 0$$

which gives us the values for a, b, c as:

$$a = 1, b = -1 \text{ and } c = 1$$

9. Let us write $\vec{\mathbf{A}} = \vec{\nabla}f$ and $\vec{\mathbf{B}} = \vec{\nabla}g$.

Then $\vec{\nabla}.(\vec{\nabla}f \times \vec{\nabla}g) = \vec{\nabla}.(\vec{\mathbf{A}}\times\vec{\mathbf{B}})$

Using Eq. (2.9e) with $\vec{\mathbf{F}} = \vec{\mathbf{A}}$ and $\vec{\mathbf{G}} = \vec{\mathbf{B}}$ we get,

$$\vec{\nabla}.(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) = \vec{\mathbf{B}}.(\nabla\times\vec{\mathbf{A}}) - \vec{\mathbf{A}}.(\nabla\times\vec{\mathbf{B}}) \qquad \text{(i)}$$

Replacing $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ by $\vec{\nabla}f$ and $\vec{\nabla}g$, respectively, in Eq. (i) we get

$$\vec{\nabla}.\,(\vec{\nabla}f \times \vec{\nabla}g) = \vec{\nabla}g.\,[\vec{\nabla}\times(\vec{\nabla}f)] - \vec{\nabla}f.[\vec{\nabla}\times(\vec{\nabla}g)] \qquad \text{(ii)}$$

But we already know that the curl of the gradient of scalar field is zero. So

$$\vec{\nabla}\times(\vec{\nabla}f) = \vec{\nabla}\times(\vec{\nabla}g) = 0 \qquad \text{(iii)}$$

Replacing from Eq. (iii) in Eq. (ii) we get: $\vec{\nabla}.(\vec{\nabla}f \times \vec{\nabla}g) = 0.$

10. i) We have defined the operator ∇^2 in Eq. (2.10a). Using that we can write:

$$\nabla^2\phi = \frac{\partial^2}{\partial x^2}\left[\ln(x^2+y^2)\right] + \frac{\partial^2}{\partial y^2}\left[\ln(x^2+y^2)\right] \qquad \text{(i)}$$

We first evaluate all the partial derivatives in Eq. (i).

$$\frac{\partial}{\partial x}\left[\ln(x^2+y^2)\right] = \frac{2x}{x^2+y^2}$$

$$\frac{\partial^2}{\partial x^2}\left[\ln(x^2+y^2)\right] = \frac{\partial}{\partial x}\left[\frac{2x}{x^2+y^2}\right] = \frac{2}{(x^2+y^2)} - \frac{4x^2}{(x^2+y^2)^2} \qquad \text{(ii)}$$

Similarly,

$$\frac{\partial}{\partial y}\left[\ln(x^2+y^2)\right] = \frac{2y}{x^2+y^2}$$

$$\frac{\partial^2}{\partial y^2}\left[\ln(x^2+y^2)\right] = \frac{\partial}{\partial y}\left[\frac{2y}{x^2+y^2}\right] = \frac{2}{(x^2+y^2)} - \frac{4y^2}{(x^2+y^2)^2} \qquad \text{(iii)}$$

Substituting from Eqs. (ii) and (iii) into Eq. (i) we get:

$$\nabla^2\phi = \frac{4}{(x^2+y^2)} - \frac{4(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{4}{(x^2+y^2)} - \frac{4}{(x^2+y^2)} = 0$$

ii) We first rewrite the function ϕ as

$$\phi = x^3yz - xy^3z + xyz^3$$

Then

$$\nabla^2\phi = \frac{\partial^2}{\partial x^2}\left[x^3yz - xy^3z + xyz^3\right] + \frac{\partial^2}{\partial y^2}\left[x^3yz - xy^3z + xyz^3\right]$$

$$+ \frac{\partial^2}{\partial z^2}\left[x^3yz - xy^3z + xyz^3\right] \qquad \text{(i)}$$

Next we evaluate all the partial derivative of ϕ in Eq. (i).

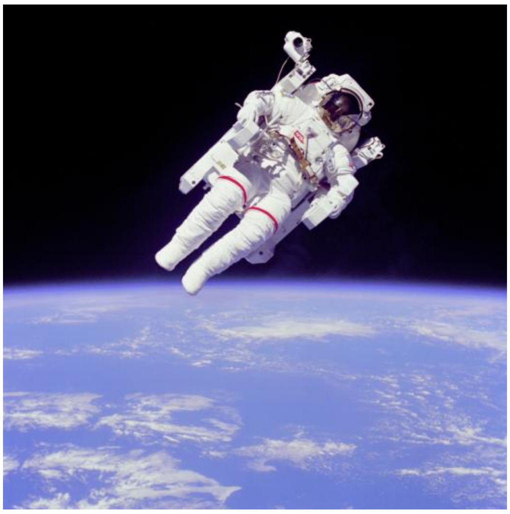
$$\frac{\partial\phi}{\partial x} = 3x^2yz - y^3z + yz^3, \quad \frac{\partial^2\phi}{\partial x^2} = 6xyz \qquad \text{(ii)}$$

$$\frac{\partial\phi}{\partial y} = x^3z - 3y^2xz + xz^3, \frac{\partial^2\phi}{\partial y^2} = -6xyz \qquad \text{(iii)}$$

$$\frac{\partial\phi}{\partial z} = x^3y - xy^3 + 3xyz^2, \frac{\partial^2\phi}{\partial z^2} = 6xyz \qquad \text{(iv)}$$

Substituting the second order partial derivatives from Eqs. (ii), (iii) and (iv) in Eq. (i), we get:

$$\therefore \quad \nabla^2\phi = 6xyz - 6xyz + 6xyz = 6xyz$$



How do we determine the work done by a variable force such as the force of gravitation? We need to solve line integrals.

UNIT 3

INTEGRATION OF VECTOR FUNCTIONS AND LINE INTEGRALS

Structure

STUDY GUIDE

In this unit, you will learn how to integrate vector functions of a scalar variable and solve line integrals. Line integrals are a generalization of ordinary integrals that you have studied in school. In order to learn these concepts better, you should revise integral calculus that you have studied in school. You must also revise the concepts of scalar and vector products, the basic concepts of vector functions of a scalar variable and how to differentiate them, all of which you have studied in Unit 2 of BPHCT-131.

"The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift ..."

Eugene Paul Wigner

3.1 INTRODUCTION

In Unit 2 of BPHCT-131 and Units 1 and 2 of this course, you have studied vector functions, scalar and vector fields, and their properties. You have learnt how to differentiate vector functions and scalar and vector fields. You have studied the concepts of the gradient of a scalar field, and the divergence and curl of vector fields. These are differential operations on scalar and vector fields that find many applications in physics. In this unit, you will learn how to determine the integrals of vector functions, and scalar and vector fields. You will also learn how to evaluate line integrals of vector fields.

There are several problems in physics where we need to calculate the integrals of vector functions and vector fields. For example, we may want to know what path a cricket ball will take after it leaves the bowler's hands with a given acceleration. Finding the path of the cricket ball involves solving a differential equation and integrating vector functions. The actual integration is essentially the same as in ordinary calculus which you have studied as a part of your school curriculum. However, integrals of vector functions and fields are different in the way in which the integrand is handled, as well as in the physical meanings of the quantities obtained. This will become clear as you study this unit.

In Sec. 3.2, you will learn how to integrate a vector function and apply it to solve some simple problems in physics. In this section you will also learn how to integrate the scalar and vector products of vector functions and some applications in physics.

In this unit you will learn how to evaluate line integrals. The line integral is a generalization of an ordinary integral over a single variable. In a line integral the path of integration is not a straight line but an arbitrary curve in space. Line integrals are used extensively in physics. One of the most important applications of the line integral is to determine the work done by a variable force. Suppose an object moves along an arbitrary curve in space, (instead of a straight line) under the action of a force. How would you calculate the work done by the force in moving the object between any two points on this path? The work done is the integral of the scalar product of the force field and an infinitesimal displacement along the path of the object. This is an example of a line integral.

In Sec. 3.3, you will learn how to evaluate line integrals in which the integrand is the scalar product of a vector field and a displacement along an arbitrary path in space. You will also study other types of line integrals of scalar and vector fields. In Sec. 3.4, you will study about conservative vector fields. You will see that line integrals can be used to define conservative force fields, an important concept in physics.

The integrals of vector functions being taken up in this unit involve integration over a single variable. In physics we often need to evaluate integrals over arbitrary surfaces and volumes. These involve integrals over two and three variables. In Unit 4, you will study about surface and volume integrals of a vector field. A brief introduction to integration over two variables is given in

Appendix A2 of this block. You should read Appendix A2 after completing your study of this unit.

Expected Learning Outcomes

After studying this unit, you should be able to:

- evaluate the integral of a vector function with respect to a scalar;
- evaluate the integrals of scalar and vector products of scalar functions; and
- evaluate line integrals of scalar and vector fields.

3.2 INTEGRATION OF A VECTOR FUNCTION

Let us begin our study by asking: **How do we integrate a vector function with respect to a scalar?**

We lay down the basic rules for the integration of a vector function with respect to a scalar. Consider a vector $\vec{\mathbf{a}}$ which is a function of a scalar *t*. Let

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}(t) = a_1(t)\hat{\mathbf{i}} + a_2(t)\hat{\mathbf{j}} + a_3(t)\hat{\mathbf{k}} \tag{3.1a}$$

where $a_1(t)$, $a_2(t)$ and $a_3(t)$ are the *x*, *y* and *z* components of $\vec{\mathbf{a}}(t)$, respectively. If

$$\frac{d\vec{\mathbf{a}}}{dt} = \vec{\mathbf{b}}(t) \tag{3.1b}$$

then the (indefinite) integral of $\vec{\mathbf{b}}(t)$ with respect to *t* is $\vec{\mathbf{a}}(t) + \vec{\mathbf{c}}$, where $\vec{\mathbf{c}}$ is an arbitrary constant vector. Symbolically, we write:

$$\int \vec{\mathbf{b}}(t)\,dt = \vec{\mathbf{a}}(t) + \vec{\mathbf{c}} \tag{3.2}$$

In physics, we deal with quantities that generally have dimensions. Therefore, $\vec{\mathbf{c}}$ is a vector whose dimension is the same as that of $\vec{\mathbf{a}}$. In a physical problem, $\vec{\mathbf{c}}$ can be determined by using given initial conditions.

You have studied integration in school and you know that integration is the reverse process of differentiation. This is also true for the integration of vector functions relative to a scalar.

In order to evaluate the integral of a vector function such as the one in Eq. (3.2), we express the vector $\vec{\mathbf{b}}$ in its component form:

$$\vec{\mathbf{b}}(t) = b_1(t)\hat{\mathbf{i}} + b_2(t)\hat{\mathbf{j}} + b_3(t)\hat{\mathbf{k}} \tag{3.3}$$

where $b_1(t)$, $b_2(t)$ and $b_3(t)$ are the *x*, *y* and *z* components of $\vec{\mathbf{b}}(t)$, respectively. We can now write the integral of the vector function $\vec{\mathbf{b}}(t)$ as:

$$\int \vec{\mathbf{b}}(t)\,dt = \hat{\mathbf{i}}\int b_1(t)\,dt + \hat{\mathbf{j}}\int b_2(t)\,dt + \hat{\mathbf{k}}\int b_3(t)\,dt \tag{3.4}$$

Note that since $\dfrac{d\vec{\mathbf{a}}}{dt} = \vec{\mathbf{b}}(t)$, we also have:

$$\frac{da_1(t)}{dt} = b_1(t), \quad \frac{da_2(t)}{dt} = b_2(t) \quad \text{and} \quad \frac{da_3(t)}{dt} = b_3(t) \tag{3.5}$$

From our knowledge of calculus, using Eq. (3.2), we can also write,

$$\int b_1(t)dt = a_1(t) + c_1, \quad \int b_2(t)dt = a_2(t) + c_2, \quad \text{and} \quad \int b_3(t)dt = a_3(t) + c_3 \qquad (3.6)$$

where c_1, c_2 and c_3 are the constants of integration.

So to evaluate $\int \vec{\mathbf{b}}(t)dt,$ we only need to integrate the scalar functions $b_1(t), b_2(t)$ and $b_3(t)$ with respect to the scalar *t,* as in ordinary calculus. Note that, we leave the unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ outside the integrals as these are constant and do not depend on *t*. In the same way, we can write the expression for the **definite** integral of a vector function in the interval $[t_1, t_2]$ as follows:

$$\int_{t_1}^{t_2} \vec{\mathbf{b}}(t)dt = \hat{\mathbf{i}} \int_{t_1}^{t_2} b_1(t)dt + \hat{\mathbf{j}} \int_{t_1}^{t_2} b_2(t)dt + \hat{\mathbf{k}} \int_{t_1}^{t_2} b_3(t)dt \qquad (3.7)$$

The integration of the two-dimensional vector function with respect to scalar is also carried out in the same way. So, let us now write down the formal definitions of the integral of a vector function $\vec{\mathbf{b}}(t)$ in two and three-dimensions:

INTEGRAL OF A VECTOR FUNCTION

1. For a vector function in three dimensions defined as $\vec{\mathbf{b}}(t) = b_1(t)\hat{\mathbf{i}} + b_2(t)\hat{\mathbf{j}} + b_3(t)\hat{\mathbf{k}}$ where $b_1(t), b_2(t)$ *and* $b_3(t)$ are continuous over the interval $[t_1, t_2]$, the **indefinite** integral of $\vec{\mathbf{b}}(t)$ with respect to *t* is given by:

$$\int \vec{\mathbf{b}}(t)dt = \hat{\mathbf{i}} \int b_1(t)dt + \hat{\mathbf{j}} \int b_2(t)dt + \hat{\mathbf{k}} \int b_3(t)dt \qquad (3.4)$$

   The definite integral of $\vec{\mathbf{b}}(t)$ over the interval $[t_1, t_2]$ is:

$$\int_{t_1}^{t_2} \vec{\mathbf{b}}(t)dt = \hat{\mathbf{i}} \int_{t_1}^{t_2} b_1(t)dt + \hat{\mathbf{j}} \int_{t_1}^{t_2} b_2(t)dt + \hat{\mathbf{k}} \int_{t_1}^{t_2} b_3(t)dt \qquad (3.7)$$

2. For a vector function in two dimensions, $\vec{\mathbf{b}}(t) = b_1(t)\hat{\mathbf{i}} + b_2(t)\hat{\mathbf{j}}$ where $b_1(t)$ and $b_2(t)$ are continuous over the interval $[t_1, t_2]$, the **indefinite** integral of $\vec{\mathbf{b}}(t)$ with respect to *t* is given by

$$\int \vec{\mathbf{b}}(t)dt = \hat{\mathbf{i}} \int b_1(t)dt + \hat{\mathbf{j}} \int b_2(t)dt \qquad (3.8)$$

   The **definite** integral of $\vec{\mathbf{b}}(t)$ with respect to *t* over the interval $[t_1, t_2]$ is

$$\int \vec{\mathbf{b}}(t)dt = \hat{\mathbf{i}} \int_{t_1}^{t_2} b_1(t)dt + \hat{\mathbf{j}} \int_{t_1}^{t_2} b_2(t)dt \qquad (3.9)$$

We now write down a few properties of the integrals of vector functions.

PROPERTIES OF INTEGRALS OF VECTOR FUNCTIONS

1. For a vector function $\vec{\mathbf{f}}(t)$ and a constant α:

$$\int \alpha \vec{\mathbf{f}}(t) dt = \alpha \int \vec{\mathbf{f}}(t) dt \quad (3.10)$$

2. For any two vector functions $\vec{\mathbf{f}}(t)$ and $\vec{\mathbf{g}}(t)$ and constants α and β:

$$\int [\alpha \vec{\mathbf{f}}(t) + \beta \vec{\mathbf{g}}(t)] dt = \alpha \int \vec{\mathbf{f}}(t) dt + \beta \int \vec{\mathbf{g}}(t) dt \quad (3.11)$$

3. For a vector function $\vec{\mathbf{f}}(t)$ and a constant vector $\vec{\mathbf{a}}$:

$$\int \vec{\mathbf{a}} . \vec{\mathbf{f}}(t) dt = \vec{\mathbf{a}} . \int \vec{\mathbf{f}}(t) dt \quad (3.12)$$

4. For a vector function $\vec{\mathbf{f}}(t)$ and a constant vector $\vec{\mathbf{a}}$:

$$\int \vec{\mathbf{a}} \times \vec{\mathbf{f}}(t) dt = \vec{\mathbf{a}} \times \int \vec{\mathbf{f}}(t) dt \quad (3.13)$$

Let us now work out a simple example on integration of vector functions.

EXAMPLE 3.1: POSITION VECTOR

Determine the position vector of a particle $\vec{\mathbf{r}}(t)$ given that its velocity function is:

$$\vec{\mathbf{v}}(t) = \sin t\, \hat{\mathbf{i}} - \cos t\, \hat{\mathbf{j}} + t^2 \hat{\mathbf{k}}$$

and the initial position of the particle (position vector of the particle at $t = 0$) is $\vec{\mathbf{r}}(t = 0) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

SOLUTION ■ Using the definition of velocity, we can write the position vector of the particle as the integral of its velocity as follows:

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}(t)}{dt} \Rightarrow \vec{\mathbf{r}}(t) = \int \vec{\mathbf{v}}(t) dt \quad (3.14)$$

We write the integral in terms of the components of the vector function $\vec{\mathbf{v}}(t)$, as defined in Eq. (3.4):

$$\vec{\mathbf{r}}(t) = \hat{\mathbf{i}} \int \sin t\, dt - \hat{\mathbf{j}} \int \cos t\, dt + \hat{\mathbf{k}} \int t^2\, dt$$

$$= -\cos t\, \hat{\mathbf{i}} - \sin t\, \hat{\mathbf{j}} + \frac{t^3}{3} \hat{\mathbf{k}} + \vec{\mathbf{C}} \quad \text{(i)}$$

where $\vec{\mathbf{C}}$ is an arbitrary constant vector.

To determine $\vec{\mathbf{C}}$ we use the given initial condition. Substituting $t = 0$ in Eq. (i) we get

$$\vec{\mathbf{r}}(t = 0) = -\hat{\mathbf{i}} + \vec{\mathbf{C}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} \quad \text{(ii)}$$

From this we get: $\vec{\mathbf{C}} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ (iii)

Substituting for $\vec{\mathbf{C}}$ in Eq. (i), we can now write the position vector as a function of time as:

$$\vec{\mathbf{r}}(t) = -\cos t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}} + \frac{t^3}{3}\hat{\mathbf{k}} + 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$= (2 - \cos t)\,\hat{\mathbf{i}} + (1 - \sin t)\,\hat{\mathbf{j}} + (1 - \frac{t^3}{3})\hat{\mathbf{k}} \qquad \text{(iv)}$$

Before we go further, let us summarize what you have studied so far:

Recap

INTEGRATION OF A VECTOR FUNCTION

- The integral of a vector function is defined as the integral of each scalar component of the function.
- This definition holds for both definite and indefinite integrals of vector functions.

You may now like to work out an SAQ on what you have studied so far.

SAQ 1 **– Integrating a vector function**

a) Evaluate $\int\left[\left(\frac{4}{1+t^2}\right)\hat{\mathbf{i}} + \left(\frac{2t}{1+t^2}\right)\hat{\mathbf{j}}\right] dt$

A table of standard integrals is given at the end of this block.

b) The acceleration of an object is $\vec{\mathbf{a}} = -10\hat{\mathbf{k}}$. Obtain its position as a function of time *t* if its initial velocity is $\vec{\mathbf{v}}(t=0) = \hat{\mathbf{i}} - \hat{\mathbf{k}}$ and its initial position is $\vec{\mathbf{r}}(t=0) = 2\hat{\mathbf{k}}$.

In Unit 2 of BPHCT-131, you have learnt that many physical quantities can be expressed as the scalar or vector products of vectors. We now study the integrals of scalar and vector products of vector functions.

3.2.1 Integrals involving Scalar and Vector Products of Vectors

Let $\vec{\mathbf{a}}(t)$ and $\vec{\mathbf{b}}(t)$ be two vector functions of a scalar *t*. Then for evaluating the integrals $I_1 = \int\left[\vec{\mathbf{a}}(t).\vec{\mathbf{b}}(t)\right]dt$ and $I_2 = \int\left[\vec{\mathbf{a}}(t)\times\vec{\mathbf{b}}(t)\right]dt$, we first compute the scalar and vector products in the integrands. Recall from Sec. 1.4 of Unit 1, BPHCT-131 that I_1 will reduce to an integral of a scalar function of *t* with respect to *t*. Similarly, I_2 will be the integral of a vector function of *t* with respect to *t*. Let us take an example to discuss the evaluation of I_1. After that you can work out another example.

EXAMPLE 3.2: INTEGRAL OF A SCALAR PRODUCT

In free space a transverse electromagnetic (EM) wave propagating in the x-direction has an electric field $\vec{\mathbf{E}} = E_0 \cos\frac{2\pi}{\lambda}(ct - x)\hat{\mathbf{j}}$ and a magnetic field $\vec{\mathbf{B}} = B_0 \cos\frac{2\pi}{\lambda}(ct - x)\hat{\mathbf{k}}$. Here c and λ are, respectively, the velocity and the wavelength of the EM wave and $E_0 = B_0 c$. The energy flowing through a volume V per unit time is given by

$$U = \frac{V}{2}(\vec{\mathbf{E}}\,.\,\vec{\mathbf{D}} + \vec{\mathbf{B}}.\,\vec{\mathbf{H}}),$$

where $\vec{\mathbf{D}} = \varepsilon_0 \vec{\mathbf{E}}$ and $\vec{\mathbf{B}} = \mu_0 \vec{\mathbf{H}}$.

Here ε_0 and μ_0 are permittivity and the magnetic permeability, respectively, of free space and $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$. Compute the total energy flowing through V in one complete cycle of EM wave if its time period is T.

SOLUTION ■ The energy flow during time dt is given by $U\,dt$. So the total energy will be the definite integral of U from $t = 0$ to $t = T$, i.e.

$$U_0 = \int_0^T U\,dt = \frac{V}{2}\int_0^T (\mathbf{E}.\mathbf{D} + \mathbf{B}.\mathbf{H})\,dt = \frac{V}{2}(I_E + I_B) \qquad \text{(i)}$$

where $I_E = \int_0^T \vec{\mathbf{E}}.\vec{\mathbf{D}}\,dt$ and $I_B = \int_0^T \vec{\mathbf{B}}.\vec{\mathbf{H}}\,dt$.

Both I_E and I_B are integrals of the type I_1. So we shall first evaluate the scalar products. Given that

$$\vec{\mathbf{E}} = E_0 \cos\frac{2\pi}{\lambda}(ct - x)\hat{\mathbf{j}} \qquad \text{(ii)}$$

$$\vec{\mathbf{D}} = \varepsilon_0 \vec{\mathbf{E}} = \varepsilon_0 E_0 \cos\frac{2\pi}{\lambda}(ct - x)\,\hat{\mathbf{j}} \qquad \text{(iii)}$$

We get $$\vec{\mathbf{E}}.\vec{\mathbf{D}} = \varepsilon_0 E_0^2 \cos^2\frac{2\pi}{\lambda}(ct - x) \qquad \text{(iv)}$$

Similarly, you can show that

$$\vec{\mathbf{B}}.\vec{\mathbf{H}} = \frac{B_0^2}{\mu_0}\cos^2\frac{2\pi}{\lambda}(ct - x) \qquad \text{(v)}$$

Substituting from Eq. (iv) and Eq. (v) into Eq. (i) we get

$$U_0 = \frac{V}{2}\left(\varepsilon_0 E_0^2 + \frac{B_0^2}{\mu_0}\right) I \qquad \text{(vi)}$$

where (see margin remark) $I = \int_0^T \cos^2\frac{2\pi}{\lambda}(ct - x)dt = \frac{T}{2}$

$\frac{2\pi c}{\lambda} = \frac{2\pi}{T}\ (\because \lambda = cT)$

$\cos^2\frac{2\pi c}{\lambda}(ct - x)$

$= \cos^2\left(\frac{2\pi t}{T} - kx\right)$,

where $k = \frac{2\pi}{\lambda}$

$= \frac{1}{2}\left\{\cos\left[2\left(\frac{2\pi t}{T} - kx\right)\right] + 1\right\}$

$\therefore \int_0^T \cos^2\frac{2\pi}{\lambda}(ct - x)dt$

$= \frac{1}{2}\int_0^T \cos\left(\frac{4\pi t}{T} - 2kx\right)dt$

$+ \frac{1}{2}\int_0^T dt$

$= \frac{1}{2}\frac{T}{4\pi}\left|\sin\left(\frac{4\pi t}{T} - 2kx\right)\right|_0^T$

$+ \frac{T}{2}$

$= \frac{T}{8\pi}[\sin(4\pi - 2kx)$

$- \sin(-2kx)] + \frac{T}{2}$

$= \frac{T}{8\pi}(-\sin 2kx + \sin 2kx) + \frac{T}{2}$

$= \frac{T}{2}$

$$\therefore \qquad U_0 = \frac{VT}{4}\left(\varepsilon_0 E_0^2 + \frac{B_0^2}{\mu_0}\right) \qquad \text{(vii)}$$

Again $B_0^2 = \frac{E_0^2}{c^2} = \varepsilon_0\mu_0 E_0^2 \left(\because c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}\right)$

$$\therefore \qquad \frac{B_0^2}{\mu_0} = \varepsilon_0 E_0^2 \qquad \text{(viii)}$$

Hence $U_0 = \frac{VT}{2}\varepsilon_0 E_0^2$

The method will be the same for integrating vector products expressed in their component form.

You may like to solve an SAQ before studying further.

***SAQ 2* - Integrals of scalar and vector products**

Given two vector functions $\vec{\mathbf{a}}(t) = t\,\hat{\mathbf{i}} + (1-t)\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$ and $\vec{\mathbf{b}}(t) = 3t^2\,\hat{\mathbf{i}} - t\,\hat{\mathbf{j}}$, evaluate the integrals:

a) $\int_0^1 \left[\vec{\mathbf{a}}(t).\,\vec{\mathbf{b}}(t)\right]dt$ and b) $\int_0^1 \left[\vec{\mathbf{a}}(t)\times\vec{\mathbf{b}}(t)\right]dt$

We now discuss line integrals of scalar and vector fields.

3.3 LINE INTEGRAL OF A VECTOR FIELD

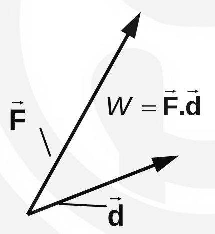


Fig. 3.1: **Work done by a force when the force and displacement are not along the same direction.**

In Unit 2 of BPHCT-131, you have studied that for a constant force, when the displacement is not along the force (Fig. 3.1), the work done is the scalar product of force and displacement:

$$W = \vec{\mathbf{F}}.\vec{\mathbf{d}} = (F\cos\theta)d \qquad (3.15)$$

In your school physics, you have learnt about work done by a constant force and variable force. You may recall that when a variable force *F(x*) is applied on an object along the *x*-axis, the work done in moving the object between any two points x_1 and x_2 is an integral given by

$$W = \int_{x_1}^{x_2} F(x)dx \qquad (3.16)$$

A well-known example of this is the work done in stretching a spring by a length *d.* The spring force is a restoring force: *F*(*x*) = – *kx*, where *k* is the spring constant. The work done is:

$$W = \int_0^d (-kx)dx \qquad (3.17)$$

Let us now consider the most general case: a **variable** force applied on an object moving along an **arbitrary path** in space. What is the work done by the force? Refer to Fig. 3.2. A planet is moving around the Sun in an elliptical orbit under the gravitational force. How will you calculate the work done for such systems?

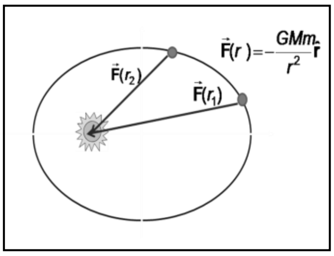


Fig. 3.2: **A planet moves around the Sun in an elliptical orbit. The force of gravitation on the planet is a variable force.**

Consider an object moving along an arbitrary path in space between the points *P* and *Q*. Note that the path is a curve and the force $\vec{\mathbf{F}} = \vec{\mathbf{F}}(x, y, z)$ is a variable force (Fig. 3.3a). Let us calculate the work done by the force in moving the object from *P* to *Q* along the path shown in Fig. 3.3a. We first divide the path *PQ* in *n* tiny segments as shown in Fig. 3.3b. We define the displacement of the object for each of these segments as $\Delta\vec{\mathbf{l}}_1, \Delta\vec{\mathbf{l}}_2, .., \Delta\vec{\mathbf{l}}_i, ... \Delta\vec{\mathbf{l}}_n$, respectively. Let $\Delta\vec{\mathbf{l}}_i$ be the displacement for the i^{th} segment. The magnitude of the displacement for each segment of the curve is almost equal to its length (read the margin remark) (inset of Fig. 3.3b).

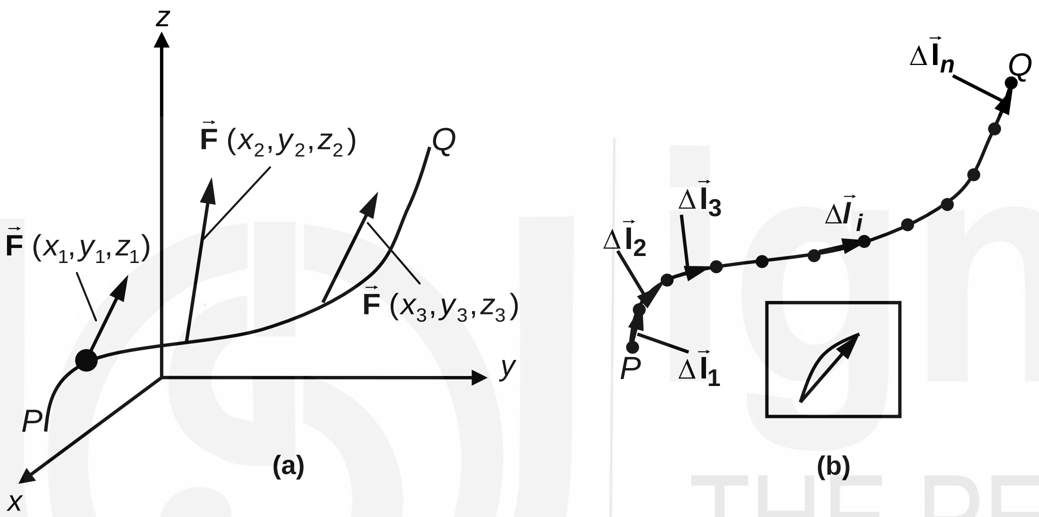


Fig. 3.3: **a) An object moves under a variable force along the path *PQ*. The force is different at different points along the path; b) the path is divided into *n* segments and the displacement is defined for each segment.**

The displacement for each segment of the path has its tail at the starting point of the segment and its head at the final point of the segment as you can see in the inset of Fig. 3.3b.

Although the force is actually different at different points of the path, we **assume** that it is **constant** over each of these segments.

Let the force acting on the object be $\vec{\mathbf{F}}_1$ for the first segment, $\vec{\mathbf{F}}_2$ for the second segment, and so on. Let us consider the *i*th segment. What is the work done by the force $\vec{\mathbf{F}}_i$ for the displacement $\Delta\vec{\mathbf{l}}_i$? From Eq. (3.15), it is $\Delta W_i = \vec{\mathbf{F}}_i . \Delta\vec{\mathbf{l}}_i$. The total work done in moving the object over the entire path is the sum of the work done in moving the object over each segment of the path. We can write it as:

$$W = \vec{\mathbf{F}}_1 . \Delta\vec{\mathbf{l}}_1 + \vec{\mathbf{F}}_2 . \Delta\vec{\mathbf{l}}_2 + ... + \vec{\mathbf{F}}_i . \Delta\vec{\mathbf{l}}_i + ... \vec{\mathbf{F}}_n . \Delta\vec{\mathbf{l}}_n = \sum_{i=1}^{n} \vec{\mathbf{F}}_i . \Delta\vec{\mathbf{l}}_i \quad (3.18a)$$

If the number of segments *n* is large, we can approximate the length of the curve by summing over the magnitude of the displacements.

In the limit as $n \to \infty$, we express the sum in Eq. (3.18a) as an **integral** along the path between *P* and *Q*:

$$W = \int_C \vec{\mathbf{F}} . d\vec{\mathbf{l}} \quad (3.18b)$$

This is an example of a **line integral** along a **path of integration** *C***.** It is the path between the points *P* and *Q* along which the object moves. It should be a

smooth curve. We will explain what is meant by a smooth curve in the next section.

Here we have defined the **line integral** in order to calculate the work done by a force field in moving an object along an arbitrary path. We can define such a line integral for any arbitrary vector field $\vec{\mathbf{A}}$ along a path of integration *C* as $\int_C \vec{\mathbf{A}}.d\vec{\mathbf{l}}$.

The line integral is a generalization of the concept of a definite integral. In a definite integral $\int_a^b f(x)\,dx$, we integrate a function *f(x)* along the *x*- axis between two points, *a* and *b.* The function is defined at every point in the interval [*a, b*]. In a line integral, we integrate along a curve *C* and the integrand ($\vec{\mathbf{F}}.d\vec{\mathbf{l}}$ in Eq. 3.18b) is a function defined at every point on the curve. Note that the path of integration can be any straight line or curve, in space or in a plane.

We now discuss how to calculate this integral. Let us write the force field $\vec{\mathbf{F}}$ in terms of its component functions as $\vec{\mathbf{F}} = F_1(x,y,z)\hat{\mathbf{i}} + F_2(x,y,z)\hat{\mathbf{j}} + F_3(x,y,z)\hat{\mathbf{k}}$, and the displacement along the path as $d\vec{\mathbf{l}} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$. The line integral of Eq. (3.18b) is then given by:

$$W = \int_C \vec{\mathbf{F}}.\,d\vec{\mathbf{l}} = \int_C [F_1 dx + F_2 dy + F_3 dz] \tag{3.19a}$$

If the force field is two-dimensional and the object is moving in the *xy* plane, we can write the line integral as:

$$W = \int_C \vec{\mathbf{F}}.\,d\vec{\mathbf{l}} = \int_C [F_1 dx + F_2 dy] \tag{3.19b}$$

Note that in general, F_1, F_2 and F_3 are functions of *x*, *y* and *z*. However, the integrals are over either *x* or *y* or *z*. Therefore, **you must express each integral in terms of a single variable.** This means, for example, to evaluate the integral $\int_C F_1(x,y,z)\,dx,$ we must express *y* and *z* in terms of *x*, so that F_1 is a function of only *x*.

This is what you will learn about in the next section.

3.3.1 Representation of a Curve

In a plane, a curve can be described by an equation of the form:

$$y = f(x) \tag{3.20a}$$

To write the equation of the circle in the form of Eq. (3.20a), we write it as:

$$y = \sqrt{a^2 - x^2}$$

For example, $y = 4x^2$ is the equation of a parabola and $x^2 + y^2 = a^2$ is the equation of a circle of radius *a* with its origin at the centre. The coordinates of a point on the curve described by Eq. (3.20a) are given by (*x, f*(*x*)).

In three-dimensional space, we may describe a curve using a set of equations

$$y = f(x); \quad z = g(x) \qquad (3.20b)$$

The coordinates of each point on the curve are $(x, f(x), g(x))$. This is also called an **explicit** representation. We may also describe the curve as an intersection of two surfaces:

$$F(x,y,z) = 0; \quad G(x,y,z) = 0 \qquad (3.20c)$$

This is called an **implicit** representation. Note that both $F(x,y,z) = 0$ and $G(x,y,z) = 0$ represent surfaces in space.

In the following example, we use the definition of line integral in Eqs. (3.19b) and the representation of a curve in a plane given by Eq. (3.20a) to calculate the work done.

Note that in all the representations of a curve, **there is only one independent variable.** This is important, because the line integral, unlike a double integral or a triple integral, is an integration over one variable.

EXAMPLE 3.3: LINE INTEGRAL OF A VECTOR FIELD IN A PLANE

Calculate the work done by a force field $\vec{\mathbf{F}} = 2xy\hat{\mathbf{i}} - y^2\hat{\mathbf{j}}$ in moving an object along the curve $y = x^2$ in the xy plane from (0,0) to (2,4).

SOLUTION ■ Using Eq. (3.19b) for the work done by a 2-dimensional force field in moving an object in the xy plane with $F_1 = 2xy$ and $F_2 = -y^2$ we can write:

$$W = \int_C \left(2xy\,dx - y^2 dy\right) \qquad \text{(i)}$$

The equation of the curve $y = x^2$ tells us how x and y are related along the path C. Using this in Eq. (i) we get:

$$W = \int_C \left[2x(x^2)dx - y^2 dy\right] \qquad \text{(ii)}$$

Since the coordinates of the initial and final points of the path are (0,0) and (2,4) we can write the limits on x and y along the path as:

$$0 \le x \le 2\,; \quad 0 \le y \le 4 \qquad \text{(iii)}$$

And the integral of Eq. (ii) reduces to:

$$W = \int_0^2 2x^3 dx - \int_0^4 y^2 dy$$

These can be evaluated as ordinary integrals:

$$\therefore \quad W = \left[\frac{2x^4}{4}\right]_0^2 - \left[\frac{y^3}{3}\right]_0^4 = -\frac{40}{3} \qquad \text{(iv)}$$

Note that each of the integrals in Eq. (ii) is over a single variable.

SAQ 3 – Work done by a force

Calculate the line integral of the force field $\vec{\mathbf{F}} = xy\hat{\mathbf{i}} + (x^2+1)\hat{\mathbf{j}}$ from (0,0) to (1,1) along the three paths labeled I, II and III in Fig. 3.4.

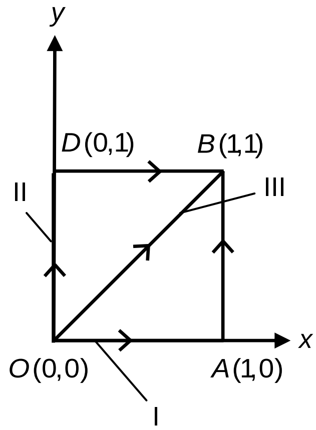


Fig. 3.4

In the next section we discuss another representation of a curve in space which is useful for evaluating line integrals.

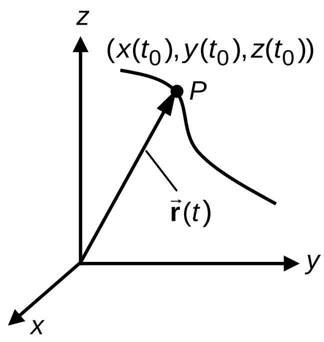


Fig. 3.5: **Parametric representation of a curve. At the point** *P*, **the value of the parameter is** t_0, **the position vector is** $\vec{\mathbf{r}}(t_0)$ **and the coordinates are** $(x(t_0), y(t_0), z(t_0))$.

3.3.2 Parametric Representation

There is yet another representation of the space curve called the **parametric** representation. In a Cartesian coordinate system, we may represent a curve using the position vector function $\vec{\mathbf{r}}(t)$ and a real parameter *t,* as follows:

$$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}} \tag{3.21a}$$

$\vec{\mathbf{r}}(t)$ is the position vector of a point on the curve, as you can see in Fig. 3.5.

As the value of *t* changes, the head of the vector traces out a curve in space.

A point on the curve has the coordinates [$x(t)$, $y(t)$, $z(t)$]. The coordinates are functions of the parameter *t* and for each value of *t*, we get a different point on the curve.

Let us now learn how to evaluate line integrals using the parametric representation of the path of integration. Sometimes, it is convenient to use the parametric representation rather than Eqs. (3.19a or 3.19b) as you will see in Example 3.4.

Let us first write down the path of integration in the parametric representation. The parametric representation of the path of integration *C* between two points *P* and *Q* (Fig. 3.6a) is,

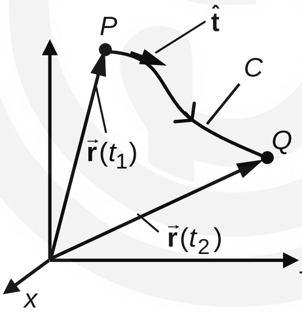
$$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}, \qquad t_1 \le t \le t_2 \tag{3.21b}$$

where t_1 and t_2 are the values of the parameter *t* at *P* and *Q*, respectively.

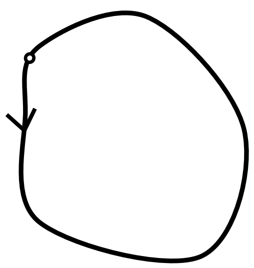
The coordinates of *P* and *Q* are *P* [$x(t_1)$, $y(t_1)$, $z(t_1)$] and *Q* [$x(t_2)$, $y(t_2)$, $z(t_2)$].

Remember that we have said earlier in this section that the path of integration in a line integral should be a **smooth** curve. You may now like to know: **When can we say that** *C* **is a smooth curve**? *C* is said to be a **smooth curve** if

- $\vec{\mathbf{r}}(t)$ as defined in Eq. (3.21b) has a continuous derivative $\vec{\mathbf{r}}'(t) = \dfrac{d\vec{\mathbf{r}}(t)}{dt}$ which is not equal to zero anywhere on *C* $(t_1 \le t \le t_2)$, and
- $\vec{\mathbf{r}}'(t)$ is directed along the tangent to the curve at every point (Fig. 3.6a).



(a)



(b)

Fig. 3.6: **a) Parametric representation of the path of integration; and b) a closed path.**

The unit tangent vector at each point on the curve is:

$$\hat{\mathbf{t}} = \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|} \tag{3.22}$$

Since we are integrating **from** *P* **to** *Q*, the path of integration also has a **specific direction** (is **oriented**). We take the direction from *P* to *Q* as the positive direction along the curve (Fig. 3.6a). We mark the positive direction on the curve by an arrow. If the path is such that the initial and final points of the curve coincide, as in Fig. 3.6b, $[\vec{\mathbf{r}}(t_1) = \vec{\mathbf{r}}(t_2)]$, then the curve is a closed curve or closed contour. When the integration is over a closed path *C*, the symbol of integration $\int_C$ is replaced by $\oint_C$.

Before you learn how to evaluate the line integral using the parametric representation, we illustrate the parametric representation of a few simple curves.

EXAMPLE 3.4: PARAMETRIC REPRESENTATION OF CURVES

Write down the parametric representation for the following:

a) A straight line between the points (0,0) and (1,2).

b) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

c) The circle $x^2+y^2=a^2$

d) A circular helix

SOLUTION ■ In all four parts, we will express the equations of the curves in terms of a single parameter, say *t*.

a) From school mathematics, you know that the equation of the straight line between any two points (x_1,y_1) and (x_2,y_2) is:

$$y-y_1=\left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1)$$

or $$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \qquad \text{(i)}$$

The LHS of Eq. (i) is a function of only *y* and the RHS is a function of only *x*. We can, therefore, equate this to a parameter *t*. Then

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}=t$$

or $$y(t)=y_1+(y_2-y_1)t \quad \text{and} \quad x(t)=x_1+(x_2-x_1)t \qquad \text{(ii)}$$

Eqs. (i) and (ii) are the parametric equations for *x* and *y*. Thus in general

$$\vec{\mathbf{r}}(t)=[x_1+(x_2-x_1)t]\,\hat{\mathbf{i}}+[y_1+(y_2-y_1)t]\,\hat{\mathbf{j}} \qquad (3.23)$$

Using $(x_1,y_1)=(0,0)$ and $(x_2,y_2)=(1,2)$ in Eq. (ii), we get

$$x(t)=t; \qquad y(t)=2t \qquad \text{(iii)}$$

To get the end points of the straight line in terms of *t*, we use Eq. (iii) as follows:

Let $t = t_1$ for the point (0, 0) and $t = t_2$ for the point (1, 2). Then since $x(t) = t$ and $y(t) = 2t$, we get

$$x_1=x(t_1)=t_1=0, \quad y_1=y(t_1)=2t_1=0 \quad \Rightarrow t_1=0$$

and $$x_2=x(t_2)=t_2=1, \quad y_2=y(t_2)=2t_2=2 \quad \Rightarrow t_2=1$$

Therefore, in terms of the parameter *t*, the initial point of the straight line is $t_1=0$ and the final point is $t_2=1$. The parametric representation of the straight line between (0,0) and (1,2) is:

$$\vec{\mathbf{r}}(t)=t\,\hat{\mathbf{i}}+2t\,\hat{\mathbf{j}}; \quad 0\le t\le 1$$

b) Note that for $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the values of both $\frac{x}{a}$ and $\frac{y}{b}$ should lie between -1 and 1. This suggests (see margin remark) that we can use the identity $\cos^2 t+\sin^2 t=1$ to write the parametric representation:

The values of sin *t* and cos *t* lie between -1 and 1.

$$\frac{x}{a} = \cos t; \qquad \frac{y}{b} = \sin t$$

$$\Rightarrow \qquad x(t) = a\cos t \quad \text{and} \quad y = b\sin t$$

So, an ellipse with its centre at the origin and semi-major and semi-minor axes *a* and *b* respectively, has the parametric representation (Fig. 3.7a):

$$\vec{\mathbf{r}}(t) = a\cos t\,\hat{\mathbf{i}} + b\sin t\,\hat{\mathbf{j}} \qquad 0 \le t < 2\pi \qquad (3.24)$$

The parameter *t* is the angle the position vector $\vec{\mathbf{r}}(t)$ makes with the *x*-axis. As *t* changes from 0 to 2π, the tip of the position vector traces the entire ellipse starting from the point *A* on the *x*-axis. The coordinate of each point on the ellipse is (*a* cos*t*, *b* sin*t*).

Note that if you want to take only a part of the ellipse, you have to choose the range of *t* accordingly. For example, for the part of ellipse in the first quadrant we write;

$$\vec{\mathbf{r}}(t) = a\cos t\,\hat{\mathbf{i}} + b\sin t\,\hat{\mathbf{j}} \qquad 0 < t < \pi/2$$

c) Substituting *a* = *b* in Eq. 3.24, we get the parametric equation of a circle $x^2 + y^2 = a^2$ (Fig. 3.7b):

$$\vec{\mathbf{r}}(t) = a\cos t\,\hat{\mathbf{i}} + a\sin t\,\hat{\mathbf{j}} \qquad 0 \le t < 2\pi \qquad (3.25)$$

The coordinate of each point on the circle is (*a* cos *t*, *a* sin *t*).

d) The parametric equation for a circular helix (Fig. 3.7c) is:

$$\vec{\mathbf{r}}(t) = a\cos t\,\hat{\mathbf{i}} + a\sin t\,\hat{\mathbf{j}} + bt\,\hat{\mathbf{k}};\ b \ne 0,\ 0 \le t \le 2\pi \qquad (3.26)$$

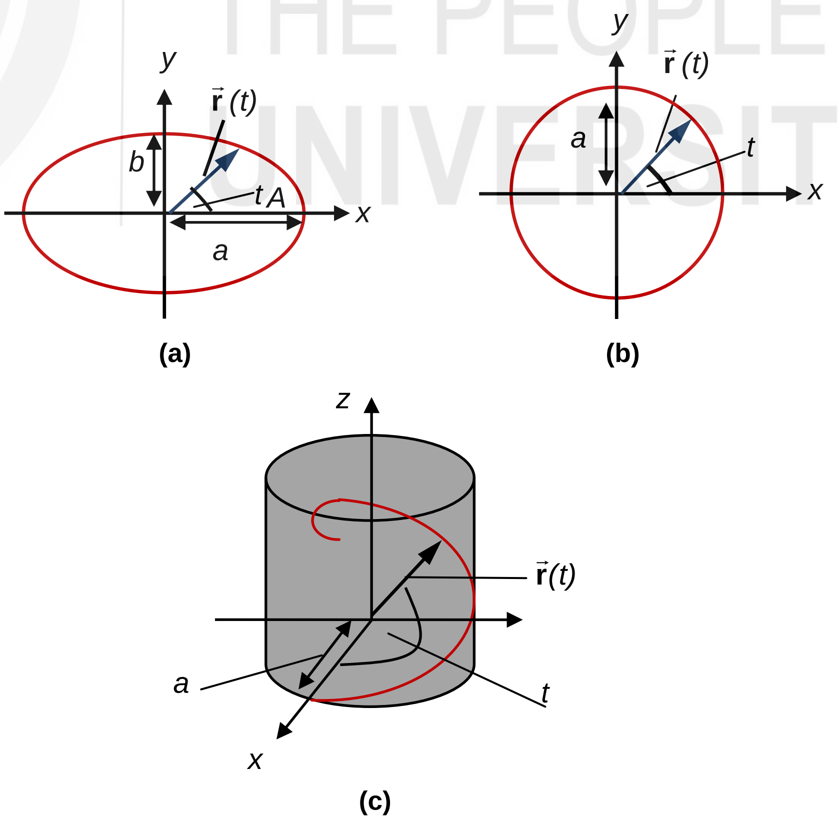


Fig. 3.7: **Parametric representation of the a) ellipse; b) circle; c) right circular helix, in which the curve lies on the cylinder** $x^2 + y^2 = a^2$.

SAQ 4 - Parametric representation of a parabola

Write down the parametric representation for the parabola $y = x^2$ between the points (0,0) and (2,4).

The parametric representation of a curve has several applications. In Mechanics the parameter t in Eq. (3.21b) may be used to represent time and we can use the vector function $\vec{\mathbf{r}}(t)$ to determine the velocity and acceleration of an object moving along a curve. We now use the parametric representation of the path of integration to define the line integral of a vector function along the path as:

$$W = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_{t_1}^{t_2} \left[\vec{\mathbf{F}}[\vec{\mathbf{r}}(t)].\frac{d\vec{\mathbf{r}}(t)}{dt}\right]dt \qquad (3.27)$$

Usually in Physics we use the symbol $\vec{\mathbf{F}}$ to denote force fields and $d\vec{\mathbf{r}}$ to indicate displacement. Here we use the $d\vec{\mathbf{l}}$ instead merely to highlight that we are talking about an infinitesimal displacement along a curve.

$\vec{\mathbf{F}}(\vec{\mathbf{r}}(t))$ is a vector function , $\vec{\mathbf{r}}(t)$ is defined in Eq. (3.21b), t_1 and t_2 are the end points of the path.

This is now the definite integral of a **scalar** function. We can write

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}\left[x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}}\right]$$

$$= \frac{dx(t)}{dt}\,\hat{\mathbf{i}} + \frac{dy(t)}{dt}\,\hat{\mathbf{j}} + \frac{dz(t)}{dt}\,\hat{\mathbf{k}} \qquad (3.28)$$

By replacing x,y,z in the vector function $\vec{\mathbf{F}} = F_1(x,y,z)\hat{\mathbf{i}} + F_2(x,y,z)\hat{\mathbf{j}} + F_3(x,y,z)\hat{\mathbf{k}}$ by the parametric functions $x = x(t)$; $y = y(t); z = z(t)$, we can write the vector function as a function of the parameter t.

Using $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = F_1(t)\hat{\mathbf{i}} + F_2(t)\,\hat{\mathbf{j}} + F_3(t)\hat{\mathbf{k}}$ (see margin remark) and Eq. (3.27) we get:

$$\int_{t_1}^{t_2}\left[\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\right]dt = \int_{t_1}^{t_2}\left[F_1(t)\frac{dx(t)}{dt} + F_2(t)\frac{dy(t)}{dt} + F_3(t)\frac{dz(t)}{dt}\right]dt \qquad (3.29a)$$

For a two-dimensional force field $\vec{\mathbf{F}} = F_1(t)\,\hat{\mathbf{i}} + F_2(t)\,\hat{\mathbf{j}}$, we can write the line integral as:

$$\int_{t_1}^{t_2}\left[\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\right]dt = \int_{t_1}^{t_2}\left[F_1(t)\frac{dx(t)}{dt} + F_2(t)\frac{dy(t)}{dt}\right]dt \qquad (3.29b)$$

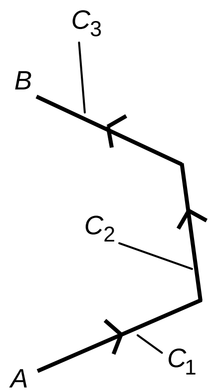


Fig. 3.8: **The curve between** A **and** B **is piecewise smooth. It is made up of the smooth curves** C_1,C_2 **and** C_3.

Note that the quantity in the bracket in Eq. (3.29b) is a scalar function of a single variable t. We can say that the integral is along the t-axis, in the direction of increasing t. It exists when C is a smooth curve or even a piecewise smooth curve. In Fig. 3.8 you can see an example of a curve which is **piecewise smooth**.

Let us now write down a formal definition of the line integral of a vector field using the parametric representation of the path of integration.

LINE INTEGRAL OF A VECTOR FIELD

If a vector field $\vec{\mathbf{F}}$ is continuous on a curve C which has a parametric representation $\vec{\mathbf{r}}(t)$ with $t_1 \le t \le t_2$ and $\vec{\mathbf{r}}(t)$ is differentiable, we define the line integral of the vector field $\vec{\mathbf{F}}$ along the curve C as:

$$W = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_C \left[\vec{\mathbf{F}}[(t)].\frac{d\vec{\mathbf{r}}(t)}{dt}\right] dt \qquad (3.30)$$

Remember that there can be more than one way of parametrizing a curve.

For example, a circle $x^2 + y^2 = a^2$ can be represented either as

$$\vec{\mathbf{r}}(t) = a\cos t\,\hat{\mathbf{i}} + a\sin t\,\hat{\mathbf{j}} \quad \text{or} \quad \vec{\mathbf{r}}(t) = a\sin t\,\hat{\mathbf{i}} + a\cos t\,\hat{\mathbf{j}}$$

The value of the line integral does not depend on the chosen parametric representation of the path of integration.

In the following example, we calculate the line integral for a two-dimensional vector field.

EXAMPLE 3.5: LINE INTEGRAL OF A VECTOR FIELD

Calculate the line integral of the vector field $\vec{\mathbf{F}}(x, y) = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$ over the curve $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}$ with $0 \le t \le \pi$.

SOLUTION ■ We use Eq. (3.30) to calculate the line integral. Let us write down the steps of this calculation.

Step 1: Calculate $\frac{d\vec{\mathbf{r}}}{dt}$.

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}[\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}] = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} \qquad \text{(i)}$$

Step 2: Write $\vec{\mathbf{F}}[\vec{\mathbf{r}}(t)]$ in terms of the parameter t.

$\vec{\mathbf{F}}$ is the vector field $\vec{\mathbf{F}}(x, y) = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$. We write $\vec{\mathbf{F}}$ in terms of the parameter t by replacing x and y in $\vec{\mathbf{F}}(x, y)$ by $x = x(t) = \cos t$, $y = y(t) = \sin t$.

$$\therefore \qquad \vec{\mathbf{F}} = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} \qquad \text{(ii)}$$

Step 3: Determine $\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}$.

Using Eqs. (i) and (ii), we can write :

$$\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt} = \left[-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}\right].\left[-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}\right] = \sin^2 t + \cos^2 t = 1 \qquad \text{(iii)}$$

Step 4: Evaluate $\int_{t_1}^{t_2}\left[\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\right]dt$.

The limits of integration are the limits of the parameter t for the path of integration. These are given as $t_1 = 0$ and $t_2 = \pi$. So using Eq. (iii), we get:

$$\int_{t_1}^{t_2}\left[\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\right]dt = \int_0^{\pi} dt = \pi$$

Let us now work out another example of a line integral of a vector field. We calculate the work done by a three-dimensional force field in moving an object along a given path.

EXAMPLE 3.6: WORK DONE BY A FORCE FIELD

Determine the work done by the force field $\vec{\mathbf{F}}(x,y,z) = xy\,\hat{\mathbf{i}} + yz\,\hat{\mathbf{j}} + zx\hat{\mathbf{k}}$ in moving an object along the curve $\vec{\mathbf{r}}(t) = t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + t^3\hat{\mathbf{k}}$ from (0,0,0) to (2,4,8).

SOLUTION ■ We use Eq. (3.29a) to calculate the work done by the force field. Comparing the expression for $\vec{\mathbf{r}}(t)$ with Eq. (3.21b), we can write:

$$x(t) = t,\;\; y(t) = t^2,\;\; z(t) = t^3 \qquad \text{(i)}$$

Note that we have to determine the limits t_1 and t_2 of t for the path of integration as these are not given in the problem. The coordinates of the starting and ending points of the path are (0,0,0) and (2,4,8). Putting these values in the parametric expressions for the coordinates in Eq. (i) we can determine t_1 and t_2 as follows:

$$x(t_1) = t_1 = 0, \quad y(t_1) = t_1^2 = 0, \quad z(t_1) = t_1^3 = 0 \Rightarrow t_1 = 0 \qquad \text{(ii)}$$

and

$$x(t_2) = t_2 = 2, \quad y(t_2) = t_2^2 = 4, \quad z(t_2) = t_2^3 = 8 \Rightarrow t_2 = 2 \qquad \text{(iii)}$$

To calculate the work done we now have to evaluate the line integral

$$W = \int_0^2 \vec{\mathbf{F}} . \frac{d\vec{\mathbf{r}}}{dt}\, dt \qquad \text{(iv)}$$

following the steps outlined in Example 3.5. Here

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}[t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t^3\hat{\mathbf{k}}] = \hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}} + 3t^2\,\hat{\mathbf{k}} \qquad \text{(v)}$$

We next write $\vec{\mathbf{F}}$ terms of the parameter t by substituting x, y, z from Eq. (i) to get:

$$\vec{\mathbf{F}}[\vec{\mathbf{r}}(t)] = t^3\,\hat{\mathbf{i}} + t^5\,\hat{\mathbf{j}} + t^4\,\hat{\mathbf{k}} \qquad \text{(vi)}$$

Using Eqs. (v) and (vi), we calculate:

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)).\frac{d\vec{\mathbf{r}}}{dt} = (t^3\,\hat{\mathbf{i}} + t^5\,\hat{\mathbf{j}} + t^4\,\hat{\mathbf{k}}).(\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}} + 3t^2\,\hat{\mathbf{k}}) = t^3 + 5t^6 \qquad \text{(vii)}$$

The work done is:

$$W = \int_0^2 \left(t^3 + 5t^6\right) dt = \left[\frac{t^4}{4} + 5\frac{t^7}{7}\right]_0^2$$

$$= \frac{668}{7}\text{ units}$$

It is convenient to use the parametric representation when the path of integration is a circle, an ellipse, a helix or a parabola. However, it is not always necessary to use a parametric representation to evaluate a line integral. In Example 3.4 the integral was evaluated using Eq. (3.19b). In some questions, as in SAQ 3, the path of integration may be along the *x, y* or *z*-axes or a combination of all these. In that case, using Eq. (3.19a or b) to evaluate the line integral will be more convenient than using Eq. (3.30).

In evaluating line integrals we can use any of the equations: 3.19a, 3.19b, 3.29a, 3.29b or 3.30.

SAQ 5 - Line integral of a vector field

Calculate the line integral of the vector field $\vec{\mathbf{F}} = -\vec{\mathbf{r}}/r^3$ along the curve $\vec{\mathbf{r}}(t) = t\hat{\mathbf{i}} + t\hat{\mathbf{j}} + t\hat{\mathbf{k}}$, with $1 \le t \le 3$.

Before you study further, you should learn some properties of line integrals.

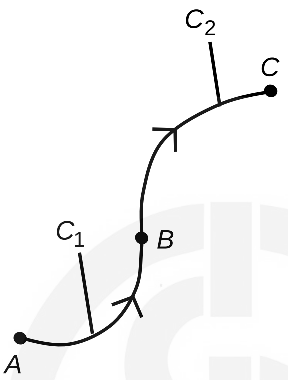


Fig. 3.9: **The curve *C* between points *A* and *C* is made up of the curves C_1 between *A* and *O* and C_2 between *O* and *C*.**

PROPERTIES OF LINE INTEGRALS

The line integral of a vector field $\vec{\mathbf{F}}$ along a curve *C* has the following general properties:

1. For a constant α,

$$\int_C \alpha \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \alpha \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} \tag{3.31}$$

2. $$\int_C \left[\vec{\mathbf{F}} + \vec{\mathbf{G}}\right].d\vec{\mathbf{l}} = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} + \int_C \vec{\mathbf{G}}.d\vec{\mathbf{l}} \tag{3.32}$$

   where $\vec{\mathbf{G}}$ is another vector field which is continuous over the curve *C*.

3. If the curve *C* is made up of two curves C_1 and C_2 as shown in Fig. 3.9, we have:

$$\int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_{C_1} \vec{\mathbf{F}}.d\vec{\mathbf{l}} + \int_{C_2} \vec{\mathbf{F}}.d\vec{\mathbf{l}} \tag{3.33}$$

   Note that the orientation of the curve is the same in all the three integrals. If the orientation of the path is reversed in any line integral, as in Fig. 3.10, the integral gets multiplied by a negative sign.

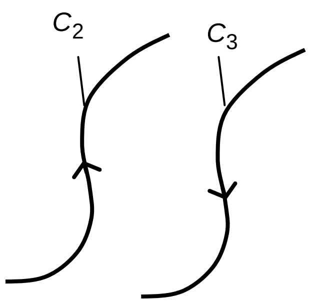


Fig. 3.10: **The line integral over the path C_2 will be the negative of the line integral over the path C_3**
$\int_{C_2} \vec{\mathbf{F}}.d\vec{\mathbf{l}} = -\int_{C_3} \vec{\mathbf{F}}.d\vec{\mathbf{l}}$

So far we have discussed line integrals of the form $\int_C \vec{\mathbf{A}}.d\vec{\mathbf{l}}$. There are other types of line integrals. Here we only state these forms.

3.3.3 Other Types of Line Integrals

There are mainly two other types of line integrals that you may need to use. These are:

i) $\int_C f\, dl$

and

ii) $\int_C \vec{\mathbf{A}} \times d\vec{\mathbf{l}}$

where f and $\vec{\mathbf{A}}$ represent a scalar and vector field, respectively. While (i) gives a scalar, (ii) gives a vector.

In the next section we discuss conservative vector fields, which are an important concept in physics. In your mechanics course BPHCT-131 you have studied about central conservative forces which are an example of a conservative vector field.

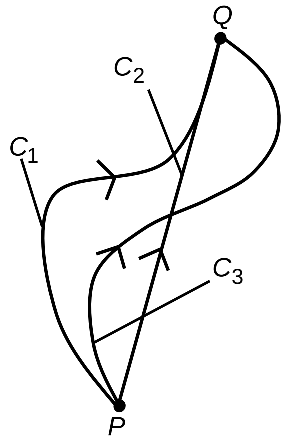


Fig. 3.11: **Three different paths of integration between two points *P* and *Q*, C_1, C_2 and C_3. If the line integral of a vector field $\vec{\mathbf{F}}$ has the same value for all these paths then $\vec{\mathbf{F}}$ is a conservative vector field.**

3.4 CONSERVATIVE VECTOR FIELDS

From the examples you have worked out so far, you have seen that the equation of the path of integration (either in a parametric form or in terms of the Cartesian coordinates) is used to evaluate the line integral. In general, then, the value of the line integral depends on the path (as in SAQ 3). However you will find that in some cases the value of the line integral of a vector field between any two points **does not depend on the path of integration between these points**. This notion of path independence of the line integral of a vector field is used to define a conservative vector field:

A vector field $\vec{\mathbf{F}}$, for which the line integral $\left(\int \vec{\mathbf{F}}.d\vec{\mathbf{l}}\right)$ between any two points *P* and *Q*, has the same value for all paths that begin at the point *P* and end at the point *Q* is called a conservative vector field.

In other words, **the line integral of a conservative force is path independent** (Fig. 3.11).

If the line integral of $\vec{\mathbf{F}}$ depends on the path between the two points, then it is called a **non-conservative** vector field.

The force of gravity is an example of a conservative force field. You know that the work done in lifting an object of mass m to a height is the same. Irrespective of the path taken, the work done is $(-mgh)$. Thus, the force of gravity is a conservative force. The electrostatic force field is also conservative, as you have also studied in Unit 10 of BPHCT-131.

There are **three different ways** of saying that a vector field $\vec{\mathbf{F}}$ is conservative. And ***all of these are equivalent to saying that the line integral of the vector field is path independent***. These are as follows:

1. The vector field can be written as the gradient of a scalar field Φ:

$$\vec{\mathbf{F}} = \vec{\nabla}\Phi \qquad (3.34)$$

2. The curl of the vector field is zero or the vector field is **irrotational**:

$$\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}} \qquad (3.35)$$

3. The line integral of the vector field along a closed path is zero:

$$\oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = 0 \qquad (3.36)$$

The line integral of a vector field over a closed path is also called a ***closed contour integral*** or a ***loop integral*.** It is denoted by a small circle superimposed on the sign of the integral as shown below:

$$\oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} \qquad (3.37)$$

For any vector field $\vec{\mathbf{F}}$ the closed contour integral along a curve *C* is also called the **circulation of the vector $\vec{\mathbf{F}}$** around the path *C*.

SAQ 6 - Circulation of a vector field

Calculate the circulation of a vector field $\vec{\mathbf{A}} = xy\,\hat{\mathbf{i}} + (3x^2 + y)\,\hat{\mathbf{j}}$ around the circle $x^2 + y^2 = 4$.

Note that we can add a constant V_0 to the scalar potential *V*, to find another potential function, $V + V_0$. This is because for any constant V_0, $\vec{\nabla} V_0 = 0$ and therefore we can write $\vec{\mathbf{F}} = -\vec{\nabla}(V + V_0)$. So the scalar potential is arbitrary up to an additive constant.

Let us now introduce another concept which is used very often in physics, that of the scalar potential associated with a conservative force.

3.4.1 Scalar Potential

In mechanics we define the potential energy as the negative of the work done in a process. For example, if we lift a mass *m* to a height *z* the work done by the force of gravity is $W = \Phi = -mgz$. However, the potential energy of the mass increases, and if the potential energy on the surface of the Earth is taken to be zero, the increase in the potential energy $V = mgz$. In other words, the potential energy is the negative of the work done. So,

$$V = -W = -\Phi = -\int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} \qquad (3.38)$$

For every conservative force $\vec{\mathbf{F}}$, we, therefore, define a function *V* which is the scalar potential function $V = -\Phi$ such that $\vec{\mathbf{F}} = -\vec{\nabla} V$.

Let us now work out an example in which we determine the scalar potential for a vector field by evaluating the line integral.

EXAMPLE 3.7: SCALAR POTENTIAL FOR A CONSERVATIVE FORCE FIELD

Determine the scalar potential for an electric field due to a point charge *q* placed at the origin.

SOLUTION ■ The electric field due to a charge *q* placed at the origin of the coordinate system at a point *P* (*x*, *y*, *z*) which is at a distance *r* from the origin is the force on the unit charge placed at that point and is given by:

$\hat{\mathbf{r}}$ is the unit vector along the position vector $\vec{\mathbf{r}}$ from the origin to the point *P*.

$$\vec{\mathbf{E}} = \frac{q}{r^2}\hat{\mathbf{r}} = \frac{q\,\vec{\mathbf{r}}}{r^3} = \frac{q(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})}{(x^2 + y^2 + z^2)^{3/2}}$$

We can check that the electric field is conservative by calculating the curl of the the field. Using Eq. (2.7a) for the curl, we get:

$$\vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{(x^2+y^2+z^2)^{3/2}} & \dfrac{y}{(x^2+y^2+z^2)^{3/2}} & \dfrac{z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left[\frac{\partial}{\partial y}\left\{\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\right\} - \frac{\partial}{\partial z}\left\{\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}\right\}\right]$$

$$+ \hat{\mathbf{j}}\left[\frac{\partial}{\partial z}\left\{\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}\right\} - \frac{\partial}{\partial x}\left\{\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\right\}\right]$$

$$+ \hat{\mathbf{k}}\left[\frac{\partial}{\partial x}\left\{\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}\right\} - \frac{\partial}{\partial y}\left\{\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}\right\}\right] \qquad \text{(i)}$$

Calculating the partial derivatives in the first term in Eq. (i) we get:

$$\frac{\partial}{\partial y}\left\{\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\right\} = -\frac{3yz}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial z}\left\{\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}\right\} = -\frac{3yz}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\therefore \quad \frac{\partial}{\partial y}\left\{\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\right\} - \frac{\partial}{\partial y}\left\{\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}\right\} = 0$$

Similarly, the remaining two terms in Eq. (i) are also zero.

$$\therefore \quad \vec{\nabla}\times\vec{\mathbf{E}} = \vec{\mathbf{0}}$$

To determine the scalar potential associated with the field we calculate the negative of the work done in bringing the unit charge from infinity to the point P, which is:

$$V = -\int_{\infty}^{r} \vec{\mathbf{E}}.\, d\vec{\mathbf{r}} = -\int_{\infty}^{r} \frac{q}{r^2}\hat{\mathbf{r}}.\, dr\,\hat{\mathbf{r}} = -\int_{\infty}^{r} \frac{q}{r^2}\, dr$$

$$= \left[\frac{q}{r}\right]_{\infty}^{r} = \frac{q}{r}$$

You will learn about electric potential in detail in Units 8 and 9.

You have seen that when a vector field is irrotational (curl of the vector field is zero), it can be written as the gradient of a scalar function, which we call the scalar potential. What if the vector field were to be solenoidal? This brings us to the concept of a vector potential, which finds many applications in Physics. Let us now study about this.

A vector field with a zero divergence is called a **solenoidal** vector field.

3.4.2 Vector Potentials

Consider a solenoidal vector field $\vec{\mathbf{F}}$. So $\vec{\nabla}.\vec{\mathbf{F}}=0$. Recall that you have studied in Unit 2 that for any vector field $\vec{\mathbf{A}}$, $\vec{\nabla}.\left(\vec{\nabla}\times\vec{\mathbf{A}}\right)=0$. Therefore we can write:

$$\vec{\nabla}.\vec{\mathbf{F}}=0 \Rightarrow \vec{\mathbf{F}}=\vec{\nabla}\times\vec{\mathbf{A}} \tag{3.39}$$

$\vec{\mathbf{A}}$ is called the **vector potential** associated with a solenoidal vector field $\vec{\mathbf{F}}$. Just as the scalar potential for a conservative field is not unique and you can add an arbitrary constant to it, similarly the vector potential for a solenoidal field is also not unique. You can add the gradient of an arbitrary function, $\vec{\nabla}f\,(x, y, z)$ to the vector potential, and the result would not change because the curl of a gradient of a scalar field is zero ($\vec{\nabla}\times(\vec{\nabla}f)=0$) .So:

$$\left[\vec{\nabla}\times\left(\vec{\mathbf{A}}+\vec{\nabla}f\right)\right]=\vec{\nabla}\times\vec{\mathbf{A}}=\vec{\mathbf{F}} \tag{3.40}$$

3.5 SUMMARY

Concept — **Description**

Integral of a vector function

- For a vector function in three dimensions defined as $\vec{\mathbf{b}}(t)=b_1(t)\hat{\mathbf{i}}+b_2(t)\hat{\mathbf{j}}+b_3(t)\hat{\mathbf{k}}$ the indefinite integral of $\vec{\mathbf{b}}(t)$ is given by:

$$\int\vec{\mathbf{b}}(t)\,dt = \hat{\mathbf{i}}\int b_1(t)\,dt + \hat{\mathbf{j}}\int b_2(t)\,dt + \hat{\mathbf{k}}\int b_3(t)\,dt$$

The definite integral of $\vec{\mathbf{b}}(t)$ over the interval $[t_1, t_2]$is:

$$\int_{t_1}^{t_2}\vec{\mathbf{b}}(t)dt = \hat{\mathbf{i}}\int_{t_1}^{t_2} b_1(t)dt + \hat{\mathbf{j}}\int_{t_1}^{t_2} b_2(t)dt + \hat{\mathbf{k}}\int_{t_1}^{t_2} b_3(t)\,dt$$

- For a vector function in two dimensions defined as $\vec{\mathbf{b}}(t)=b_1(t)\hat{\mathbf{i}}+b_2(t)\hat{\mathbf{j}}$, the indefinite integral of $\vec{\mathbf{b}}(t)$ is given by

$$\int\vec{\mathbf{b}}(t)\,dt = \hat{\mathbf{i}}\int b_1(t)\,dt + \hat{\mathbf{j}}\int b_2(t)\,dt$$

The definite integral of $\vec{\mathbf{b}}(t)$ over the interval $[t_1, t_2]$ is

$$\int_{t_1}^{t_2}\vec{\mathbf{b}}(t)dt = \hat{\mathbf{i}}\int_{t_1}^{t_2} b_1(t)dt + \hat{\mathbf{j}}\int_{t_1}^{t_2} b_2(t)dt$$

Properties of integrals of vector functions

- For any two vector functions $\vec{\mathbf{f}}(t)$ and $\vec{\mathbf{g}}(t)$ we can write

$$\int\left[\vec{\mathbf{f}}(t)+\vec{\mathbf{g}}(t)\right]dt = \int\vec{\mathbf{f}}(t)dt + \int\vec{\mathbf{g}}(t)\,dt$$

- For the product of a vector function $\vec{\mathbf{f}}(t)$ and a constant α we can write

$$\int\alpha\vec{\mathbf{f}}(t)\,dt = \alpha\int\vec{\mathbf{f}}(t)\,dt$$

- For a vector function $\vec{\mathbf{f}}(t)$ and a constant vector $\vec{\mathbf{a}}$, we can write

$$\int\vec{\mathbf{a}}.[\vec{\mathbf{f}}(t)]\,dt = \vec{\mathbf{a}}\int\vec{\mathbf{f}}(t)\,dt$$

$$\int\vec{\mathbf{a}}\times[\vec{\mathbf{f}}(t)(t)]\,dt = \vec{\mathbf{a}}\times\int\vec{\mathbf{f}}(t)(t)\,dt$$

Integrals of the scalar and vector products of vector functions

- For any two vector functions of a scalar *t*, $\vec{\mathbf{a}}(t)$ and $\vec{\mathbf{b}}(t)$, to evaluate the integrals $I_1 = \int \left[\vec{\mathbf{a}}(t).\vec{\mathbf{b}}(t)\right]dt$ and $I_2 = \int \left[\vec{\mathbf{a}}(t)\times\vec{\mathbf{b}}(t)\right]dt$, we first compute the scalar and vector products in the integrands. We then integrate the result.

Line integral

- A line integral of a scalar or a vector field is a generalization of the single integral where the path of integration may be any curve in space. It can appear in three forms:

$$\int_C f\,dl, \int_C \vec{\mathbf{A}}.d\vec{\mathbf{l}} \text{ and } \int_C \vec{\mathbf{A}}\times d\vec{\mathbf{l}}$$

Work done by a force field $\vec{\mathbf{F}}$

- The work done by the force field $\vec{\mathbf{F}}$ in moving an object along a path *C* between the points *P* and *Q* is given by the line integral

$$W = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}}$$

Line integral in the component form

- The line integral of a three-dimensional force field $\vec{\mathbf{F}} = F_1(x,y,z)\hat{\mathbf{i}} + F_2(x,y,z)\hat{\mathbf{j}} + F_3(x,y,z)\hat{\mathbf{k}}$ along a path *C* in space can be written in terms of its component functions as:

$$W = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_C [F_1dx + F_2dy + F_3dz]$$

- The line integral of a two-dimensional force field $\vec{\mathbf{F}} = F_1(x,y)\hat{\mathbf{i}} + F_2(x,y)\hat{\mathbf{j}}$ along a path *C* in the *xy* plane can be written as:

$$W = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_C [F_1dx + F_2dy]$$

Line integral of a vector field using the parametric representation of the path

- The line integral of the vector field $\vec{\mathbf{F}}$ along the curve *C* which has a parametric representation $\vec{\mathbf{r}}(t)$ with $t_1 \le t \le t_2$ where $\vec{\mathbf{r}}(t)$ is differentiable is:

$$W = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_{t_1}^{t_2}\left[\vec{\mathbf{F}}[\vec{\mathbf{r}}(t)].\frac{d\vec{\mathbf{r}}(t)}{dt}\right]dt$$

Properties of the line integral

- For a constant α,

$$\int_C \alpha\vec{\mathbf{F}}.d\vec{\mathbf{l}} = \alpha\int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}}$$

- $\int_C \left[\vec{\mathbf{F}}+\vec{\mathbf{G}}\right].d\vec{\mathbf{l}} = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} + \int_C \vec{\mathbf{G}}.d\vec{\mathbf{l}}$ for two vector fields $\vec{\mathbf{G}}$ and $\vec{\mathbf{F}}$.

- If the path of integration *C* is split into two curves C_1 and C_2

$$\int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_{C_1} \vec{\mathbf{F}}.d\vec{\mathbf{l}} + \int_{C_2} \vec{\mathbf{F}}.d\vec{\mathbf{l}}$$

- If the orientation of the path of integration is reversed in any line integral, the integral gets multiplied by a negative sign.

Circulation of a vector field

- For any vector field $\vec{\mathbf{F}}$ the closed contour integral along a curve *C* $\oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}}$ is also called the **circulation of the vector $\vec{\mathbf{F}}$** around the path *C*.

Conservative vector fields

- There are three different ways of saying that a vector field $\vec{\mathbf{F}}$ is conservative or that the line integral of the vector field is path independent:
  - The vector field can be written as the gradient of a scalar field Φ: $\vec{\mathbf{F}} = \vec{\nabla}\Phi$
  - The curl of the vector field is zero: $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$
  - The **circulation** of the vector field is zero: $\oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = 0$

3.6 TERMINAL QUESTIONS

1. Evaluate the following integrals:

   i) $I = \int_0^{\pi} \left[4\sin t\,\hat{\mathbf{i}} - \cos t\,\hat{\mathbf{j}} + (2-t)\hat{\mathbf{k}}\right] dt$

   ii) $I = \int_1^{2} \left[t^2\hat{\mathbf{i}} + te^t\,\hat{\mathbf{j}} + \ln t\,\hat{\mathbf{k}}\right] dt$

2. Obtain a function $\vec{\mathbf{a}}(t)$ which satisfies the relation $\frac{d\vec{\mathbf{a}}(t)}{dt} = \sqrt{t}\,\hat{\mathbf{i}} + (\cos \pi t)\,\hat{\mathbf{j}} + \left(\frac{4}{t}\right)\hat{\mathbf{k}}$, given that $\vec{\mathbf{a}}(1) = 2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$.

3. Evaluate $\int_1^2 \left[\vec{\mathbf{a}}(t).\frac{d\vec{\mathbf{a}}(t)}{dt}\right] dt$ given that $\vec{\mathbf{a}}(2) = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and $\vec{\mathbf{a}}(1) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$.

4. Evaluate $\int_0^1 \left[\vec{\mathbf{a}}(t) \times \frac{d^2\vec{\mathbf{a}}(t)}{dt^2}\right] dt$ given that $\vec{\mathbf{a}}(t) = 2t\,\hat{\mathbf{i}} + (1-t)\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$.

5. A two-dimensional force field is defined as $\vec{\mathbf{F}} = \frac{k\left(x\hat{\mathbf{j}} - y\hat{\mathbf{i}}\right)}{x^2 + y^2}$, where k is a constant. Compute the work done by this force in taking a particle from point $P(1,0)$ to $Q(0, 1)$ along a straight line.

6. Determine the work done by a force $\vec{\mathbf{F}} = (x - 3y)\hat{\mathbf{i}} + (2x - y)\hat{\mathbf{j}}$ in moving a particle along a curve in the xy plane given by $x = 2t; y = 3t^2$ from $t = 0$ to $t = 2$.

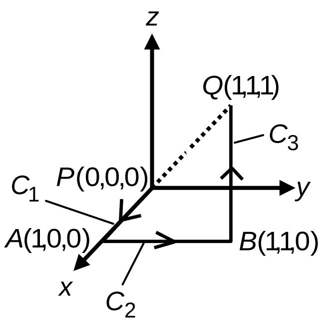


Fig. 3.12: **The path of integration between the points *P* and *Q* for TQ 7.**

7. Calculate the line integral of the vector field $\vec{\mathbf{F}} = (6x^2 + 6y)\,\hat{\mathbf{i}} - 14yz\,\hat{\mathbf{j}} + 10xz^2\,\hat{\mathbf{k}}$ over the path C ($PABQ$) between the points $P(0,0,0)$ and $Q(1,1,1)$ defined by three straight line segments PA, AB and BQ shown in Fig. 3.12.

8. An object of mass m moves along a curve $\vec{\mathbf{r}}(t) = t^2\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + \sin t\,\hat{\mathbf{k}},\ 0 \le t \le 1.$ Calculate the total force acting on the object and the work done by the force.

9. Show that the line integral of the vector field $\vec{\mathbf{A}} = (2xy + 1)\hat{\mathbf{i}} + (x^2 - 2y)\,\hat{\mathbf{j}}$ between the points (0, 0) and (2,1) is independent of the path between these points.

10. Calculate the circulation of the vector field $\vec{\mathbf{F}} = y^2\hat{\mathbf{i}} + xy\hat{\mathbf{j}}$ around the closed path along the parabola $y = 2x^2$ from (0,0) to (1,2) and back from (1, 2) to (0, 0) along the straight line $y = 2x$ as shown in Fig. 3.13.

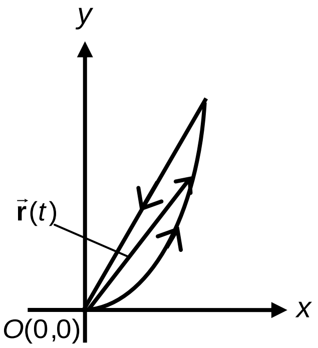


Fig. 3.13: **Figure for TQ 10.**

3.7 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. a) $I = \hat{\mathbf{i}}\int \frac{4}{1+t^2}\,dt + \hat{\mathbf{j}}\int \frac{2t}{1+t^2}dt$

$$= \left(4\tan^{-1}t\right)\hat{\mathbf{i}} + \ln\left(1+t^2\right)\hat{\mathbf{j}} + \vec{\mathbf{C}}$$

Let $u = 1 + t^2$ then

$$\frac{du}{dt} = 2t\,dt$$

and $\int \frac{2t}{1+t^2}dt = \int \frac{du}{u}$

$$\Rightarrow \ln u = \ln(1+t^2)$$

b) We use Eq. (3.4) to write down the expression for the velocity of the object as:

$$\vec{\mathbf{v}}(t) = \int \vec{\mathbf{a}}\,dt = -\int 10\hat{\mathbf{k}}\,dt = -10t\,\hat{\mathbf{k}} + \vec{\mathbf{C}}_1 \quad \text{(i)}$$

To determine $\vec{\mathbf{C}}_1$ (the constant vector) we use the initial condition on the velocity $\vec{\mathbf{v}}(t=0) = \hat{\mathbf{i}} - \hat{\mathbf{k}}$. Substituting $t = 0$ in Eq. (i) we get:

$$\vec{\mathbf{v}}(t=0) = \vec{\mathbf{C}}_1 = \hat{\mathbf{i}} - \hat{\mathbf{k}} \quad \text{(ii)}$$

Substituting for $\vec{\mathbf{C}}_1$ from Eq. (ii) into Eq. (i) we get

$$\vec{\mathbf{v}}(t) = \hat{\mathbf{i}} - (1+10t)\hat{\mathbf{k}}$$

To determine the position vector $\vec{\mathbf{r}}(t)$ we use Eq. (3.4) to write:

$$\vec{\mathbf{r}}(t) = \int \vec{\mathbf{v}}(t)\,dt = \int\left[\hat{\mathbf{i}} - (1+10t)\hat{\mathbf{k}})\right]dt$$

$$= t\,\hat{\mathbf{i}} - t\,\hat{\mathbf{k}} - 5t^2\,\hat{\mathbf{k}} + \vec{\mathbf{C}}_2 \quad \text{(iii)}$$

To evaluate $\vec{\mathbf{C}}_2$ we substitute $t = 0$ in Eq. (iii) and using the given initial position vector $\vec{\mathbf{r}}(t=0) = 2\hat{\mathbf{k}}$ we get:

$$\vec{\mathbf{r}}(t=0) = \vec{\mathbf{C}}_2 = 2\hat{\mathbf{k}} \quad \text{(iv)}$$

Substituting for $\vec{\mathbf{C}}_2$ from Eq. (iv) into Eq. (iii) we get the position vector of the object:

$$\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + (2 - t - 5t^2)\hat{\mathbf{k}}$$

2. a) $\vec{\mathbf{a}}(t).\vec{\mathbf{b}}(t) = [t\,\hat{\mathbf{i}} + (1-t)\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}]\,.\,[3t^2\hat{\mathbf{i}} - t\,\hat{\mathbf{j}}] = 3t^3 - t(1-t) = 3t^3 + t^2 - t$

$$\therefore \int_0^1 \left[\vec{\mathbf{a}}(t).\vec{\mathbf{b}}(t)\right]dt = \int_0^1 \left(3t^3 + t^2 - t\right)dt = \left[\frac{3t^4}{4} + \frac{t^3}{3} - \frac{t^2}{2}\right]_0^1 = \frac{7}{12}$$

b) $\vec{\mathbf{a}}(t) \times \vec{\mathbf{b}}(t) = \left[t\,\hat{\mathbf{i}} + (1-t)\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}\right] \times \left[3t^2\hat{\mathbf{i}} - t\,\hat{\mathbf{j}}\right] = t^3\,\hat{\mathbf{i}} + 3t^4\,\hat{\mathbf{j}} + (3t^3 - 4t^2)\hat{\mathbf{k}}$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t & (1-t) & t^2 \\ 3t^2 & -t & 0 \end{vmatrix}$$

$$\therefore \int_0^1 \left[\vec{\mathbf{a}}(t) \times \vec{\mathbf{b}}(t)\right]dt = \int_0^1 \left[t^3\,\hat{\mathbf{i}} + 3t^4\hat{\mathbf{j}} + (3t^3 - 4t^2)\hat{\mathbf{k}}\right]dt$$

$$= \left[\frac{t^4}{4}\hat{\mathbf{i}} + \frac{3t^5}{5}\hat{\mathbf{j}} + \left(\frac{3t^4}{4} - \frac{4t^3}{3}\right)\hat{\mathbf{k}}\right]_0^1 \quad \text{(i)}$$

or $\int_0^1 \left[\vec{\mathbf{a}}(t)\times\vec{\mathbf{b}}(t)\right]dt = \frac{1}{4}\hat{\mathbf{i}} + \frac{3}{5}\hat{\mathbf{j}} - \frac{7}{12}\hat{\mathbf{k}}$

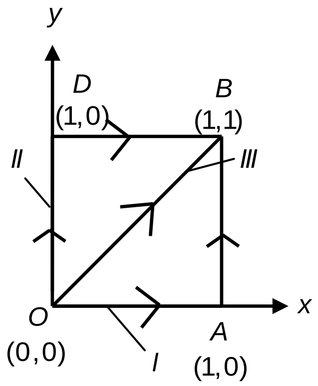


Fig. 3.14: **Path of integration for SAQ 3.**

3. We evaluate these integrals using Eq. (3.19b) with $F_1 = xy$ and $F_2 = x^2 + 1$

Along the path *I* the integral is the sum of the integrals along the straight line segments *OA* and *AB* (see Fig. 3.14):

$$I_I = \int_I \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_{OA} \vec{\mathbf{F}}.\,d\vec{\mathbf{l}} + \int_{AB} \vec{\mathbf{F}}.\,d\vec{\mathbf{l}} = \int_{OA} \left[F_x dx + F_y dy\right] + \int_{AB} \left[F_x dx + F_y dy\right]$$

$$= \int_{OA} \left[xydx + (x^2+1)dy\right] + \int_{AB} \left[xydx + (x^2+1)dy\right] \qquad \text{(i)}$$

Along *OA*,

$$0 \le x \le 1;\, y = 0 \Rightarrow dy = 0 \qquad \text{(ii)}$$

Along *AB*

$$0 \le y \le 1;\, x = 1 \Rightarrow dx = 0 \qquad \text{(iii)}$$

So substituting from Eqs. (ii) and (iii) into Eq.(i) we get

$$I_I = \int_{AB} \left[(x^2+1)dy\right] = \int_0^1 (1+1)\,dy = \left[2y\right]_0^1 = 2$$

Along the path *II* the integral is the sum of the integrals along the straight line segments *OD* and *DB*:

$$I_{II} = \int_{II} \vec{\mathbf{F}}.\,d\vec{\mathbf{l}} = \int_{OD} \vec{\mathbf{F}}.\,d\vec{\mathbf{l}} + \int_{DB} \vec{\mathbf{F}}.\,d\vec{\mathbf{l}} = \int_{OD} \left[F_x dx + F_y dy\right] + \int_{DB} \left[F_x dx + F_y dy\right]$$

$$= \int_{OD} \left[xydx + (x^2+1)dy\right] + \int_{DB} \left[xydx + (x^2+1)dy\right] \qquad \text{(iv)}$$

Along *OD*,

$$0 \le y \le 1;\, x = 0 \Rightarrow dx = 0 \qquad \text{(v)}$$

Along *DB*,

$$0 \le x \le 1;\, y = 1 \Rightarrow dy = 0 \qquad \text{(vi)}$$

So substituting from Eqs. (v) and (vi) into Eq.(iv) we get

$$I_{II} = \int_{OD} dy + \int_{DB} xdx = \int_0^1 dy + \int_0^1 xdx = \left[y\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1 = \frac{3}{2}.$$

Along the path *III* the integral is the integral along the straight line segment *OB*:

$$I_{III} = \int_{III} \vec{\mathbf{F}}.\,d\vec{\mathbf{l}} = \int_{OB} \vec{\mathbf{F}}.\,d\vec{\mathbf{l}} = \int_{OB} \left[F_x dx + F_y dy\right] = \int_{OB} \left[xydx + (x^2+1)dy\right] \qquad \text{(vii)}$$

The equation of the straight line *OB* is $y = x$. The limits on x and y are

$$0 \le x \le 1; \quad 0 \le y \le 1 \qquad \text{(viii)}$$

So substituting from Eqs. (viii) and $y = x$ into Eq.(vii) and using the methods of Example 3.3 we get:

$$I_{III} = \int_{OB} [xydx + (x^2+1)dy] = \int_0^1 x^2 dx + \int_0^1 (y^2+1)dy$$

On evaluating these integrals we get

$$I_{III} = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{y^3}{3} + y\right]_0^1 = \frac{1}{3} + \frac{4}{3} = \frac{5}{3}$$

As you can see, the value of the line integral along each of these paths is different.

4. The parametric equation of the parabola $y = x^2$ (Fig. 3.15) is:

$$x(t) = t, y(t) = t^2$$

You can check that this satisfies the equation $y = x^2$. To obtain the end points, we write

$$x(t_1) = t_1 = 0; \quad y(t_1) = t_1^2 = 0 \Rightarrow t_1 = 0$$

and

$$x(t_2) = t_2 = 2; \quad y(t_2) = t_2^2 = 4 \Rightarrow t_2 = 2$$

So the parametric representation is

$$\vec{r}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}; \qquad 0 \le t \le 2$$

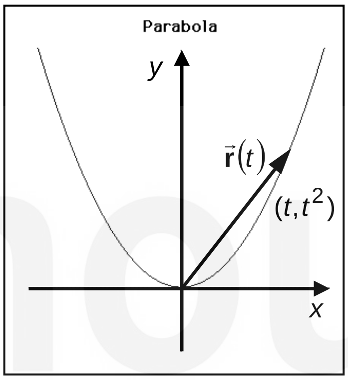


Fig. 3.15

5. We use Eq. (3.27) to evaluate the line integral with:

$$\vec{\mathbf{F}} = -\frac{\vec{\mathbf{r}}}{r^3} = -\frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}; \vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}\,; x(t) = y(t) = z(t) = t;$$

and $\quad t_1 = 1; t_2 = 3$

The derivative of $\vec{\mathbf{r}}$ is:

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}\left[t\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}\right] = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} \qquad \text{(i)}$$

In terms of t, we can write $\vec{\mathbf{F}}$ as:

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = -\frac{\left[t\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}\right]}{(t^2+t^2+t^2)^{3/2}} = -\frac{\left[t\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}\right]}{(3t^2)^{3/2}} == -\frac{1}{3\sqrt{3}\,t^2}\left[\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}\right] \text{(ii)}$$

Using the results of Eqs. (i) and (ii) in Eq. (3.27) we get:

$$I = \int_1^3 \left[\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\right] dt = -\int_1^3 \left[\frac{(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})}{3\sqrt{3}\,t^2}.(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\right] dt = -\int_1^3 \frac{1}{\sqrt{3}\,t^2} dt = -\frac{2\sqrt{3}}{9}$$

6. Using Eq. (3.25) we write the parametric equation for the circle C $x^2 + y^2 = 4$ as:

$$\vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}}, \; 0 \le t \le 2\pi \qquad \text{(i)}$$

Writing down $\vec{\mathbf{A}}$ in terms of t using $x(t) = 2\cos t; y(t) = 2\sin t$ we get:

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}(t)) = 4\cos t \sin t\,\hat{\mathbf{i}} + (12\cos^2 t + 2\sin t)\hat{\mathbf{j}} \qquad \text{(ii)}$$

$\int_0^{2\pi} \sin^2 t \cos t\, dt$

$= \int_0^0 u^2 du = 0$ (using $u = \sin t$ and $du = \cos t\, dt$)

$\int_0^{2\pi} \cos t\; dt = -\sin t\Big|_0^{2\pi}$

$= 0$

$\int_0^{2\pi} \sin t \cos t\, dt = \dfrac{\sin^2 t}{2}\Big|_0^{2\pi}$

$= 0$

Differentiating Eq. (i) w.r.t. t we get

$$\frac{d\vec{\mathbf{r}}}{dt} = -2\sin t\,\hat{\mathbf{i}} + 2\cos t\,\hat{\mathbf{j}}$$

Using Eq. (3.27), with $\vec{\mathbf{F}} = \vec{\mathbf{A}}$, we get the circulation of $\vec{\mathbf{A}}$ as (read the see margin remark):

$$\oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \int_0^{2\pi} \left[4\cos t \sin t\,\hat{\mathbf{i}} + (12\cos^2 t + 2\sin t)\hat{\mathbf{j}}\right].\left[-2\sin t\,\hat{\mathbf{i}} + 2\cos t\,\hat{\mathbf{j}}\right]dt$$

$$= \int_0^{2\pi} \left[-8\sin^2 t\cos t + 24\cos^3 t + 4\sin t\cos t\right] dt$$

$$= \int_0^{2\pi} \left[-8\sin^2 t\cos t + 24(1-\sin^2 t)\cos t + 4\sin t\cos t\right] dt$$

$$= \int_0^{2\pi} [-32\sin^2 t\cos t + 4\sin t\cos t + 24\cos t]\; dt = 0$$

$$\therefore \qquad \oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}} = 0$$

The circulation of the vector field is zero.

Terminal Questions

1. i) $I = \hat{\mathbf{i}}\int_0^{\pi} 4\sin t\, dt - \hat{\mathbf{j}}\int_0^{\pi}\cos t\, dt + +\hat{\mathbf{k}}\int_0^{\pi}(2-t)dt$

$$= \hat{\mathbf{i}}[-4\cos t]_0^{\pi} - \hat{\mathbf{j}}[\sin t]_0^{\pi} + \hat{\mathbf{k}}\left[2t - \frac{t^2}{2}\right]_0^{\pi}$$

$$= 8\,\hat{\mathbf{i}} + \left(2\pi - \frac{\pi^2}{2}\right)\hat{\mathbf{k}}$$

ii) $I = \int_1^2 \left[t^2\,\hat{\mathbf{i}} + t\,e^t\,\hat{\mathbf{j}} + \ln t\,\hat{\mathbf{k}}\right]dt$

$$= \left[\frac{t^3}{3}\right]_1^2 \hat{\mathbf{i}} + \left[te^t - e^t\right]_1^2 \hat{\mathbf{j}} + [t\ln t - t]_1^2\hat{\mathbf{k}} = \frac{7}{3}\hat{\mathbf{i}} + e^2\hat{\mathbf{j}} + [2\ln 2 - 1]\hat{\mathbf{k}}$$

2. Using Eq. (3.4) with $\vec{\mathbf{b}}(t) = \sqrt{t}\,\hat{\mathbf{i}} + (\cos\pi t)\hat{\mathbf{j}} + \left(\frac{4}{t}\right)\hat{\mathbf{k}}$ we can write:

$$\vec{\mathbf{a}}(t) = \int\left[\sqrt{t}\,\hat{\mathbf{i}} + (\cos\pi t)\,\hat{\mathbf{j}} + \left(\frac{4}{t}\right)\hat{\mathbf{k}}\right]dt + \vec{\mathbf{C}}$$

where $\vec{\mathbf{C}}$ is a constant vector. Then

$$\vec{\mathbf{a}}(t) = \frac{2}{3}t^{3/2}\,\hat{\mathbf{i}} + \frac{\sin \pi t}{\pi}\,\hat{\mathbf{j}} + 4\ln t\,\hat{\mathbf{k}} + \vec{\mathbf{C}} \qquad \text{(i)}$$

Substituting $t = 1$ in Eq. (i) and given that $\vec{\mathbf{a}}(1) = 2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ we get:

$$\vec{\mathbf{a}}(t = 1) = \frac{2}{3}\,\hat{\mathbf{i}} + \vec{\mathbf{C}} \qquad \text{(ii)}$$

$$= 2\,\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

$$\Rightarrow \qquad \vec{\mathbf{C}} = \frac{4}{3}\,\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}} \qquad \text{(iii)}$$

Substituting for $\vec{\mathbf{C}}$ in Eq. (i) we get:

$$\vec{\mathbf{a}}(t) = \frac{2}{3}(t^{3/2} + 2)\,\hat{\mathbf{i}} + \left(\frac{\sin \pi t}{\pi} + 3\right)\hat{\mathbf{j}} + (4\ln t + 4)\,\hat{\mathbf{k}}$$

3. For any vector $\vec{\mathbf{a}}(t)$ we can write:

$$\frac{d}{dt}\left[\vec{\mathbf{a}}(t).\vec{\mathbf{a}}(t)\right] = \vec{\mathbf{a}}(t).\frac{d\vec{\mathbf{a}}(t)}{dt} + \frac{d\vec{\mathbf{a}}(t)}{dt}.\vec{\mathbf{a}}(t) = 2\left[\vec{\mathbf{a}}(t).\frac{d\vec{\mathbf{a}}(t)}{dt}\right] \qquad \text{(i)}$$

or

$$\vec{\mathbf{a}}(t).\frac{d\vec{\mathbf{a}}(t)}{dt} = \frac{1}{2}\frac{d}{dt}\left[\vec{\mathbf{a}}(t).\vec{\mathbf{a}}(t)\right] \qquad \text{(ii)}$$

Then we can write:

$$\int_1^2 \left[\vec{\mathbf{a}}(t).\frac{d\vec{\mathbf{a}}(t)}{dt}\right]dt = \int_1^2 \frac{1}{2}\frac{d}{dt}(\vec{\mathbf{a}}(t).\vec{\mathbf{a}}(t))\,dt = \frac{1}{2}\int_1^2 d\left[\vec{\mathbf{a}}(t).\vec{\mathbf{a}}(t)\right] = \frac{1}{2}\left[\vec{\mathbf{a}}(t).\vec{\mathbf{a}}(t)\right]_1^2$$

Using $\vec{\mathbf{a}}(2) = 2\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}}$ and $\vec{\mathbf{a}}(1) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$, we get:

$$\int_1^2 \left[\vec{\mathbf{a}}(t).\frac{d\vec{\mathbf{a}}(t)}{dt}\right]dt = \frac{1}{2}\left[\vec{\mathbf{a}}(2).\vec{\mathbf{a}}(2) - \vec{\mathbf{a}}(1).\vec{\mathbf{a}}(1)\right] = \frac{1}{2}[29 - 27] = 1$$

4. For any vector $\vec{\mathbf{a}}(t)$ we can write:

$$\frac{d}{dt}\left[\vec{\mathbf{a}}(t)\times\frac{d\vec{\mathbf{a}}(t)}{dt}\right] = \frac{d\vec{\mathbf{a}}(t)}{dt}\times\frac{d\vec{\mathbf{a}}(t)}{dt} + \vec{\mathbf{a}}(t)\times\frac{d^2\vec{\mathbf{a}}(t)}{dt^2} = \vec{\mathbf{a}}(t)\times\frac{d^2\vec{\mathbf{a}}(t)}{dt^2} \qquad \text{(i)}$$

as $\dfrac{d\vec{\mathbf{a}}(t)}{dt}\times\dfrac{d\vec{\mathbf{a}}(t)}{dt} = \vec{\mathbf{0}}$. So we can write:

$$\vec{\mathbf{a}}(t)\times\frac{d^2\vec{\mathbf{a}}(t)}{dt^2} = \frac{d}{dt}\left[\vec{\mathbf{a}}(t)\times\frac{d\vec{\mathbf{a}}(t)}{dt}\right] \qquad \text{(ii)}$$

Therefore,

$$\int_0^1 \left[\vec{\mathbf{a}}(t)\times\frac{d^2\vec{\mathbf{a}}(t)}{dt^2}\right]dt = \int_0^1 \frac{d}{dt}\left[\vec{\mathbf{a}}(t)\times\frac{d\vec{\mathbf{a}}(t)}{dt}\right]dt = \int_0^1 d\left[\vec{\mathbf{a}}(t)\times\frac{d\vec{\mathbf{a}}(t)}{dt}\right] \qquad \text{(iii)}$$

The integral is then:

$$\int_0^1 \left[\vec{\mathbf{a}}(t)\times\frac{d^2\vec{\mathbf{a}}(t)}{dt^2}\right]dt = \left[\vec{\mathbf{a}}(t)\times\frac{d\vec{\mathbf{a}}(t)}{dt}\right]_0^1 \qquad \text{(iv)}$$

Given that $\vec{\mathbf{a}}(t) = 2t\,\hat{\mathbf{i}} + (1-t)\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$ we can write:

$$\vec{\mathbf{a}}(t)\times\frac{d\vec{\mathbf{a}}}{dt} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2t & 1-t & t^2 \\ 2 & -1 & 2t \end{vmatrix}$$

$$\frac{d\vec{\mathbf{a}}(t)}{dt} = 2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}$$

$$\vec{\mathbf{a}}(t)\times\frac{d\vec{\mathbf{a}}(t)}{dt} = \left(2t\,\hat{\mathbf{i}} + (1-t)\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}\right)\times\left(2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}\right) = (2t - t^2)\,\hat{\mathbf{i}} - 2t^2\hat{\mathbf{j}} - 2\hat{\mathbf{k}} \qquad \text{(v)}$$

$$\int_0^1 \left[\vec{\mathbf{a}}(t)\times\frac{d^2\vec{\mathbf{a}}(t)}{dt^2}\right]dt = \left[\vec{\mathbf{a}}(t)\times\frac{d\vec{\mathbf{a}}(t)}{dt}\right]_0^1$$

$$= \hat{\mathbf{i}} - 2\,\hat{\mathbf{j}}$$

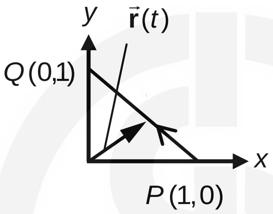


Fig. 3.16: **Figure for TQ 5.**

5. In order to evaluate the integral we have to express $d\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ as a function of the same parameter, say t. The equation of PQ (Fig. 3.16)as explained in Example 3.4 is:

$$x + y = 1 \Rightarrow y = 1 - x \qquad \text{(i)}$$

This can be expressed in the parametric form as $x(t) = t\,;\, y(t) = 1 - t$, where t goes from 1 to 0. Following the steps in Example 3.5, we first write :he position vector:

$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = t\,\hat{\mathbf{i}} + (1-t)\,\hat{\mathbf{j}} \quad \text{and} \quad \frac{d\vec{\mathbf{r}}}{dt} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$$

Next we write $\vec{\mathbf{F}} = \vec{\mathbf{F}}(t)$

$$\vec{\mathbf{F}} = k\,\frac{t\,\hat{\mathbf{j}} + (t-1)\hat{\mathbf{i}}}{t^2 + (1-t)^2}$$

$$\therefore \qquad \vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt} = \frac{k\left[(t-1)\hat{\mathbf{i}} + t\hat{\mathbf{j}}\right].\left[\hat{\mathbf{i}} - \hat{\mathbf{j}}\right]}{t^2 + (1-t)^2} = k\,\frac{(t-1) - t}{2t^2 - 2t + 1} = -\frac{k}{2t^2 - 2t + 1}$$

The work done is calculated using Eq. (3.30) as:

$$W = -k\int_1^0 \frac{dt}{2t^2 - 2t + 1}$$

$$= -\frac{k}{2}\int_1^0 \frac{dt}{t^2 - t + \frac{1}{2}} = -\frac{k}{2}\int_1^0 \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \frac{1}{4}} \qquad \text{(ii)}$$

$$= -\frac{k}{2}(-\pi) = \frac{k\pi}{2} \qquad \text{(read the margin remark)}$$

Let

$$u = t - \frac{1}{2} \Rightarrow du = dt$$

$$\therefore \int_1^0 \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \frac{1}{4}}$$

$$= \int_{1/2}^{-1/2} \frac{du}{u^2 + \frac{1}{4}}$$

$$= \left[2\tan^{-1}(2u)\right]_{1/2}^{-1/2} = -\pi$$

Alternative Method

The integral can be evaluated using Eq. (3.19b) as well, as follows:

$$F_1 = \frac{-y}{x^2+y^2}; \quad F_2 = \frac{x}{x^2+y^2}$$

as : $$\int_{PQ} \vec{\mathbf{F}}.d\vec{\mathbf{r}} = \int_{PQ}\left[-\frac{ky}{x^2+y^2}\right]dx + \int_{PQ}\left[\frac{kx}{x^2+y^2}\right]dy \qquad \text{(iii)}$$

The equation of the straight line PQ is $x + y = 1$

$$\therefore \quad y = 1 - x \text{ and } dy = -dx, \qquad \text{(iv)}$$

$$x^2 + y^2 = x^2 + (1-x)^2 = 2x^2 - 2x + 1 \qquad \text{(v)}$$

Substituting from Eqs. (iv) and (v) into Eq.(iii) we get (see margin remark):

$$\int_{PQ} \vec{\mathbf{F}}.d\vec{\mathbf{r}} = \int_{PQ} \frac{-xdx + (x-1)dx}{2x^2 - 2x + 1} = -k\int_{x=1}^{0} \frac{dx}{2x^2-2x+1} = \frac{k\pi}{2} \qquad \text{(vi)}$$

Note that the integral evaluated in Eq. (vi) is the same as the integral you evaluated in Eq. (ii).

6. We use Eq. (3.29b) to evaluate the line integral with:

$$\vec{\mathbf{F}} = (x-3y)\,\hat{\mathbf{i}} + (2x-y)\,\hat{\mathbf{j}},\ x(t) = 2t, y(t) = 3t^2,\ t_1 = 0\ ;\ t_2 = 2 \qquad \text{(i)}$$

From Eq. (i) we write:

$$x'(t) = 2,\ y'(t) = 6t \qquad \text{(ii)}$$

In terms of t, we can write the components of $\vec{\mathbf{F}}$ as:

$$F_1 = (x-3y) = 2t - 9t^2,\ F_2 = (2x-y) = 4t - 3t^2 \qquad \text{(iii)}$$

Using the results of Eqs. (i) and (ii) in Eq. (3.29b) we get:

$$I = \int_0^2 (F_1 x'(t) + F_2 y'(t))dt = \int_0^2 (4t - 18t^2 + 24t^2 - 18t^3)dt$$

$$= \left[2t^2 + 2t^3 - \frac{9t^4}{2}\right]_0^2 = -48$$

7. We calculate the line integral of the vector field using Eq. (3.19a) with: $F_x = (6x^2 + 6y),\ F_y = -14yz,\ F_z = 10xz^2$. Then

$$I = \int_C \left[(6x^2+6y)dx - (14yz)dy + (10xz^2)dz\right]$$

We use the path C between P and Q shown in Fig. 3.17. It consists of the straight line C_1 from $P(0,0,0)$ to $A(1,0,0)$, then the straight line C_2 from $A(1,0,0)$ to $B(1,1,0)$ and finally the straight line C_3 from $B(1,1,0)$ to $Q(1,1,1)$. Using the property of the line integral given in Eq. (3.33), we can write the line integral along the path C as:

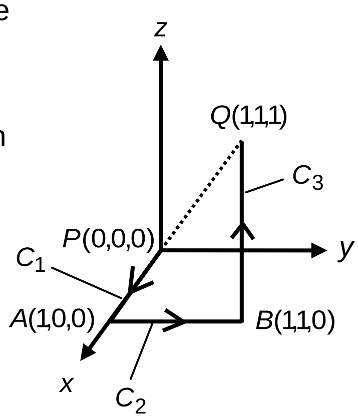


Fig. 3.17: **The path of integration between the points** ***P*** **and** ***Q*** **for TQ 7.**

$$I = I_{PA} + I_{AB} + I_{BQ}$$

$$= \int_{PA}\left[(6x^2+6y)dx - (14yz)dy + (10xz^2)dz\right]$$

$$+ \int_{AB}\left[(6x^2+6y)dx - (14yz)dy + (10xz^2)dz\right] \qquad \text{(i)}$$

$$+ \int_{BQ}\left[(6x^2+6y)dx - (14yz)dy + (10xz^2)dz\right]$$

Along PA, $0 \le x \le 1,\ y = z = 0 \Rightarrow dy = dz = 0$

$$\therefore I_{PA} = \int_{x=0}^{1} 6x^2 dx = \left[\frac{6x^3}{3}\right]_0^1 = 2 \qquad \text{(ii)}$$

Along AB: $0 \le y \le 1,\ x = 1,\ z = 0 \Rightarrow dx = dz = 0$

$$\therefore I_{AB} = -\int_{y=0}^{1} 14yz\,dy = 0 \qquad \text{(iii)}$$

Along $BQ, 0 \le z \le 1,\ x = 1, y = 1 \Rightarrow dx = dy = 0$

And $$I_{BQ} = \int_{z=0}^{1} 10xz^2 dz = \left[\frac{10z^3}{3}\right]_0^1 = \frac{10}{3} \qquad \text{(iv)}$$

$$\therefore I = 2 + 0 + \frac{10}{3} = \frac{16}{3}$$

8. We first derive an expression for the acceleration of the object: $\vec{\mathbf{a}} = \dfrac{d^2\vec{\mathbf{r}}}{dt^2}$

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}\left[t^2\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + \sin t\,\hat{\mathbf{k}}\right] = 2t\hat{\mathbf{i}} - \sin t\ \hat{\mathbf{j}} + \cos t\ \hat{\mathbf{k}} \qquad \text{(i)}$$

$$\frac{d^2\vec{\mathbf{r}}}{dt^2} = \frac{d}{dt}\left[2t\hat{\mathbf{i}} - \sin t\ \hat{\mathbf{j}} + \cos t\ \hat{\mathbf{k}}\right] = 2\hat{\mathbf{i}} - \cos t\ \hat{\mathbf{j}} - \sin t\ \hat{\mathbf{k}}$$

The force acting on the object is:

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} = m(2\hat{\mathbf{i}} - \cos t\,\hat{\mathbf{j}} - \sin t\ \hat{\mathbf{k}}) \qquad \text{(ii)}$$

Using Eq. (3.30), the work done is:

$$W = \int_0^1 \left[\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\right] dt \qquad \text{(iii)}$$

Using the results of Eqs. (i) and (ii) in Eq.(iii):

$$W = m\int_0^1 \left[2\hat{\mathbf{i}} - \cos t\,\hat{\mathbf{j}} - \sin t\,\hat{\mathbf{k}}\right].\left[2t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}} + \cos t\,\hat{\mathbf{k}}\right] dt$$

$$= m\int_0^1 \left[4t + \sin t \cos t - \sin t \cos t\ \right]\ dt = m\int_0^1 [4t]\,dt = m\left[2t^2\right]_0^1 = 2m$$

9. Refer to Fig. 3.18. Let us calculate the line integral of the field $\vec{\mathbf{A}}$ between the points $A(0,0)$ and $B(2,1)$, along two different paths: One is the straight line AB and the other is ACB. Let us first consider the path of integration AB. The equation of the straight line AB is $y = \dfrac{x}{2}$ (read the margin remark).

> The equation of a straight line between two points (x_1,y_1) and (x_2,y_2) in the xy plane is:
>
> $$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$
>
> For the line AB, we get
>
> $$y = \frac{1}{2}x$$
>
> $(\because x_1 = 0, y_1 = 0, x_2 = 2, y_2 = 1)$

We use Eq. (3.19b) for the line integral along AB with

$$\vec{\mathbf{F}} = \vec{\mathbf{A}} \quad \text{and} \quad F_1 = 2xy + 1;\ F_2 = x^2 - 2y \qquad \text{(i)}$$

We get the integral of $\vec{\mathbf{A}}$ along AB as:

$$I_{AB} = \int_{AB} \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \int_{AB} (2xy+1)dx + \int_{AB} (x^2 - 2y)dy \qquad \text{(ii)}$$

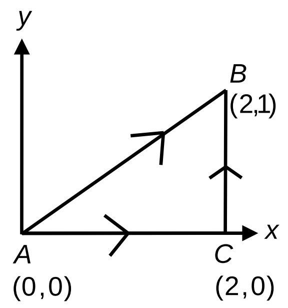


Fig. 3.18: **Paths of integration for TQ 9.**

The limits on x and y are as follows:

$$0 \le x \le 2;\ 0 \le y \le 1 \qquad \text{(iii)}$$

To evaluate the line integral over AB, we need to write each one of the integrals in Eq. (ii) as an integral over one variable. So we write (read the margin remark):

$$I_{AB} = \int_0^2 (2xy+1)dx + \int_0^1 (x^2 - 2y)dy$$

$$= \int_0^2 (x^2+1)dx + \int_0^1 (4y^2 - 2y)dy \qquad \text{(iv)}$$

$$= \left[\frac{x^3}{3} + x\right]_0^2 + \left[\frac{4y^3}{3} - y^2\right]_0^1 = 5$$

Note that the integration is along the line AB given by $y = \frac{x}{2}$ and not along the x or y axes. Therefore, when we evaluate Eq. (ii), to integrate over x, we must write y in terms of x (i.e $y = \frac{x}{2}$) in the integrand. Similarly, when we integrate over y, we write x in terms of y (i.e., $x = 2y$).

Next we evaluate the integral along ACB, which is the sum of the line integrals over AC and CB.

$$\therefore\ I_{ACB} = \int_{ACB} \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \int_{AC} \vec{\mathbf{A}}.d\vec{\mathbf{l}} + \int_{CB} \vec{\mathbf{A}}.d\vec{\mathbf{l}} \qquad \text{(v)}$$

Along AC, the value of y is a constant (y = 0) and therefore dy = 0.

$$\int_{AC} \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \int_0^2 (2xy+1)dx = \int_0^1 (2x(0)+1)dx = [x]_0^2 = 2 \qquad \text{(vi)}$$

Along CB, the value of x is constant (x = 2), so dx = 0.

$$\therefore \quad \int_{CB} \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \int_0^1 (x^2 - 2y)dy = \int_0^1 (4-2y)dy = \left[4y - y^2\right]_0^1 = 3 \qquad \text{(vii)}$$

Substituting from Eq. (vi) and (vii) into Eq. (v), we get:

$$\therefore \quad I_{ACB} = 2 + 3 = 5. \qquad \text{(viii)}$$

Since the value of the integral is same for two different paths AB and ACB, we can say that the line integral is path independent.

10. The closed path of integration C is made up of the curves C_1 and C_2 between the points O(0,0)and A (1,2) (see Fig. 3.14 reproduced here as Fig. 3.19). C_1 is described by the parabola $y = 2x^2$ between the points O and A. C_2 is the straight line $y = 2x$ from A to O, so the circulation of $\vec{\mathbf{F}}$ is:

$$I = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_{C_1} \vec{\mathbf{F}}.d\vec{\mathbf{l}} + \int_{C_2} \vec{\mathbf{F}}.d\vec{\mathbf{l}}$$

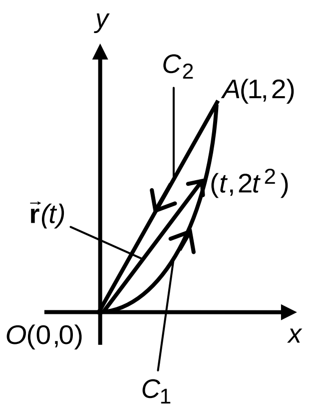


Fig. 3.19: **Figure for TQ 10.**

We parameterize the parabola $y = 2x^2$ as :

$$\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + 2t^2\,\hat{\mathbf{j}};\ x(t) = t\ ;\ y(t) = 2t^2\ ;\ 0 \le t \le 1$$

Therefore $\frac{d\vec{\mathbf{r}}}{dt} = \hat{\mathbf{i}} + 4t\,\hat{\mathbf{j}},\ \vec{\mathbf{F}} = y^2\hat{\mathbf{i}} + xy\,\hat{\mathbf{j}} = 4t^4\hat{\mathbf{i}} + 2t^3\,\hat{\mathbf{j}}$ (i)

Using Eq. (3.30) we then get:

$$I_1 = \int_{C_1}\left[\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\right]dt = \int_0^1\left[4t^4\hat{\mathbf{i}} + 2t^3\hat{\mathbf{j}}\right].\left[\hat{\mathbf{i}} + 4t\,\hat{\mathbf{j}}\right]dt = \int_0^1\left[4t^4 + 8t^4\right]dt$$

$$= \int_0^1\left[12t^4\right]dt = \left[\frac{12t^5}{5}\right]_0^1 = \frac{12}{5}$$

Here we have used the parametric representation to evaluate the integral along AO. Alternatively we can write, using Eq. (3.19b) and $y = 2x$:

$$I_2 = \int_{C_2}\left(y^2dx + xydy\right)$$

$$= \int_{C_2}\left(4x^2dx + \frac{y^2}{2}dy\right)$$

$$= \int_1^0 4x^2dx + \int_2^0 \frac{y^2}{2}dy$$

$$= -\frac{8}{3}$$

We next calculate $I_2 = \int_{C_2}\vec{\mathbf{F}}.d\vec{\mathbf{l}}$. The parametric representation for the straight line C_2 is

$$\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}};\ x(t) = t,\ y(t) = 2t,\ 1 \le t \le 0$$

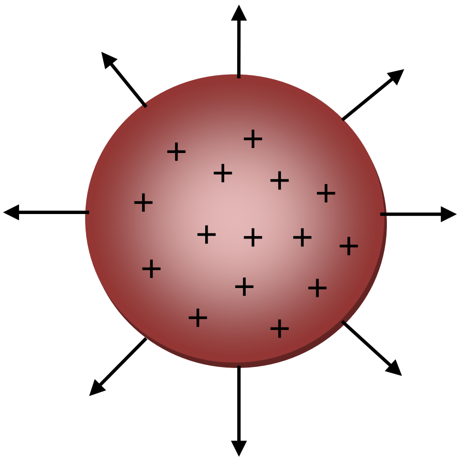
Then, $\frac{d\vec{\mathbf{r}}}{dt} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}},\ \vec{\mathbf{F}} = y^2\hat{\mathbf{i}} + xy\hat{\mathbf{j}} = 4t^2\hat{\mathbf{i}} + 2t^2\hat{\mathbf{j}}$ (ii)

Using Eq. (3.30) we get:

$$I_2 = \int_{C_2}\left[\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\right]dt = \int_1^1\left[4t^2\hat{\mathbf{i}} + 2t^2\hat{\mathbf{j}}\right].\left[\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}}\right]dt = \int_1^0\left[4t^2 + 4t^2\right]dt$$

$$= \int_1^0\left[8t^2\right]dt = \left[\frac{8t^3}{3}\right]_1^0 = -\frac{8}{3}$$

Finally, adding I_1 and I_2 we get: $I = I_1 + I_2 = \frac{12}{5} - \frac{8}{3} = -\frac{4}{15}$



How do we calculate the electric field of a spherical charge distribution? We need to solve a volume integral for this.

UNIT 4

SURFACE AND VOLUME INTEGRALS

Structure

STUDY GUIDE

In this unit, you will study surface integrals and volume integrals. You should study Appendix A2 of this block thoroughly before you start studying this unit so that you understand the methods of evaluating double integrals. Surface integrals are evaluated by reducing them to double integrals. Volume integrals are integrations over three variables. Line integrals are used in this unit in the applications of Stokes' theorem. Therefore, revise how to evaluate line integrals from Unit 3.

"Everyone now agrees that a physics lacking all connection with mathematics .... would only be an historical amusement, fitter for entertaining the idle, than occupying the mind of a philosopher."

Franz Karl Achard

4.1 INTRODUCTION

The real world is three-dimensional and as such, most physical functions depend on all the three spatial variables (x,y,z), as you have seen in Units 1 and 2. You have already studied how to integrate vector functions and fields with respect to one variable in Unit 3. However, in physics you often have to integrate functions of two and three variables, over planes and arbitrary surfaces and volumes in space. Such integrals are called multiple integrals. In this unit you will study multiple integrals and their applications in physics. You will also study two important theorems of vector integral calculus, namely, Stokes' theorem and Gauss's divergence theorem.

In Appendix A2 of this block, you have learnt how to evaluate double integrals which are integration of functions of two variables and the regions of integration are on the coordinate planes. At the beginning of this unit in Sec. 4.2, we discuss some applications of double integrals in physics, like determining the volume of objects and their centre of mass, etc.

In Unit 3, you have studied line integrals. Recall that in a line integral, the integration is over a single independent variable but the path may be an arbitrary curve in space. In Sec. 4.3 of this unit, you will study the surface integral of a vector field, in which the integration is over a two-dimensional surface in space. Surface integrals are a generalisation of double integrals. You will learn how to evaluate a special type of surface integral which is the **flux** of a vector field across a surface. This is used extensively in physics, e.g., in electromagnetic theory. You will learn about some other types of surface integrals as well. In Sec. 4.4, you will study Stokes' theorem and its applications. Stokes' theorem tells us how to transform a line integral into a surface integral and vice versa.

In Sec. 4.5, you will learn how to evaluate a volume integral in which the integrand is a function of three variables. This is the same as triple integral. In Sec. 4.6 you will study Gauss's divergence theorem and its application. The divergence theorem tells us how to transform a surface integral into a volume integral and vice versa.

With this unit we will complete our study of Vector Analysis. In the remaining blocks of the course you will study the basic principles of electricity, magnetism and electromagnetic theory, where you will use the concepts and techniques of vector analysis covered in this block.

Expected Learning Outcomes

After studying this unit, you should be able to:

- ❖ use double integrals to evaluate physical quantities;
- ❖ calculate the flux of a vector field;
- ❖ evaluate volume integrals of scalar and vector fields;
- ❖ state Stokes' theorem and Gauss's divergence theorem and write them in a mathematical form; and
- ❖ solve problems based on these theorems and apply them to simple physical situations.

4.2 APPLICATIONS OF DOUBLE INTEGRALS

In Appendix A2 you have studied that a double integral can be used to determine the area of a region and volume of a solid. In the example below, you will use the techniques for evaluating double integrals explained in A2.2 and A2.3 to calculate area and volume.

EXAMPLE 4.1 : AREA AND VOLUME USING DOUBLE INTEGRALS

i) Determine the area of the region R on xy plane bounded by the curves $y = x + 2$ and $y = x^2$ by evaluating a double integral.

ii) Calculate the volume of the solid below the surface defined by the function $f(x, y) = 4 + \cos x + \cos y$, above the region R on the xy plane (z = 0), bounded by the curves $x = 0, x = \pi, y = 0$ and $y = \pi$ by evaluating a double integral.

SOLUTION ■ i) To determine the area of the region R, we have to evaluate $\iint_R dx\, dy$ where R is the region bounded by the curves $y = x + 2$ and $y = x^2$ (Eq. A2.7). To carry out the double integration we first obtain the limits of integration for the variables x and y in the region R.

To obtain the bounds (limits) on x, we solve the system of equations $y = x^2$ and $y = x + 2$, to get

$$x^2 = x + 2 \Rightarrow x = -1, 2$$

The region of integration R is then defined by the conditions $x^2 \le y \le x + 2,\ -1 \le x \le 2$ (read the margin remark) and we write

Note that for y we write $x^2 \le y \le x + 2$, and not $x + 2 \le y \le x^2$. This is because in the range $-1 \le x \le 2,$, $x^2 \le x + 2$.

$$\text{Area of } R = \int_{-1}^{2}\left[\int_{x^2}^{x+2} dy\right] dx = \int_{-1}^{2} [y]_{x^2}^{x+2}\, dx$$

$$= \int_{-1}^{2} [x + 2 - x^2]\, dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$

ii) The volume of the solid bound by the surface $f(x, y) = 4 + \cos x + \cos y$ and the region R defined by $0 \le x \le \pi;\ \ 0 \le y \le \pi$ is (Eq. A2.3)

$$V = \int_0^{\pi}\int_0^{\pi} (4 + \cos x + \cos y)\, dy\, dx \qquad \text{(i)}$$

Integrating (i) over y we get:

$$= \int_0^{\pi} [4y + y\cos x + \sin y]_0^{\pi}\, dx = \int_0^{\pi} [4\pi + \pi\cos x]_0^{\pi}\, dx \qquad \text{(ii)}$$

Next, integrating over x, we get

$$= [4\pi x + \pi\sin x]_0^{\pi} = 4\pi^2 \qquad \text{(iii)}$$

SAQ 1 - Determining area and volume using double integrals

a) Calculate the area of the region R bounded by the curves $y = x^2$ and $y = x^3$ for $x > 0;\ y > 0$.

b) Find the volume of the solid that lies below the surface of the curve $f(x,y) = x^4$ and above the region in the xy plane bounded by the curves $y = x^2$ and $y = 1$.

In physics, we also use double integrals to calculate several other quantities. We could use the double integral to determine the mass of an object like a planar lamina with a density function. We can also find the centre of mass of a laminar object or its moment of inertia about an arbitrary axis.

Before you solve an example on the applications of double integrals, let us summarize some important applications:

Recap

APPLICATIONS OF DOUBLE INTEGRALS

- Centre of mass (x_{cm}, y_{cm}) of a body with a density $\gamma(x,y)$ over a region R

$$x_{cm} = \frac{\iint_R x\,\gamma(x,y)\,dx\,dy}{m};\quad y_{cm} = \frac{\iint_R y\,\gamma(x,y)\,dx\,dy}{m} \qquad (4.1)$$

- Mass m of a body with a density (mass/area) $\gamma(x,y)$ over a region R

$$m = \iint_R \gamma(x,y)\,dx\,dy \qquad (4.2)$$

- Moment of inertia of a body with a density $\gamma(x,y)$ over a region R about the x-axis, I_x and the y-axis I_y

$$I_x = \iint_R y^2\,\gamma(x,y)\,dx\,dy;\quad I_y = \iint_R x^2\,\gamma(x,y)\,dx\,dy \qquad (4.3)$$

- The average value μ of a continuous function $f(x,y)$ over a closed region R in the xy-plane is:

$$\mu = \frac{\iint_R f(x,y)\,dx\,dy}{\iint_R dx\,dy};\quad \iint_R dx\,dy = \text{Area of the region of integration } R \qquad (4.4)$$

We study one of these applications in the following example, where we determine the mass of an object using double integrals.

EXAMPLE 4.2 : APPLICATION OF DOUBLE INTEGRAL

A rectangular plate covers the region $0 \le x \le 4;\ 0 \le y \le 3$ and has the mass density $\gamma(x,y) = x + y$. Calculate the mass of the plate.

SOLUTION ■ We use Eq. (4.2) to determine the mass of the body with the density function $\gamma(x,y) = x + y$. R is defined by the equations $0 \le x \le 4;\ 0 \le y \le 3,$. So the mass

$$m = \iint_R (x+y)\,dx\,dy = \int_{x=0}^{4}\int_{y=0}^{3}(x+y)\,dxdy$$

$$= \left[\int_{x=0}^{4} x dx\right]\left[\int_{y=0}^{3} dy\right] + \left[\int_{x=0}^{4} dx\right]\left[\int_{y=0}^{3} y dy\right]$$

$$= \left[\frac{x^2}{2}\right]_0^4 [y]_0^3 + [x]_0^4 \left[\frac{y^2}{2}\right]_0^3$$

$$= 42 \text{ units}$$

In the following example we study one more application of a double integral in physics.

EXAMPLE 4.3 : AVERAGE VALUE USING DOUBLE INTEGRALS

The temperature distribution at a point on a flat rectangular metal plate is $T(x,y) = 20 - 4x^2 - y^2$ °C. Calculate the average temperature on the plate, if the dimensions of the plate are described by $0 \le x \le 2;\ 0 \le y \le 1$.

SOLUTION ■ Using Eq. (4.4) we can write the average temperature on the plate as:

$$T_{avg} = \frac{\iint_R T(x,y)\,dx\,dy}{\iint_R dx\,dy}; \quad R:\ 0 \le x \le 2\,;\, 0 \le y \le 1 \qquad \text{(i)}$$

Note that $\iint_R dx\,dy = \text{Area of the rectangular plate} = 2 \text{ units}$. To evaluate the integral in the numerator of Eq. (i), we write:

$$\iint_R T(x,y)\,dx\,dy = \int_{x=0}^{2}\int_{y=0}^{1}\left(20 - 4x^2 - y^2\right) dxdy \qquad \text{(ii)}$$

$$= 20\int_{x=0}^{2}\int_{y=0}^{1} dxdy - 4\int_{x=0}^{2}\int_{y=0}^{1} x^2 dxdy - \int_{x=0}^{2}\int_{y=0}^{1} y^2\,dxdy$$

$$= 20\left[\int_{x=0}^{2}\int_{y=0}^{1} dxdy\right] - 4\left[\int_{x=0}^{2} x^2 dx\right]\left[\int_{y=0}^{1} dy\right] - \left[\int_{x=0}^{2} dx\right]\left[\int_{y=0}^{1} y^2\,dxdy\right] \text{(iii)}$$

$$= 20[2] - 4\left[\frac{x^3}{3}\right]_0^2 [y]_0^1 - [x]_0^2 \left[\frac{y^3}{3}\right]_0^1 = \frac{86}{3}$$

Using Eqs. (i) and (iii), the average temperature is:

$$T_{avg} = \frac{43}{3}\,°\text{C}$$

In the next section, you will study surface integrals of vector fields. Just as line integrals are integrals along a curve, for surface integrals the region of integration is a surface. Surface integrals have several applications in physics.

4.3 SURFACE INTEGRALS

In physics, we come across many types of surface integrals. The commonest example of a surface integral is that of flux. You may recall the concept of electromagnetic induction from school physics. If we move a bar magnet M towards a circular coil C (Fig. 4.1), we know that an electromotive force is induced in the coil. This happens because the **magnetic flux** linked with the coil changes with time. The question is: How do we calculate the magnetic flux linked with the coil at a particular position?

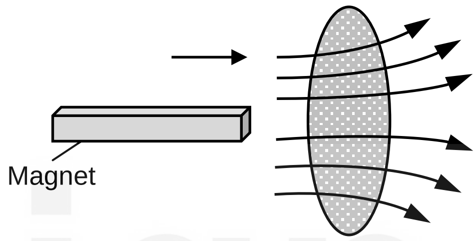


Fig. 4.1: **Magnetic flux.**

To determine the magnetic flux, we have to **integrat**e the magnetic field vector over the area enclosed by the coil. It is given by

$$\phi_B = \iint_S \vec{\mathbf{B}} . d\vec{\mathbf{S}} \qquad (4.5)$$

Here $\vec{\mathbf{B}}$ is the magnetic field due to the magnet at the position of the element of area $d\vec{\mathbf{S}}$ of the coil. Here S is the area of the coil (the shaded region in Fig. 4.1).

This type of integral is called a **surface integral**. This involves the integral of a vector field over a surface. This is one type of surface integral. You will come across different types of surface integrals in physics as given below.

Types of Surface Integrals

Analogous to line integrals, surface integrals may appear in the following different forms:

i) $$\iint_S \phi \, d\vec{\mathbf{S}} \qquad (4.6a)$$

ii) $$\iint_S \vec{\mathbf{A}} . d\vec{\mathbf{S}} \qquad (4.6b)$$

iii) $$\iint_S \vec{\mathbf{A}} \times d\vec{\mathbf{S}} \qquad (4.6c)$$

where ϕ is a scalar field and $\vec{\mathbf{A}}$, a vector field.

Type (ii) is the most common form of surface integrals in physics. In this unit, we focus on this type of surface integral. It is the flux of vector field $\vec{\mathbf{A}}$ through surface S.

4.3.1 Flux of a Vector Field

Let us consider a region of space in which we have a constant vector field $\vec{\mathbf{A}}(x,y,z) = A_0\hat{\mathbf{i}}$. Recall what you have studied about flux in Unit 1 of BPHCT-131. You saw that the flux of rainwater can be expressed as a scalar product of the vector field representing the flow of rain and an area vector representing the top surface of the bucket.

The word *flux* is derived from the Latin word "fluxus" which means *flow.* The concept of flux is easier to understand in the context of fluid flow. You can of course determine the flux of any vector field.

Let us now learn how, in general, the flux of any vector field can be written as a surface integral. Suppose that $\vec{\mathbf{A}}$ is a vector field associated with fluid flowing through any region. Let the magnitude A_0 of the vector field be the amount of fluid that crosses unit area in unit time. Then **by definition, the flux of the field $\vec{\mathbf{A}}$ through any area is the amount of fluid that flows through that area in unit time**.

Since the loop (in *yz* plane) in Fig. 4.2a is perpendicular to fluid flow (along *x*-axis), fluid flows through it. Since the loop in Fig. 4.2b is parallel to the fluid flow, no fluid flows through it.

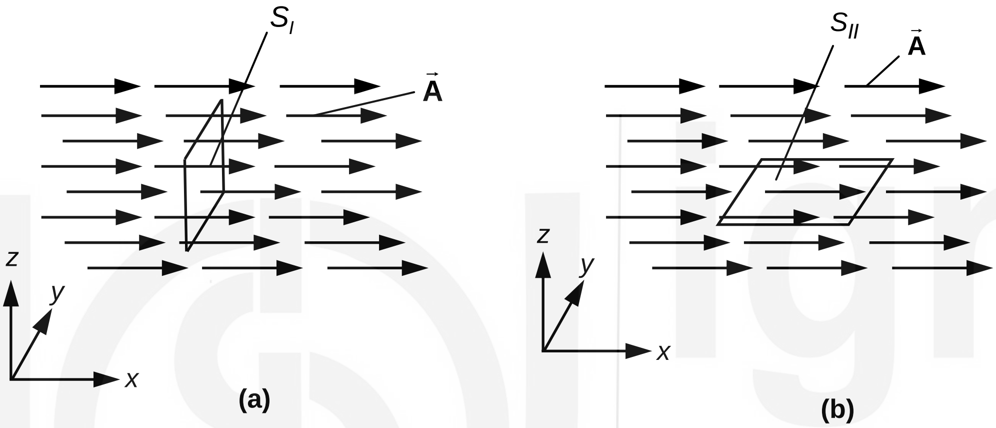


Fig. 4.2: **Flux of a vector field $\vec{\mathbf{A}}$ $(= A_x\,\hat{\mathbf{i}})$ through a surface a)** S_{I} **perpendicular to $\vec{\mathbf{A}}$; b)** S_{II} **parallel to it.**

Thus, the flux of $\vec{\mathbf{A}}$ through an imaginary square loop of area $(S = a^2)$ placed in the *yz* plane (Fig. 4.2a) is defined as

$$\Phi_{\text{I}} = A_0\, a^2 \qquad (4.7)$$

The flux of $\vec{\mathbf{A}}$ through the same area element placed in the *xy* plane (Fig. 4.2b) is

$$\Phi_{\text{II}} = 0 \qquad (4.8)$$

What happens if this imaginary loop is placed at an arbitrary angle to $\vec{\mathbf{A}}$ (Fig. 4.3a)? That is, it is neither parallel nor perpendicular to the flow.

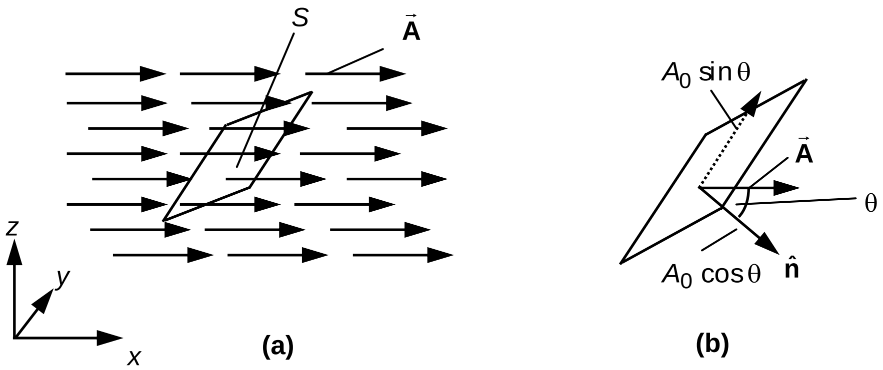


Fig. 4.3: **Flux of a vector field $\vec{\mathbf{A}}$ through a surface** *S*. **The normal to the surface makes an angle** θ **with the vector field.**

Let θ be the angle between the normal $\hat{\mathbf{n}}$ to the area element and the vector field $\vec{\mathbf{A}}$ (Fig. 4.3b). We can resolve the vector field $\vec{\mathbf{A}}$ into two components,

- one perpendicular to *S*: $(A_0 \cos\theta)$ and
- one parallel to it: $(A_0 \sin\theta)$.

The angle between $\hat{\mathbf{n}}$ and $\vec{\mathbf{A}}$ is θ. So $\vec{\mathbf{A}}.\hat{\mathbf{n}} = A_0 \cos\theta$, using the definition of the scalar product. Note that if we draw unit normal vectors to the surfaces S_I and S_{II} as well and use the expression $(\vec{\mathbf{A}}.\hat{\mathbf{n}})a^2$ we can get back Eqs. (4.7 and 4.8) because $\theta = 0$ for S_I and $\theta = \pi/2$ for S_{II}.

The only contribution to the flux is from the component of the field which is perpendicular to *S*, i.e. $A_0 \cos\theta$. So the flux is

$$\Phi_{III} = a^2 A_0 \cos\theta$$

In vector notation, we can write this flux as the following scalar product:

$$\Phi = (\vec{\mathbf{A}}.\hat{\mathbf{n}})S \qquad (4.9)$$

where $\hat{\mathbf{n}}$ is the unit normal to the surface *S* (Fig. 4.3b).

We can write the area itself in terms of the normal vector $\hat{\mathbf{n}}$ as $\vec{\mathbf{S}} = S\hat{\mathbf{n}}$. Then, the flux Φ of the vector field $\vec{\mathbf{A}}$ is:

$$\Phi = \vec{\mathbf{A}}.\vec{\mathbf{S}} \qquad (4.10)$$

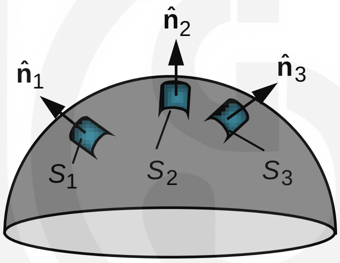


Fig. 4.4: **The unit normals to the three surface elements, S_1, S_2 and S_3 are $\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2$ and $\hat{\mathbf{n}}_3$.**

Here both the vector field $\vec{\mathbf{A}}$ and the unit normal are constant over the entire area element $(\vec{\mathbf{S}})$ over which we are defining the flux of the vector field. In general, the vector field may be a function of position (*x, y, z*). Also the surface itself may not be a plane, so the unit normal would point in different directions at different points on the surface. For example consider a part of the surface of a sphere (Fig. 4.4). In Fig. 4.4, we show the normal to this surface at different points. Note that their directions are different. How do we determine the flux in such cases?

This is where we need the concept of a surface integral.

4.3.2 Flux of a Vector Field as a Surface Integral

Let us determine the flux of a vector field $\vec{\mathbf{A}}$ (*x, y, z*) over the surface *S* shown in Fig. 4.5.

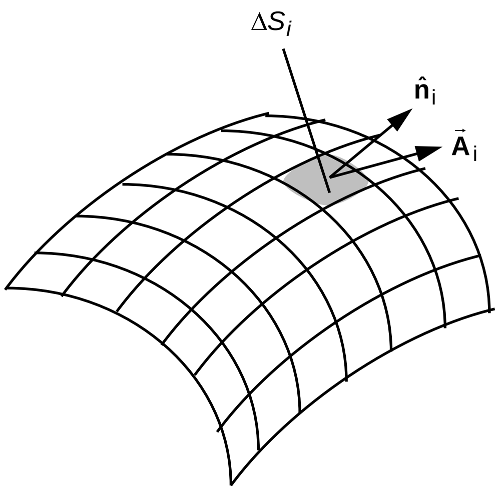


Fig. 4.5: **A surface *S* divided into *n* tiny area elements. The area of the i^{th} element is ΔS_i, it has a unit normal $\hat{\mathbf{n}}_i$ and the vector field over this area element is a constant equal to $\vec{\mathbf{A}}_i$.**

We carry out the following steps:

1. We divide the surface into n tiny elements of area. The i^{th} area element is $\Delta\vec{\mathbf{S}}_i = \Delta S_i \hat{\mathbf{n}}_i$ where $\hat{\mathbf{n}}_i$ is the unit normal to the surface for the area element ΔS_i (Fig 4.5).
2. Assume that the vector field over each such area element is a constant $\vec{\mathbf{A}}_i$.
3. The flux through each element of area is $\Delta\Phi_i = \vec{\mathbf{A}}_i \,.\, \Delta\vec{\mathbf{S}}_i$.
4. The flux through the entire surface is then the sum of the flux through each of these elements of area. It is

$$\Phi = \vec{\mathbf{A}}_1 . \Delta\vec{\mathbf{S}}_1 + \vec{\mathbf{A}}_2 . \Delta\vec{\mathbf{S}}_2 + ... + \vec{\mathbf{A}}_n . \Delta\vec{\mathbf{S}}_n = \sum_{i=1}^{n} \vec{\mathbf{A}}_i . \Delta\vec{\mathbf{S}}_i \qquad (4.11)$$

5. In the limit as $n \to \infty$, we can write flux as an integral over the surface S:

$$\Phi = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{\mathbf{A}}_i . \Delta\vec{\mathbf{S}}_i = \iint_S \vec{\mathbf{A}} . d\vec{\mathbf{S}} \qquad (4.12)$$

where $d\vec{\mathbf{S}} = dS\hat{\mathbf{n}}$ is the infinitesimal element of area on this surface.

If the surface is a closed surface (like that of a sphere), we put a small circle on the sign of the integral and write flux of $\vec{\mathbf{A}}$ as

$$\Phi = \oiint_S \vec{\mathbf{A}} . d\vec{\mathbf{S}} \qquad (4.13)$$

There are several physical situations in which we need to calculate the flux of a vector field. One of these is the magnetic flux through the coil given by

$$\Phi_B = \iint_S \vec{\mathbf{B}} . d\vec{\mathbf{S}}. \qquad (4.14a)$$

The current i flowing through a wire is the flux of the current density ($\vec{\mathbf{J}}$) (see margin remark) vector across a cross-section of the wire, i.e.,

$$i = \iint_S \vec{\mathbf{J}} . d\vec{\mathbf{S}} \qquad (4.14b)$$

where $d\vec{\mathbf{S}}$ is an area element of the cross-section of the wire.

$\vec{\mathbf{J}} = ne\vec{\mathbf{v}}$.
where n is the number of electrons per unit volume, e is the charge on an electron and $\vec{\mathbf{v}}$ is the average drift velocity of an electron.

The mass (m) of fluid flowing out of a volume V is the flux of the vector $\rho\vec{\mathbf{v}}$ across the closed surface S enclosing V. Here ρ is the density of the fluid and $\vec{\mathbf{v}}$ its **average flow velocity**.

$$m = \oiint_S \rho\vec{\mathbf{v}} . d\vec{\mathbf{S}} \qquad (4.15)$$

Before we actually evaluate surface integrals, we need to know the convention used for choosing the direction of $\hat{\mathbf{n}}$. We discuss this point and define the area elements for integration in the following section.

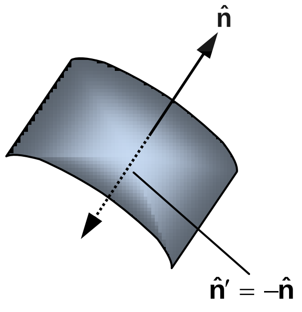


Fig. 4.6: **The unit normal to the surface of integration may point outward from the surface like $\hat{\mathbf{n}}$ or in the opposite direction as $\hat{\mathbf{n}}'$.**

4.3.3 Surface of Integration

In Fig. 4.6 you see an arbitrary surface of integration with a unit normal $\hat{\mathbf{n}}$. Note that **we could have chosen the unit normal** to be pointing downwards from the surface instead of in the upward direction, as shown by $\hat{\mathbf{n}}'$ in

Fig. 4.6. Obviously this would change the sign of the scalar product between the vector field and the unit normal in the expression for the surface integral in Eq. (4.12). How then do we decide in which direction to choose the unit normal for each area?

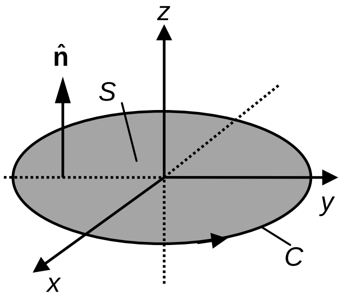


Fig. 4.7: **Direction of the normal vector for a plane surface.**

Consider a surface *S* enclosed by a closed curve *C* (Fig. 4.7) in a plane. It is an open surface lying on that plane. The direction of the normal depends on the sense in which the perimeter of this surface is traversed. If the right hand fingers are placed in the sense of travel around the perimeter, the positive normal points in the direction indicated by the thumb of the right hand (Fig. 4.7). Suppose the surface shown traversed in the sense, $+x \rightarrow +y \rightarrow -x \rightarrow -y \rightarrow +x$. The positive normal to the surface will be parallel to the positive z-axis.

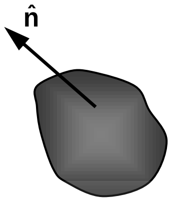


Fig. 4.8: **Outward drawn normal to a closed surface.**

If a volume is enclosed by a curved surface, it is called a closed surface (Fig. 4.8). The shell of a whole egg is an example of a closed surface. For such a surface the direction of the normal varies from point to point. However, at any point, the convention is to take the normal to the surface pointing outwards.

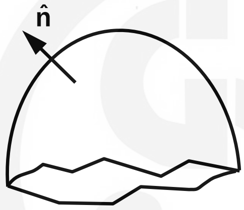


Fig. 4.9: **We choose the outer surface of the shell to be the outside and draw the outward normal.**

We may sometimes come across curved open surfaces. Examples of such surfaces are the shell of a cracked egg or a bowl (Fig. 4.9). In this case one side of the surface is chosen arbitrarily as outside and at any point the direction of the normal is outward. So we come to the general convention that:

The vector $\hat{\mathbf{n}}$ for any curved surface always points outwards from the surface.

In this unit we will study the surface integral over plane surfaces like the surface of a cube or cuboid. Surface integrals over curved surfaces are usually evaluated using non-Cartesian coordinates and this is beyond the scope of this syllabus.

Let us now describe the area element $d\vec{\mathbf{S}} = dS\,\hat{\mathbf{n}}$ for the surface of a cube or cuboid.

Area elements on the surface of a cube or cuboid

In Fig. 4.10 we show some typical area elements on the different faces of a cube/cuboid. For example, for an area element on face S_1, the outward normal is along the negative *z*-axis, so the area element is $-dxdy\hat{\mathbf{k}}$.

The integral over a closed surface like the surface of a sphere, is indicated by $\oiint_S$.

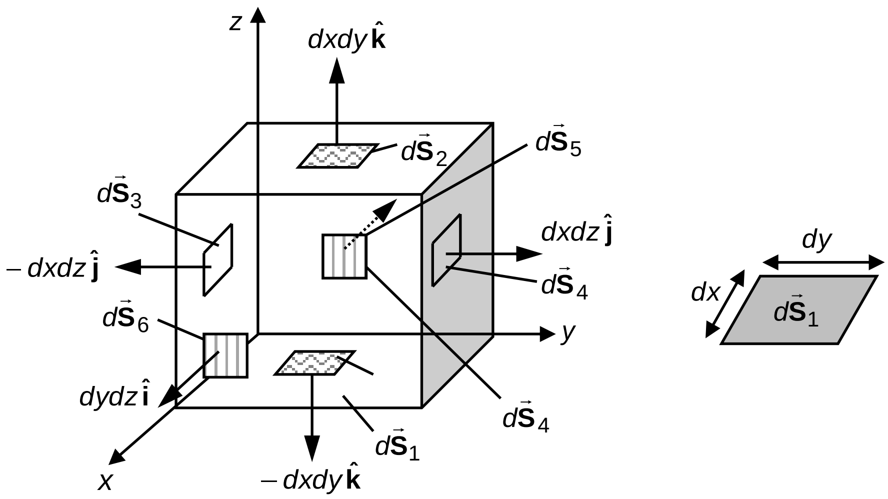


Fig. 4.10: **Surface area elements on a cube.**

You may note that the area element for the face S_6 is $dydz\,\hat{\mathbf{i}}$. You may like to write the area elements for the remaining faces. Try the following SAQ:

SAQ 2 **- Area elements on the surface of a cube/cuboid**

Write down the area element for the faces S_2, S_3, S_4 and S_5 of the cuboid in Fig. 4.10.

In many problems of surface integration, non-Cartesian coordinates are used for convenience. The choice of coordinate system is decided by the symmetry of the physical system.

4.3.4 Evaluation of Surface Integrals

A surface integral is evaluated as a double integral, over two variables. This means that we must describe both the vector field and the surface in terms of the same variables and then evaluate the double integral. In many problems on surface integrals, the choice of variables can be made by looking at the symmetry of the surface of integration.

Let us understand this by working out a few examples.

EXAMPLE 4.4 **: SURFACE INTEGRAL OF A VECTOR FIELD OVER A CUBE**

Calculate the surface integral of a vector field $\vec{\mathbf{A}} = 2xz\hat{\mathbf{i}} + 2xz\hat{\mathbf{j}} - yz\hat{\mathbf{k}}$ over the surface of a unit cube occupying the space $0 \le x \le 1;\ 0 \le y \le 1;\ 0 \le z \le 1$.

SOLUTION ■ You have learnt about the area elements for each face of a cube in Sec. 4.3.3. Integrating $\vec{\mathbf{A}}$ over the surface of the cube means that we have to integrate over each face of the cube. So

$$\oiint_S \vec{\mathbf{A}}.d\vec{\mathbf{S}} = \iint_{S_1} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_1 + \iint_{S_2} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_2 + \iint_{S_3} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_3 + \iint_{S_4} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_4 + \iint_{S_5} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_5 + \iint_{S_6} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_6 \quad \text{(i)}$$

NOTE

The LHS of Eq. (i) is an integral over the entire surface whereas each integral on the RHS is on a plane, a face of the cube.

Now let us integrate $\vec{\mathbf{A}}$ over the surface S_1 which is on the plane $z = 0$ (Fig. 4.11):

$$I_1 = \iint_{S_1} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_1 = \iint_{S_1} [2xz\hat{\mathbf{i}} + 2xz\hat{\mathbf{j}} - yz\hat{\mathbf{k}}].[-\,dxdy\hat{\mathbf{k}}] = \iint_{S_1} yzdxdy = 0\ (\because z = 0) \quad \text{(ii)}$$

We next integrate $\vec{\mathbf{A}}$ over the surface S_2 which is on the plane $z = 1$:

$$I_2 = \iint_{S_2} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_2 = \iint_{S_2} \left[2xz\hat{\mathbf{i}} + 2xz\hat{\mathbf{j}} - yz\hat{\mathbf{k}}\right]\left[dxdy\hat{\mathbf{k}}\right] = -\iint_{S_2} yzdxdy = -\iint_{S_2} ydxdy$$

$$(\because z = 1) \quad \text{(iii)}$$

We can evaluate this as a double integral on a rectangular region S_2 using Eq. (4.7) with the following limits on x and y to define the region S_2:

$$0 \le x \le 1;\ 0 \le y \le 1 \quad \text{(iv)}$$

$$I_2 = -\iint_{S_2} ydxdy = -\left[\int_0^1 dx\right]\left[\int_0^1 ydy\right] = -[x]_0^1\left[\frac{y^2}{2}\right]_0^1 = -\frac{1}{2} \quad \text{(v)}$$

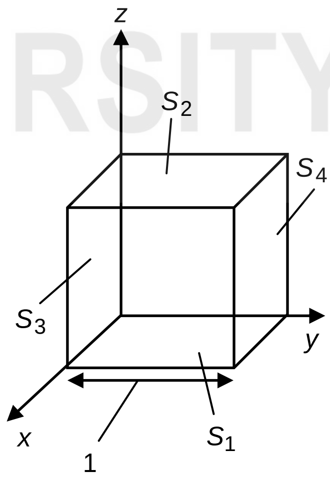


Fig. 4.11: **Unit cube of Example 10.1 with the surfaces** S_1, S_2, S_3 **and** S_4 **marked. These correspond to the planes** $z = 0$, $z = 1$, $y = 0$ **and** $y = 1$ **respectively.**

Similarly, for the surface S_3 $(0 \le x \le 1; 0 \le z \le 1)$ is on the plane $y = 0$, we have

$$I_3 = \iint_{S_3} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_3 = \iint_{S_3} \left[2xz\hat{\mathbf{i}} \; + \; 2xz\hat{\mathbf{j}} - yz\hat{\mathbf{k}}\right]\left[-\,dxdz\hat{\mathbf{j}}\right] = -\iint_{S_3} 2xzdxdz \qquad \text{(vi)}$$

Evaluating this as a double integral we can write:

$$I_3 = -2\iint_{S_3} xzdxdz = -2\left[\int_0^1 xdx\right]\left[\int_0^1 zdz\right] = -2\left[\frac{x^2}{x}\right]_0^1\left[\frac{z^2}{2}\right]_0^1 = -\frac{1}{2} \qquad \text{(vii)}$$

You may like to work out for yourself the values of the integral of *A* over the faces S_4, S_5 and S_6 of the cube (SAQ 3a). You will see that

$$I_4 = \iint_{S_4} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_4 = \frac{1}{2} \qquad \text{(viii)}$$

$$I_5 = \iint_{S_5} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_5 = 0 \qquad \text{(ix)}$$

and

$$I_6 = \iint_{S_6} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_6 = 1 \qquad \text{(x)}$$

The total flux of $\vec{\mathbf{A}}$ through the surface of the cube is found by substituting the surface integral corresponding to each surface in Eq. (i) from Eqs.(iii),(v),(vii), (viii),(ix) and (x) to get:

$$\iint_S \vec{\mathbf{A}}.\, d\vec{\mathbf{S}} = 0 - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} + 0 + 1 = \frac{1}{2}$$

SAQ 3 **- Surface integral**

a) Calculate the integrals I_4, I_5 and I_6 from Example 4.4.

b) Calculate the surface integral $\iint_S \vec{\mathbf{r}}.d\vec{\mathbf{S}}$ where $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and *S* is the surface of a disc of radius 2 units, lying in the plane $z = 5$, defined by

$$x^2 + y^2 \le 4; \quad z = 5$$

4.3.5 Solid Angle

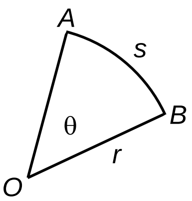


Fig. 4.12: **Angle** θ **in a plane.**

We now explain the concept of a *solid angle* which will be used in the next block. You are familiar with an angle in a plane. You know that in two-dimensions, it is the angle between two straight lines, say *AO* and *BO*, that intersect at a point *O* (Fig. 4.12). It is measured in the plane of the same lines and defined as

$$\theta = \frac{s}{r} \quad \text{(in rad)}$$

where *s* is the arc length of a circle of radius *r* passing through *A* and *B*. A solid angle is a three-dimensional analogue of the two-dimensional angle. Let us define it.

Consider an arbitrary differential area element $d\vec{\mathbf{S}}$ of a surface at a distance r from a point P to the surface element. Let $\hat{\mathbf{r}}$ denote the unit vector **from** the point P to the area element (Fig. 4.13). Then, by definition, the solid angle dΩ subtended by the surface area element $d\vec{\mathbf{S}}$ at the point P is given by

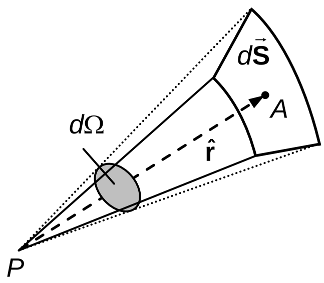


Fig. 4.13: **The solid angle** $d\Omega$ **subtended by an area element** $d\vec{\mathbf{S}}$ **at a point** O**.**

$$d\Omega = \frac{d\vec{\mathbf{S}}.\hat{\mathbf{r}}}{r^2} = \frac{dS\cos\theta}{r^2} \tag{4.16}$$

where θ is the angle between the normal to the surface element and $\hat{\mathbf{r}}$. The unit of solid angle is the *steradian* which is dimensionless.

The net solid angle subtended by the entire surface S is given by the surface integral:

$$\iint_S d\Omega = \iint_S \frac{\hat{\mathbf{r}}.d\vec{\mathbf{S}}}{r^2} \tag{4.17}$$

The solid angle of a closed surface is an important special case that we will use in Unit 6 of Block 2.

The net solid angle subtended by a closed surface S surrounding a point is given by

$$\Omega = \oint\!\!\oint_S \frac{\hat{\mathbf{r}}.d\vec{\mathbf{S}}}{r^2} = 4\pi \tag{4.18}$$

Note that for a closed surface, the vector $d\vec{\mathbf{S}}$ is always taken as the normal to the surface pointing *outwards*. The proof of Eq. (4.18) is beyond the scope of this course.

So, the net solid angle subtended by a closed surface of any shape, on a point enclosed by it, is 4π steradians.

You may now work out the following SAQ.

SAQ 4 **- Surface integral on the surface of a sphere**

Evaluate (i) $\oint\!\!\oint_S \hat{\mathbf{r}}.d\vec{\mathbf{S}}$ and (ii) $\oint\!\!\oint_S \frac{\hat{\mathbf{r}}.d\vec{\mathbf{S}}}{r^2}$ where S is a sphere of radius R.

Integral theorems allow you to transform one type of integral into another. We now study the Stokes theorem which allows us to transform surface integrals into line integrals, and conversely, line integrals into surface integrals.

4.4 STOKES' THEOREM

Stokes' theorem states that: **'The integral of the curl of a vector field over a surface** S **is equal to the line inetgral of the vector field over the closed path** C **bounding** S. It is expressed mathematically as

$$\oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \iint_S \left(\vec{\nabla} \times \vec{\mathbf{A}}\right).d\,\vec{\mathbf{S}} \tag{4.19}$$

Fig. 4.14 shows some examples of surfaces bounded by closed paths.

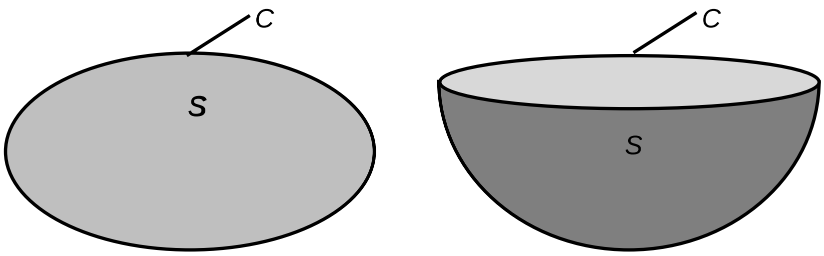


Fig. 4.14: **Surfaces bounded by closed paths.**

Let us now use Stokes' theorem to evaluate an integral.

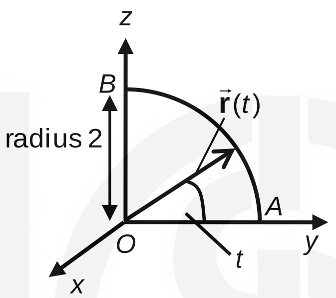


Fig. 4.15: **The contour** *C* **and surface** *S* **for Example 4.5.**

EXAMPLE 4.5 : EVALUATION OF LINE INTEGRAL USING THE STOKES' THEOREM

Verify Stokes' theorem for the vector field $\vec{\mathbf{F}} = y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}$ over the closed contour *C* enclosing the plane surface *S* shown in the Fig. 4.15. Here *AB* is the arc of the circle of radius 2 with its centre at the origin.

SOLUTION ■ To verify Stokes' theorem we must show:

$$\oint_C (y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}).d\vec{\mathbf{l}} = \iint_S \left[\vec{\nabla} \times (y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}})\right].d\vec{\mathbf{S}} \tag{i}$$

C is the closed contour *OAB* which encloses the quarter circle in the *yz* plane. The radius of the circle is 2 units. Let us first integrate the line integral on the LHS of Eq. (i). The contour *C* is made up of C_1, C_2 and C_3, C_1 is the straight line *OA* along the *y*-axis, C_2 is the arc *AB* of the circle and C_3 is the straight line *BO* along the *z*-axis. Then

$$I_1 = \oint_{C_1} (y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}).d\vec{\mathbf{l}} + \oint_{C_2} (y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}).d\vec{\mathbf{l}} + \oint_{C_3} (y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}).d\vec{\mathbf{l}} \tag{ii}$$

We first evaluate the first and third integrals on the RHS of Eq. (ii). Using Eq. (3.19a), we can write

$$\int_{C_1} (y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}).d\vec{\mathbf{l}} = \int_{C_1} (y\,dx + z\,dy + x\,dz) = \int_0^2 z\,dy \quad (\because dx = dz = 0 \text{ along } OA)$$

$$= 0 \; (\because z = 0 \text{ along } OA) \tag{iii}$$

and

$$\int_{C_3} (y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}).d\vec{\mathbf{l}} = \int_{C_3} (ydx + zdy + xdz) = \int_2^0 xdz \quad (\because dx = dy = 0 \text{ along } BO)$$

$$= 0 \; (\because x = 0 \text{ along } BO) \qquad \text{(iv)}$$

To evaluate the line integral along C_2, we parametrize the curve AB as follows:

$$\vec{\mathbf{r}}(t) = 2\cos t\, \hat{\mathbf{j}} + 2\sin t\, \hat{\mathbf{k}}; \;\; x(t) = 0, \;\; y(t) = 2\cos t, \;\; z(t) = 2\sin t; \;\; 0 \le t \le \pi/2 \qquad \text{(v)}$$

$$\therefore \vec{\mathbf{F}} = y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}} = 2\cos t\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}}; \;\; \frac{d\vec{\mathbf{r}}(t)}{dt} = -\,2\sin t\,\hat{\mathbf{j}} + 2\cos t\,\hat{\mathbf{k}} \qquad \text{(vi)}$$

Using Eq. (vi) in Eq. (3.28) for the line integral we get (see also margin remark):

$$\int_{C_2} (y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}).d\vec{\mathbf{l}} = \int_{t=0}^{\pi/2} (2\cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}}).(-2\sin t\,\hat{\mathbf{j}} + 2\cos t\,\hat{\mathbf{k}})dt$$

$$= \int_{t=0}^{\pi/2} (-4\sin^2 t)\; dt = -\pi \qquad \text{(vii)}$$

$$\int_0^{\pi/2} \sin 2t\, dt$$
$$= \int_0^{\pi/2} \frac{(1-\cos 2t)}{2} dt$$
$$= \frac{1}{2}\int_0^{\pi/2} dt - \frac{1}{2}\int_0^{\pi/2} \cos 2t\, dt$$
$$= \frac{1}{2}[t]_0^{\pi/2} - \frac{1}{2}[\sin 2t)]_0^{\pi/2}$$
$$= \pi/4$$

Adding up the contributions from each segment, the line integral over OAB is found by substituting the results of Eqs. (iii), (iv) and (vii) in Eq. (ii):

$$I_1 = 0 - \pi + 0 = -\pi \qquad \text{(viii)}$$

We next evaluate the surface integral in the RHS of Eq. (i). We first calculate the curl of the vector field (see margin remark):

$$\vec{\nabla} \times \vec{\mathbf{F}} = -\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}} \qquad \text{(ix)}$$

$$\vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$
$$= -\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$$

Note that the surface S is a plane surface on the yz plane. If we curl the fingers of our right hand around the contour in the direction of the contour, the normal to the surface is along the positive x-direction. We can consider the element of area on the yz plane to be:

$$d\vec{\mathbf{S}} = dydz\,\hat{\mathbf{i}} \qquad \text{(x)}$$

Then

$$I_2 = \iint_S \text{curl}(y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}).d\vec{\mathbf{S}} = \iint_S (-\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}).(dydz\,\hat{\mathbf{i}}) = -\iint_S dydz \qquad \text{(xi)}$$

Using the area property of the double integral we can see that:

$$\iint_S dydz = \text{Area of } S = \frac{1}{4}(\text{Area of circle of radius 2}) = \pi$$

Therefore, the integral of Eq. (xi) is just

$$I_2 = -\pi \qquad \text{(xii)}$$

The line integral of Eq. (viii) and the surface integral of Eq. (xii) both give us the same result, thereby, verifying Stokes' theorem.

You may now like to work out an SAQ on solving integrals using Stokes' theorem.

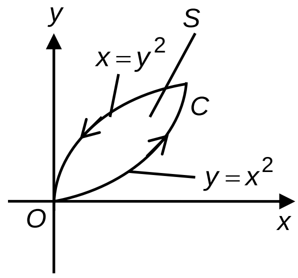


Fig. 4.16: **Figure for SAQ 5.**

SAQ 5 - Evaluation of line integral using Stokes' theorem

Using Stokes' theorem, evaluate $\oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}}$ around the closed curve C shown in Fig. 4.16 given that:

$$\vec{\mathbf{A}} = (x - y)\,\hat{\mathbf{i}} + (x + y)\,\hat{\mathbf{j}}$$

4.4.1 Applications of Stokes' Theorem

We shall now discuss an application of this theorem. The direct evaluation of $\vec{\nabla} \times \vec{\mathbf{B}}$ where $\vec{\mathbf{B}}$ is magnetic field due to a current carrying conductor is quite tedious. To obtain $\vec{\nabla} \times \vec{\mathbf{B}},$ we shall use Stokes' theorem and the circuital form of Ampere's law,

$$\oint_{C'} \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_0 i \tag{4.20}$$

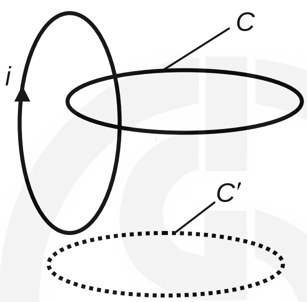


Fig. 4.17

where C is any closed path that is linked with the current i (Fig. 4.17). For a path like C', which is not linked with the current, we have

$$\oint_{C} \vec{\mathbf{B}}.d\vec{\mathbf{l}} = 0$$

Now, our task is to calculate $\vec{\nabla} \times \vec{\mathbf{B}}.$ From Stokes' theorem we get:

$$\oint_{C} \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \iint_S \left(\vec{\nabla} \times \vec{\mathbf{B}}\right).d\vec{\mathbf{S}} \tag{4.21}$$

where S is enclosed by C.

Recall that in Eq. (4.14b) we have defined current in terms of the current density $\vec{\mathbf{J}}$ as:

$$I = \iint_S \vec{\mathbf{J}}.d\vec{\mathbf{S}} \tag{4.22}$$

Hence, from Eqs. (4.20), (4.21) and (4.22), we get

$$\iint_S \left(\vec{\nabla} \times \vec{\mathbf{B}}\right).d\vec{\mathbf{S}} = \iint_S \mu_0\, \vec{\mathbf{J}}.d\vec{\mathbf{S}} \tag{4.23}$$

or

$$\iint_S \left(\vec{\nabla} \times \vec{\mathbf{B}} - \mu_0 \vec{\mathbf{J}}\right).d\vec{\mathbf{S}} = 0 \tag{4.24}$$

Since $d\vec{\mathbf{S}}$ is arbitrary, the integrand must be zero. Therefore,

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} \tag{4.25}$$

Thus we see that $\vec{\mathbf{B}}$ has a non-vanishing curl.

You have learnt in your Mechanics Course (BPHCT-131) that the curl of a conservative force field is zero. We can prove the same result using Stokes' theorem. You can work this out yourself in the following SAQ.

SAQ 6 **- Application of Stokes' theorem**

Using Stokes' theorem, prove that curl of a conservative force field is zero everywhere.

So far you have learnt how to evaluate double integrals and surface integrals, which involve successive integrations over two variables. Next we study volume integrals (also called triple integrals) which involve successive integrations over three variables.

4.5 VOLUME INTEGRALS

Let us first define volume (also known as triple integrals) integrals, where we integrate a function of three variables, $f(x, y, z)$ over a closed volume Ω in the Cartesian coordinate system. The method we follow is similar as for defining a double integral.

4.5.1 Volume Integral of the Function $f(x, y, z)$

Like the double integral, the triple or volume integral is also defined as the limit of a sum. Let us see how this is done.

1. We first partition the three dimensional volume Ω into *n* parts by drawing planes parallel to the three coordinate planes. As a result, the volume Ω is filled with boxes, which we now number from 1 to *n*. Each box has a volume $\Delta V_i = \Delta x_i \, \Delta y_i \, \Delta z_i$.
2. We choose a point $(x_{i,} y_i, z_i)$ in each of these boxes and define a sum of the form:

$$S_n = \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta V_i \qquad (4.26)$$

3. As *n* increases, the volume of the boxes becomes smaller and smaller. The volume integral of the function $f(x,y,z)$ over the region Ω is defined as the limit of the sum S_n in the limit $n \to \infty$.

The volume integral of a function $f(x,y,z)$ over a closed bounded region Ω is defined by the expression:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i = \iiint_{\Omega} f(x,y,z)dV = \iiint_{\Omega} f(x,y,z)dxdydz \quad (4.27)$$

You have seen before that the integral of the function of a single variable with respect to that variable represents an area, and a double integral of a function of two variables represents the volume under a surface. What, then, is a volume integral? We can say that it represents a summation in a hypothetical 4th dimension.

Let us try to understand this point with an example. Imagine a balloon that is being inflated. We define the surface of the balloon with the help of an equation $z = f(x, y)$. However since the size of the balloon is changing with time, each of these variables is also a function of time *t*. If we integrate with respect to *x* and *y*, we get the volume of the balloon as a function of *t*. If we put in a value of *t* we will get the value of the volume of the balloon at that instant of time. However, now we can perform the integration over *t* to sum up the volume over the entire process of inflation that would be the volume integral.

We now write down the properties of the volume integral, which are quite similar to the properties of a double integral.

4.5.2 Properties of the Volume Integral

For any two functions $f(x,y,z)$ and $g(x,y,z)$ defined over the three dimensional region Ω, the volume integral has the following properties:

Linearity:

$$\iiint_{\Omega} [\alpha f(x,y,z)+\beta g(x,y,z)]dxdydz = \alpha\iiint_{\Omega} f(x,y,z)dxdydz + \beta\iiint_{\Omega} f(x,y,z)dxdydz \tag{4.28}$$

where α and β are constants.

Additivity:

If the region Ω can be broken up into several non-overlapping regions $\Omega_1,\Omega_2,...,\Omega_n$, we can write:

$$\iiint_{\Omega} f(x,y,z)dxdydz$$
$$= \iiint_{\Omega_1} f(x,y,z)dxdydz + \iiint_{\Omega_2} f(x,y,z)dxdydz + ... + \iiint_{\Omega_n} f(x,y,z)dxdydz \tag{4.29}$$

Volume Property:

If the function $f(x,y,z)=1$, then the volume integral over the region Ω gives the volume of Ω:

$$\iiint_{\Omega} [1]dxdydz = \text{Volume of the region } \Omega \tag{4.30}$$

Let us now see how a volume integral may be evaluated by iterated integration.

4.5.3 Evaluation of a Volume Integral

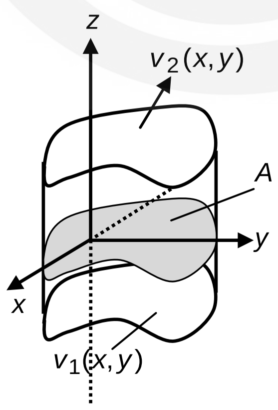


Fig. 4.18: **Limits of integration on the variable *z* in the region Ω.**

In evaluating the volume integrals we will once again perform **iterated integration**. In evaluating a double integral, where we integrate with respect to two variables, we perform a two-fold iterated integration. This, as you have seen in Sec. A2.2 of Appendix A2, can be carried out in two different ways depending on the order in which the integration over the two variables is carried out. Here we have three variables, so we carry out three-fold iterated integrations. However, in this case there are six possible ways of carrying out the repeated integral. If $f(x,y,z)$ is continuous, all the six iterated integrals are equal.

Let us consider the solid region Ω bounded below by the surface $z=v_1(x,y)$, and above by the surface $z=v_2(x,y)$, as shown in Fig. 4.18. The projection of the solid onto the *xy* plane is the region *A* (Fig. 4.18). We assume that the functions $v_1(x,y)$ and $v_2(x,y)$ are continuous in the region *A*. Then, for a function $f(x,y,z)$ continuous in the solid region Ω, we can write.

$$\iiint_{\Omega} f(x,y,z)\,dxdydz = \iint_{A}\left[\int_{z=v_1(x,y)}^{v_2(x,y)} f(x,y,z)\,dz\right]dydx \qquad (4.31)$$

Once we calculate the integral within the bracket, we are left with (in general) a double integral of a function of two variables *x* and *y* to be integrated with respect to *x* and *y*. And we can use Eqs. (A.2.9) or (A.2.11) to evaluate this double integral. So if *A* is a region in the *xy* plane defined by:

$$a \le x \le b; \quad u_1(x) \le y \le u_2(x) \qquad (4.32)$$

the volume integral reduces to:

$$\iiint_{\Omega} f(x,y,z)\,dxdydz = \int_{x=a}^{b}\left[\int_{y=u_1(x)}^{u_2(x)}\left[\int_{z=v_1(x,y)}^{v_2(x,y)} f(x,y,z)\,dz\right]dy\right]dx \qquad (4.33)$$

As for double integrals, remember that iterated integral for the volume integral can be performed in any order of variables. Here we have chosen to integrate over *z* first, then over *y*, and finally over *x*. The choice of the order of the variables of integration is to be made according to our convenience. In Example 4.6, we integrate over *y* first, then over *z* and finally over *x*. Volume integrals are used to evaluate several quantities of interest to physicists, such as the volume and mass of an object of arbitrary shape, its centre of mass and its moment of inertia. We summarize these applications below.

APPLICATIONS OF VOLUME INTEGRALS

- Volume *V* of a region Ω:

$$V = \iiint_{\Omega} dx\,dydz \qquad (4.34)$$

- Mass *m* of a body with a density $\gamma(x,y,z)$ over a region Ω:

$$m = \iiint_{\Omega} \gamma(x,y,z)\,dx\,dydz \qquad (4.35)$$

- Centre of mass of a body (x_{cm}, y_{cm}, z_{cm}) with a density $\gamma(x,y,z)$ over a region Ω:

$$x_{cm} = \frac{\iiint_{\Omega} x\,\gamma(x,y,z)\,dx\,dydz}{m};\quad y_{cm} = \frac{\iiint_{\Omega} y\,\gamma(x,y,z)\,dx\,dydz}{m};\quad z_{cm} = \frac{\iiint_{\Omega} z\,\gamma(x,y,z)\,dx\,dydz}{m} \qquad (4.36)$$

- Moment of inertia of a body with a density $\gamma(x,y,z)$ over a region Ω about the *x*-axis (I_x), about the *y*-axis (I_y) and about the z-axis (I_z):

$$I_x = \iiint_{\Omega} (y^2 + z^2)\gamma(x,y,z)\,dx\,dydz \qquad (4.37a)$$

$$I_y = \iiint_{\Omega} (x^2 + z^2)\,\gamma(x,y,z)\,dx\,dydz; \qquad (4.37b)$$

$$I_z = \iiint_{\Omega} (x^2 + y^2)\,\gamma(x,y,z)\,dx\,dydz \qquad (4.37c)$$

In the following example, we determine the moment of inertia of a cube by carrying out a volume integral.

EXAMPLE 4.6 : DETERMINING MOMENT OF INERTIA USING THE VOLUME INTEGRAL

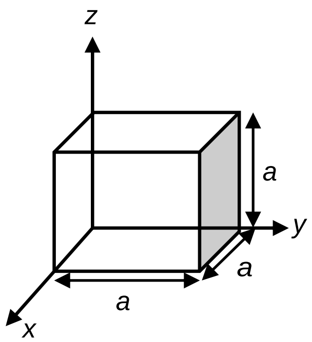


Fig. 4.19: **A cube of side *a*.**

Consider a cube with uniform density ρ and side *a*. The cube is placed such that its edges lie along the *x*, *y* and *z* axes as shown in Fig. 4.19. Determine the moment of inertia about an edge of the cube.

SOLUTION ■ To evaluate the moment of inertia about the *x*-axis, we use Eq. (4.37a). The limits of integration on the three variables are (Fig. 4.19):

$$0 \le x \le a;\ \ 0 \le y \le a;\ \ 0 \le z \le a$$

We write the moment of inertia as:

$$I_x = \rho \int_{x=0}^{a} \int_{y=0}^{a} \int_{z=0}^{a} \left(y^2 + z^2\right) dz\, dy\, dx$$

$$= \rho \int_{x=0}^{a} \int_{y=0}^{a} \left[y^2 z + \frac{z^3}{3} \right]_0^a dy\, dx \quad \text{(integrating over } z \text{ first)}$$

$$= \rho \int_{x=0}^{a} \int_{y=0}^{a} \left[y^2 a + \frac{a^3}{3} \right] dy\, dx$$

$$= \rho \int_{x=0}^{a} \left[\frac{y^3 a}{3} + \frac{a^3 y}{3} \right]_0^a dx \quad \text{(integrating over } y\text{)}$$

$$= \rho \int_0^a \left[\frac{2}{3} a^4 \right] dx \quad = \rho \left[\frac{2}{3} a^4 x \right]_0^a = \frac{2}{3} \rho a^5 \qquad \text{(i)}$$

The mass of the cube is $M = (\text{density}) \times (\text{volume}) = \rho a^3$. Substituting for ρ in Eq. (i) we get:

$$I_x = \frac{2}{3} M a^2$$

You may now like to evaluate a few integrals by this method.

SAQ 7 - Evaluating volume integrals

a) Evaluate the volume integral of the function $f(x, y, z) = (\sin x) yz$ for $0 \le x, y, z \le \pi$.

b) Determine the mass of a unit cube of density $\gamma\,(x, y, z) = x + 2y + 3z$.

4.6 VOLUME INTEGRAL OF A VECTOR FIELD

So far in this unit we have discussed the volume integrals of a scalar field. Sometimes, however, you may have to evaluate the volume integral of a vector field. The volume integral of a vector field is written as:

$$\iiint_V \vec{\mathbf{A}} dV \tag{4.38}$$

where V is the volume over which the integration is to carried out. The volume element dV is a scalar and so we can write the volume integral of the vector field $\vec{\mathbf{A}} = A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{j}} + A_3\hat{\mathbf{k}}$, as:

$$\iiint_V \vec{\mathbf{A}} dV = \hat{\mathbf{i}}\iiint_V A_1 dV + \hat{\mathbf{j}}\iiint_V A_2 dV + \hat{\mathbf{k}}\iiint_V A_3 dV \tag{4.39}$$

The integral of Eq. (4.39) reduces to a combination of integrals of scalar functions. The result of the integration is a vector quantity.

We now discuss another integral theorem. This theorem tells us how to convert a surface integral into a volume integral and vice versa.

4.7 THE DIVERGENCE THEOREM

The divergence theorem states that **'the integral of the divergence of a vector field over a volume *V* is equal to the surface integral of the vector over the closed surface bounding *V*.'**

The divergence theorem is sometimes also referred to as the Gauss's divergence theorem, Gauss's theorem or the divergence theorem of Gauss. It is expressed mathematically as

$$\oiint_S \vec{\mathbf{A}} . d\vec{\mathbf{S}} = \iiint_V \left(\vec{\nabla} . \vec{\mathbf{A}}\right) dV \tag{4.40}$$

where V is enclosed by S (Fig. 4.20).

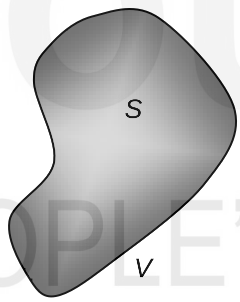


Fig. 4.20: **A closed surface *S* enclosing a volume *V*.**

Let us now work out an example to understand how to apply the divergence theorem.

EXAMPLE 4.7 : DIVERGENCE THEOREM

i) Use the divergence theorem to obtain the flux of a vector field $\vec{\mathbf{A}} = 3x\hat{\mathbf{i}} - y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$ over a cube of side $2a$. The vertices of the cube are at $(\pm a, \pm a, \pm a)$ as shown in Fig. 4.21.

ii) Use the divergence theorem to evaluate the flux of the vector field $\vec{\mathbf{F}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ over the sphere $x^2 + y^2 + z^2 = a^2$.

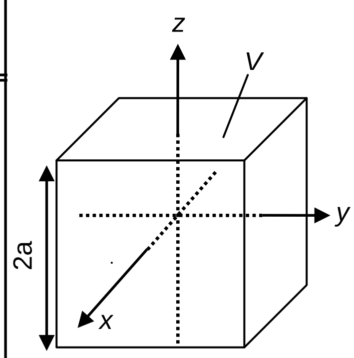


Fig. 4.21: **Cube with side $2a$. The cube is bounded by the planes** $x = \pm a$, $y = \pm a$, $z = \pm a$.

SOLUTION ■ i) Recall from Eq. (4.6b) that the flux of the vector field is defined as $\oiint_S \vec{\mathbf{A}} . d\vec{\mathbf{S}}$. Here S is the surface of the cube shown in Fig. 4.21.

Using the divergence theorem, we evaluate $\iiint_V \vec{\nabla} . \vec{\mathbf{A}} \, dV$, where V is the

In writing the final result we have used the volume property of the triple integral to write $\iiint_V dV$ as the volume of the region V, which is just the volume of the cube of side $2a$ that is $8a^3$.

region enclosed by the surface of the cube. The region V is defined by the limits:

$$-a \le x \le a;\ -a \le y \le a;\ -a \le z \le a \qquad \text{(i)}$$

Let us first evaluate $\vec{\nabla}.\vec{\mathbf{A}}$:

$$\vec{\nabla}.\vec{\mathbf{A}} = \frac{\partial(3x)}{\partial x} + \frac{\partial(-y)}{\partial y} + \frac{\partial(2z)}{\partial z} = 4 \qquad \text{(ii)}$$

Using the result of Eq. (ii) in the divergence theorem we can write the flux of the vector field $\vec{\mathbf{A}}$ as (read the margin remark):

$$\oiint_S \vec{\mathbf{A}}.d\vec{\mathbf{S}} = \iiint_V \vec{\nabla}.\vec{\mathbf{A}}\, dV = 4\iiint_V dV = 4(2a)^3 = 32a^3$$

Using the volume property of the triple integral we can see that $\iiint_V dV$ is just the volume of the sphere of radius a which is $\frac{4}{3}\pi a^3$.

ii) Using the divergence theorem for the vector field $\vec{\mathbf{F}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ we can write for the flux,

$$\Phi = \oiint_S \vec{\mathbf{F}}.d\vec{\mathbf{S}} = \oiint_S \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right).d\vec{\mathbf{S}} = \iiint_V \vec{\nabla}.\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) dV \qquad \text{(i)}$$

where V is the volume enclosed by the sphere enclosed by surface S given by $x^2 + y^2 + z^2 = a^2$. We evaluate the integral on the RHS of Eq. (i)

$$\iiint_V \vec{\nabla}.\vec{\mathbf{F}}\, dV = \iiint_V \left[\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right] dV = 3\iiint_V dV = 3\left[\frac{4}{3}\pi a^3\right] = 4\pi a^3 \qquad \text{(ii)}$$

You may now like to work out the following SAQ.

SAQ 8 – **Evaluating surface integral using the divergence theorem**

Evaluate $\oiint_S \vec{\mathbf{V}}.\hat{\mathbf{n}}\, d\vec{\mathbf{S}}$, where $\vec{\mathbf{V}} = x\cos^2 y\,\hat{\mathbf{i}} + xz\,\hat{\mathbf{j}} + z\sin^2 y\,\hat{\mathbf{k}}$ and S is the surface of a sphere with its centre at the origin and radius 3 units.

Let us now consider an application of the divergence theorem.

4.7.1 Application of the Divergence Theorem

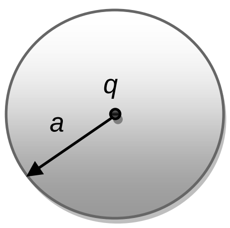


Fig. 4.22: **Electric flux due to a point charge** *q* **through a sphere of radius** *a*.

You have studied in your school physics courses that the electric field due to a point charge q, at a point whose position vector with respect to the location of q is $\vec{\mathbf{r}}$, is given by

$$\vec{\mathbf{E}} = \frac{kq}{r^3}\vec{\mathbf{r}} \qquad (r \ne 0) \qquad (4.41)$$

where k is a constant dependent on the nature of the medium.

Let us now determine the flux of $\vec{\mathbf{E}}$ through a sphere of radius a (Fig. 4.22) whose centre is at the position of the charge q.

The required surface integral is $\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}}$, where S is the surface of a sphere of radius a. Here

$$\vec{\mathbf{E}} = \frac{kq}{r^3}\vec{\mathbf{r}} = \frac{kq}{r^3}r\hat{\mathbf{r}} = \frac{kq}{r^2}\hat{\mathbf{r}} \tag{4.42}$$

where $\hat{\mathbf{r}}$ is the unit vector along the position vector $\vec{\mathbf{r}}$ .Contribution to a surface integral comes from the surface only. So we have to know $\vec{\mathbf{E}}$ on the surface of the sphere, which is $\frac{kq}{a^2}\hat{\mathbf{r}}$. Again, we know at every point on the sphere

$d\vec{\mathbf{S}} = dS\,\hat{\mathbf{r}}$ where dS is the surface element on the surface of a sphere.

Hence, the required flux = $\oint\!\!\oint_S \frac{kq}{a^2}\hat{\mathbf{r}}.dS\hat{\mathbf{r}} = \oint\!\!\oint_S \frac{kq}{a^2}dS \qquad (\because\ \hat{\mathbf{r}}.\hat{\mathbf{r}} = 1)$

$$= \frac{kq}{a^2}\oint\!\!\oint_S dS \tag{4.43}$$

because $\oint\!\!\oint dS$ is the surface area of the sphere of radius a which is $4\pi a^2$, we can write

$$\oint\!\!\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = 4\pi kq \tag{4.44}$$

where S is the surface of a sphere that encloses charge q. It can be shown that the above result is true for **any** charge distribution. Suppose that a closed surface enclosing a volume V has a continuous distribution of charge. If the charge per unit volume is ρ, then $q = \iiint_V \rho dV$.

An example of such a distribution is a charged sphere. For this distribution, we have

$$\oint\!\!\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = 4\pi k\iiint_V \rho\, dV \tag{4.45}$$

But using Eq. (4.40), we have

$$\oint\!\!\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \iiint_V \vec{\nabla}.\,\vec{\mathbf{E}}\, dV \tag{4.46}$$

From Eqs. (4.40) and (4.44),, we get

$$\iiint_V \vec{\nabla}.\,\vec{\mathbf{E}}\, dV = 4\pi k\iiint_V \rho dV$$

or

$$\iiint_V \left(\vec{\nabla}.\,\vec{\mathbf{E}} - 4\pi k\rho\right) dV = 0 \tag{4.47}$$

Since dV is an arbitrary infinitesimal volume element, the integrand in Eq. (4.47) must be zero:

$$\therefore \quad \vec{\nabla}.\,\vec{\mathbf{E}} - 4\pi k\rho = 0 \Rightarrow \vec{\nabla}.\,\vec{\mathbf{E}} = 4\pi k\rho \tag{4.48}$$

Eq. (4.48) tells us that the divergence of the electric field vector due to a continuous distribution of charge is independent of the extent of distribution. It depends only on the charge per unit volume. In charge-free space, $\rho = 0$, so that

$$\vec{\nabla}.\,\vec{\mathbf{E}} = 0 \tag{4.49}$$

The advantage of the divergence theorem is that it enables us to convert a volume integral to a surface integral and vice versa. In applications of the divergence theorem, the strategy for problem solving should be to evaluate the simpler of the two integrals.

You may now like to solve an SAQ to apply the divergence theorem.

SAQ 9 **- The divergence theorem**

a) Show that for any closed surface S the surface integral

$$\oiint_S \vec{\mathbf{r}} \cdot d\vec{\mathbf{S}} = 3V$$

where V is the volume of the region enclosed by the surface.

b) Show that for a vector $\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}}$

$$\oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0$$

4.8 SUMMARY

Concept	Description
Applications of double integrals	■ Double integrals are used in physics to evaluate the following quantities: • **Area** A of a region R $A = \iint_R dx\,dy$ • **Mass** m **of a body** with a density $\gamma(x,y)$ over a region R $m = \iint_R \gamma(x,y)\,dx\,dy$ • **Centre of mass** (x_{cm}, y_{cm}) **of a body** with a density $\gamma(x,y)$ over a region R $x_{cm} = \dfrac{\iint_R x\,\gamma(x,y)\,dx\,dy}{m};\quad y_{cm} = \dfrac{\iint_R y\,\gamma(x,y)\,dx\,dy}{m}$ • The **average value** μ of **a continuous function** *f(x,y)* over a closed region R in the *xy* plane is: $\mu = \dfrac{\iint_R f(x,y)\,dx\,dy}{\iint_R dx\,dy};\quad \iint_R dx\,dy = \text{Area of the region of integration } R$
Surface integral	■ The surface integral of a scalar or a vector field is the generalisation of the double integral where the region of integration may be any surface. Surface integrals can occur in any of the following three forms: $\iint_S \phi\, d\vec{\mathbf{S}},\quad \iint_S \vec{\mathbf{A}}.d\vec{\mathbf{S}}$ and $\iint_S \vec{\mathbf{A}} \times d\vec{\mathbf{S}}$ The element of area is $d\vec{\mathbf{S}} = dS\hat{\mathbf{n}}$.

Flux of a vector field

- The flux of a vector field $\vec{\mathbf{A}}$ over a surface S is given by the surface integral

$$\Phi = \iint_S \vec{\mathbf{A}}.d\vec{\mathbf{S}}$$

Volume/triple integral of a function

- The volume/triple integral of a function $f(x,y,z)$ over a closed bounded region Ω is written as $\iiint_\Omega f(x,y,z)dV$ or $\iiint_\Omega f(x,y,z)dxdydz$ and can be defined as the limit of a sum as follows:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i,y_i,z_i)\Delta V_i = \iiint_\Omega f(x,y,z)dV = \iiint_\Omega f(x,y,z)dxdydz$$

Applications of volume integrals

- Volume/triple integrals are used in physics to evaluate the following quantities:
  - **Volume** V of a region Ω

$$V = \iiint_\Omega dx\,dydz$$

  - **Mass** m **of a body** with a density $\gamma(x,y,z)$ over a region Ω

$$m = \iiint_\Omega \gamma(x,y,z)dx\,dydz$$

  - **Centre of mass of a body** (x_{cm}, y_{cm}, z_{cm}) with a density $\gamma(x,y,z)$ over a region Ω

$$x_{cm} = \frac{\iiint_\Omega x\,\gamma(x,y,z)dx\,dydz}{m}; \quad y_{cm} = \frac{\iiint_\Omega y\,\gamma(x,y,z)dx\,dydz}{m};$$

$$z_{cm} = \frac{\iiint_\Omega z\,\gamma(x,y,z)dx\,dydz}{m}$$

  - **Moment of inertia of a body** with density $\gamma(x,y,z)$ over a region Ω about the x-axis, I_x, about the y-axis I_y and about the z-axis, I_z.

$$I_x = \iiint_\Omega (y^2+z^2)\gamma(x,y,z)dx\,dydz$$

$$I_y = \iiint_\Omega (x^2+z^2)\,\gamma(x,y,z)dx\,dydz;$$

$$I_z = \iiint_\Omega (x^2+y^2)\,\gamma(x,y,z)dx\,dydz$$

Vector integral theorems

- The **Stokes' theorem** states that the integral of the curl of a vector field over a surface S is equal to the line integral of the vector field over the closed path bounding S and is expressed mathematically as:

$$\oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \iint_S \text{curl}\,\vec{\mathbf{A}}.d\vec{\mathbf{S}}$$

- The **divergence theorem** states that the integral of the divergence of a vector field over a volume V is equal to the surface integral of the vector field over the closed surface bounding V and is expressed mathematically as:

$$\oiint_S \vec{\mathbf{A}}.d\vec{\mathbf{S}} = \iiint_V \textbf{div}\,\vec{\mathbf{A}}\,dV$$

4.9 TERMINAL QUESTIONS

1. Use double integration to find the area of the region in the xy plane bounded by the curves $y = x$ and $y = x^3$ for $x > 0$.

2. Calculate the volume V of a solid which is bound above by the plane $z = 4 - y$ and below by the region R defined by the circle $x^2 + y^2 = 4$.

3. The product of inertia of a lamina in the xy plane about the x and y-axes is given by

$$I_{xy} = I_{yx} = \iint_R \sigma xy \, dx \, dy$$

where R is the region of space covered by the lamina and σ is the mass per unit area of the lamina. Determine I_{xy} for the lamina shown in Fig. 4.23.

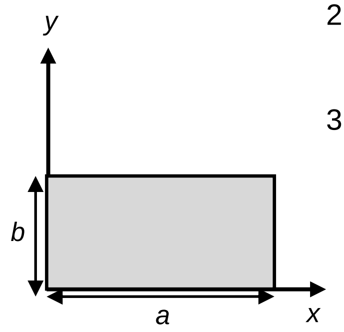


Fig. 4.23: **A rectangular lamina** $0 \le x \le a$, $0 \le y \le b$.

4. A box is bounded by the planes $x = 0$; $x = 1$; $y = 0$; $y = 1$; $z = 0$ and $z = 2$. It has a density $\gamma(x, y, z) = \left(9 - z^3\right)$ kg m^{-3}. Calculate the mass of the box.

5. Determine the flux of the vector field $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} - 2z\,\hat{\mathbf{k}}$ over the surface of a sphere S defined by the equation $x^2 + y^2 + z^2 = 1$.

6. Verify Stokes' theorem for the vector field $\vec{\mathbf{A}} = z^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}}$, where C is the path OPQ in the yz plane shown in Fig. 4.24.

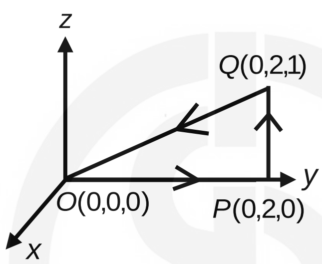


Fig. 4.24: **Path** ***OPQ*** **for TQ 6.**

7. Show that the line integral $\oint_C (yz\,dx + xz\,dy + xz\,dz)$ is zero along any closed contour C.

8. Using Stoke's Theorem evaluate $\oint_C \vec{\mathbf{F}} . d\vec{\mathbf{l}}$

$$\vec{\mathbf{F}} = x^2\hat{\mathbf{i}} + 2x\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$$

where C is the ellipse in the xy plane defined by

$$\frac{x^2}{16} + \frac{y^2}{64} = 1, \; z = 0$$

9. Using the divergence theorem, calculate the flux of a vector field $\vec{\mathbf{F}} = z\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} - x^3\hat{\mathbf{k}}$ over a sphere of radius 2 units.

10. Evaluate the flux of the vector field $\vec{\mathbf{A}} = (2y\,\hat{\mathbf{i}} + 5y^2\,\hat{\mathbf{j}} + 4z\,\hat{\mathbf{k}})$ through the surface of a unit cube which has one corner at the origin, one corner at (1, 1, 1) and all its edges are parallel to the coordinate axes.

4.10 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. a) We have to evaluate $\iint_R dx \, dy$ where R is the region bound by $y = x^2$ and $y = x^3$. Following Example 4.1, let us first determine the points of intersection of the two curves in the region $x > 0$; $y > 0$, for this we solve the equations

$$y = x^2 \quad \text{and} \quad y = x^3$$

$$\Rightarrow \qquad x^2 = x^3 \Rightarrow x^2(x-1) = 0$$

So the points of intersection are $x = 0$ and $x = 1$ and the limits on x and y are $x^3 \le y \le x^2$; $0 \le x \le 1$

> Note that the limits on y are $x^3 \le y \le x^2$ and not $x^2 \le y \le x^3$. This is because for $0 \le x \le 1$, $x^3 < x^2$.

$$\therefore \qquad A = \int_{x=0}^{1} \left[\int_{y=x^3}^{x^2} dy\, dx \right] = \int_0^1 [y]_{x^3}^{x^2} dx \Bigg| = \int_0^1 (x^2 - x^3)\, dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{12}$$

b) We use Eq. (A2.3) to evaluate the integral with $f(x,y) = x^4$. The lim on y for the region of integration on the xy plane is defined by the equation:

$$x^2 \le y \le 1$$

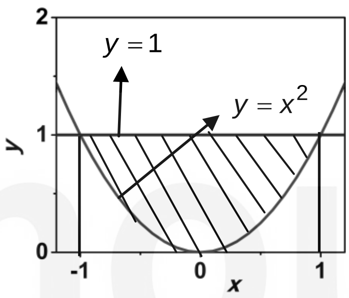


Fig. 4.25: **The region of integration for SAQ 1(b). The two curves intersect at** $x = 1$ **and**

We obtain the limits on x in the region of integration by determining the value of x at the points at which the two curves $y = x^2$ and $y = 1$ intersect, as you see in Fig. 4.25. This is found by solving for x as follows:

$$x^2 = 1 \Rightarrow x = 1, -1$$

So the integral we have to evaluate is the following:

$$I = \int_{x=-1}^{1} \int_{y=x^2}^{1} x^4\, dy\, dx$$

Integrating over y first, we get:

$$I = \int_{-1}^{1} \left[y\, x^4 \right]_{x^2}^{1} dx = \int_{-1}^{1} \left[x^4 - x^6 \right] dx$$

Integrating over x, we then get:

$$I = \left[\frac{x^5}{5} - \frac{x^7}{7} \right]_{-1}^{1} = \frac{4}{35}$$

2. From Fig. 4.10 we can see that

$$d\vec{\mathbf{S}}_2 = dxdy\, \hat{\mathbf{k}}$$

$$d\vec{\mathbf{S}}_3 = -dxdz\, \hat{\mathbf{j}}$$

$$d\vec{\mathbf{S}}_4 = dxdz\, \hat{\mathbf{j}}$$

and $\quad d\vec{\mathbf{S}}_5 = -dy\, dz\, \hat{\mathbf{i}}$

3. a) Using the results of SAQ 2, we can write the surface integral over the surface $S_4 (0 \le x \le 1; 0 \le z \le 1)$ on the plane $y = 1$ as

$$I_4 = \iint_{S_4} \vec{\mathbf{A}} . d\vec{S}_4 = \iint_{S_4} \left[2xz\hat{\mathbf{i}} + 2xz\hat{\mathbf{j}} - yz\hat{\mathbf{k}}\right]\left[dxdz\hat{\mathbf{j}}\right] = \iint_{S_4} 2xzdxdz$$

We evaluate this double integral to get:

$$I_4 = 2\iint_{S_4} xzdxdz = 2\left[\int_0^1 xdx\right]\left[\int_0^1 zdz\right] = \frac{1}{2}$$

Similarly over the surface S_5 $(0 \le y \le 1; 0 \le z \le 1)$ on the plane $x = 0$ (Fig. 4.10) we get

$$I_5 = \iint_{S_5} \vec{\mathbf{A}} . d\vec{\mathbf{S}}_5 = \iint_{S_5} \left[2xz\hat{\mathbf{i}} + 2xz\hat{\mathbf{j}} - yz\hat{\mathbf{k}}\right]\left[-dydz\,\vec{\mathbf{i}}\right]$$

$$= -\iint_{S_5} 2xzdydz = 0 \ (\because x = 0)$$

Over the surface $S_6 (0 \le y \le 1; 0 \le z \le 1)$ which is on the plane $x = 1$, we have

$$I_6 = \iint_{S_6} \vec{\mathbf{A}} . d\bar{\mathbf{S}}_6 = \iint_{S_6} \left[2xz\hat{\mathbf{i}} + 2xz\hat{\mathbf{j}} - yz\hat{\mathbf{k}}\right]\left[dydz\,\vec{\mathbf{i}}\right]$$

$$= -\iint_{S_6} 2xzdydz = \iint_{S_6} 2zdydz \ (\because x = 1)$$

We evaluate this as a double integral:

$$I_6 = 2\iint_{S_6} zdydz = 2\left[\int_0^1 dy\right]\left[\int_0^1 zdz\right] = 2[y]_0^1\left[\frac{z^2}{2}\right]_0^1 = 1$$

b) Since the disc is parallel to the xy plane, we can write as explained in Sec. 3.3,

$$d\vec{\mathbf{S}} = dxdy\,\hat{\mathbf{k}}$$

$$\therefore \iint_S \vec{\mathbf{r}} . d\vec{\mathbf{S}} = \iint_S (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})\,.dxdy\,\hat{\mathbf{k}}$$

$$= \iint_S z\,dx\,dy = 5\iint_S dx\,dy \quad \text{(since the disc lies in the plane } z = 5)$$

$$= 5\pi . 2^2 = 20\pi \ \left(\iint_S dx\,dy \text{ is the area of the circle of radius 2 units}\right)$$

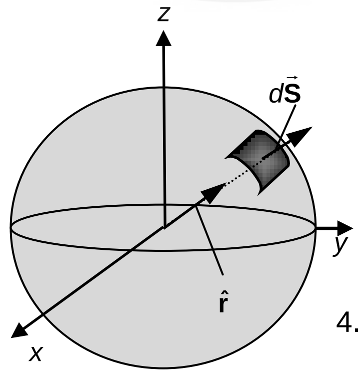


Fig. 4.26: **The unit normal for an area element on the surface of a sphere.**

4. i) Refer to Fig. 4.26. $\hat{\mathbf{r}}$ is the unit vector along the position vector $\vec{\mathbf{r}}$. Since $d\vec{\mathbf{S}}$ points along the outward drawn normal, it points along $\hat{\mathbf{r}}$ at every point on the sphere so that $d\vec{\mathbf{S}} = dS\,\hat{\mathbf{r}}$

$$\therefore \quad \hat{\mathrm{r}} . d\vec{\mathbf{S}} = \hat{\mathbf{r}} . dS\hat{\mathbf{r}} = dS(\hat{\mathbf{r}} . \hat{\mathbf{r}}) = dS \quad (\because \hat{\mathbf{r}} . \hat{\mathbf{r}} = 1)$$

Hence $\oiint_S \hat{\mathrm{r}} . d\vec{\mathbf{S}} = \oiint_S dS = S,$ which is the surface area of the sphere.

Thus $$\oiint_S \hat{\mathrm{r}} . d\vec{\mathbf{S}} = 4\pi R^2$$

ii) Similarly

$$\oiint_S \frac{\hat{\mathbf{r}}.d\vec{\mathbf{S}}}{r^2} = \frac{1}{R^2}\oiint_S dS = \frac{4\pi R^2}{R^2} = 4\pi$$

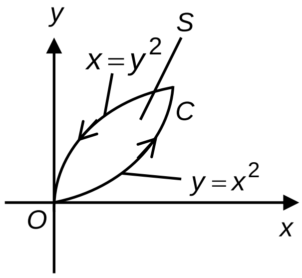


Fig. 4.27: **Figure for SAQ 5.**

5. To evaluate the line integral using Stoke's theorem as given in Eq. (4.19), we first evaluate $\vec{\nabla}\times\vec{\mathbf{A}}$ as:

$$\vec{\nabla}\times\vec{\mathrm{A}} = \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x-y) & (x+y) & 0 \end{vmatrix} = \hat{\mathrm{k}}\left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(x-y)\right] = 2\hat{\mathrm{k}}$$

The contour C and the region R are in the xy plane as shown in Fig. 4.27 (Fig. 4.16 reproduced here), therefore $d\vec{\mathbf{S}} = dx\,dy\,\hat{\mathbf{k}}$. Substituting for $\vec{\nabla}\times\vec{\mathbf{A}}$ and $d\vec{\mathbf{S}}$ into Eq. (4.19) we can write the integral as:

$$I = \oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \iint_S (\vec{\nabla}\times\vec{\mathbf{A}}).d\,\vec{\mathbf{S}} = \iint_S (2\hat{\mathbf{k}}).(dx\,dy\,\hat{\mathbf{k}}) = 2\iint_S dxdy$$

The limits on x are given by the points of intersection of the curves $x = y^2$ and $y = x^2$. By solving $\sqrt{x} = x^2$ we get the points of intersection as $x = 0$ and $x = 1$.

We define the region S (shown in Fig. 4.25) by the equations (see margin remark):

$$0 \le x \le 1;\; x^2 \le y \le \sqrt{x}$$

Then

$$I = 2\int_{x=0}^{1}\left[\int_{y=x^2}^{\sqrt{x}} dy\right]dx = 2\int_{x=0}^{1}[y]_{x^2}^{\sqrt{x}}dx = 2\int_{x=0}^{1}\left[\sqrt{x} - x^2\right]dx$$

$$= 2\left[\frac{2}{3}\left(x^{3/2}\right) - \frac{x^3}{3}\right]_0^1 = 2\left[\frac{2}{3} - \frac{1}{3}\right] = \frac{2}{3}$$

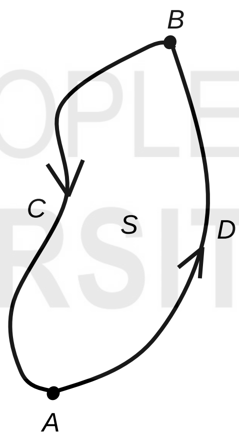


Fig. 4.28: **Diagram for the solution of SAO 6.**

6. Refer to Fig. 4.28. You have seen that for a conservative force

$$\int_{ACB}\vec{\mathbf{F}}.d\vec{\mathbf{r}} = -\int_{-ADB}\vec{\mathbf{F}}.d\vec{\mathbf{r}}$$

or

$$\int_{ACB}\vec{\mathbf{F}}.d\vec{\mathbf{r}} + \int_{-ADB}\vec{\mathbf{F}}.d\vec{\mathbf{r}} = 0$$

i.e. $$\oint_{ACBDA}\vec{\mathbf{F}}.d\vec{\mathbf{r}} = 0$$

From Stokes' theorem, we know that

$$\oint_{ACBDA}\vec{\mathbf{F}}.d\vec{\mathbf{r}} = \iint_S (\vec{\nabla}\times\vec{\mathbf{F}}).d\vec{\mathbf{S}}$$

So,

$$\iint_S (\vec{\nabla}\times\vec{\mathbf{F}}).d\vec{\mathbf{S}} = 0$$

But $d\vec{\mathbf{S}}$ is arbitrary. Hence the integrand is zero. Moreover, since the path *ACBDA* has been chosen anywhere in the field, we can write

$\vec{\nabla} \times \vec{\mathbf{F}} = 0$ everywhere in the field.

7. a) We write the volume integral as

$$I = \iiint_{\Omega} (\sin x) y\, z\, dx dy dz\,,$$

where Ω is defined by the equations:

$$0 \le x, y, z \le \pi$$

In this case, we can write the integral as:

$$I = \left[\int_0^{\pi} \sin x\, dx\right]\left[\int_0^{\pi} y\, dy\right]\left[\int_0^{\pi} z\, dz\right] = [-\cos x]_0^{\pi} \left[\frac{y^2}{2}\right]_0^{\pi} \left[\frac{z^2}{2}\right]_0^{\pi}$$

$$= \frac{\pi^4}{2}$$

b) Using Eq. (4.35) with γ (*x, y, z*) = ρ (*x, y, z*) we can write the mass of the cube as $m = \iint_{\Omega}\int \rho(x, y, z)\, dx\, dy\, dz$ where Ω is the volume of the cube. For the unit cube

$$0 \le x \le 1, \quad 0 \le y \le 1 \text{ and } 0 \le z \le 1.$$

$$\therefore \quad m = \int_0^1\int_0^1\left(\int_0^1 (x + 2y + 3z)\, dz\right) dx\, dy$$

$$= \int_0^1\left(\int_0^1 \left(xz + 2yz + \frac{3z^2}{2}\right)_0^1 dy\right) dz$$

$$= \int_0^1\left(\int_0^1 \left(x + 2y + \frac{3}{2}\right) dy\right) dx$$

$$= \int_0^1 \left(xy + y^2 + \frac{3}{2}y\right)_0^1 dx$$

$$= \int_0^1 \left(x + 1 + \frac{3}{2}\right) dx = \left(\frac{x^2}{2} + \frac{5}{2}x\right)_0^1$$

$$= 3 \text{ units}$$

8. Using Eq. (2.3), we first determine the divergence of the vector field,

$$\vec{\nabla}.\vec{\mathbf{V}} = \frac{\partial}{\partial x}(x \cos^2 y) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(z \sin^2 y)$$

$$= \cos^2 y + \sin^2 y = 1 \qquad \text{(i)}$$

Using Eq. (4.39), we write using the result of Eq. (i)

$$I = \oiint_S \vec{\mathbf{V}}.\hat{\mathbf{n}}\, d\vec{\mathbf{S}} = \iiint_{\Omega} \vec{\nabla}.\vec{\mathbf{V}}\, dV = \iiint_{\Omega} dV$$

where Ω is the sphere of radius 3 units with its centre at the origin.

$$\therefore \qquad I = \frac{4}{3}\pi(3^3) = 36\pi$$

9. a) Using Eq. (4.39) with $\vec{\mathbf{A}} = \vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ we can write

$$I = \oiint_S \vec{\mathbf{r}}.\, d\vec{\mathbf{S}} = \iiint_\Omega \left(\vec{\nabla}.\vec{\mathbf{r}}\right) dV \qquad \text{(i)}$$

and

$$\vec{\nabla}.\vec{\mathbf{r}} = \vec{\nabla}.\,(x\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

Replacing $\vec{\nabla}.\vec{\mathbf{r}} = 3$ in Eq. (i) we get:

$$I = \oiint_S \vec{\mathbf{r}}.d\vec{\mathbf{S}} = 3\left[\iiint_\Omega dV\right] \qquad \text{(ii)}$$

Using the volume property of a triple integral, the quantity in the bracket in the RHS of Eq. (ii) is just the volume of the region of integration which is V.

$$\therefore \quad I = \oiint_S \vec{\mathbf{r}}.\, d\vec{\mathbf{S}} = 3V$$

b) Using the divergence theorem we can write:

$$\oiint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}} = \left[\iiint_V \vec{\nabla}.\vec{\mathbf{B}}\; dV\right] \qquad \text{(i)}$$

Given that $\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}}$, we can write:

$$\vec{\nabla}.\vec{\mathbf{B}} = \vec{\nabla}.\;(\vec{\nabla} \times \vec{\mathbf{A}}) = 0$$

This is because the divergence of the curl of a vector field is always zero, as you have studied in Unit 2.

$$\therefore \quad \oiint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}} = 0$$

Terminal Questions

1. We use the area property of the double integral given in Eq. (A.2.7) to find the area. Following Example 4.1, the range of x is decided by determining the points of intersection of the curves $y = x$ and $y = x^3$ (Fig. 4.29). We solve as follows:

$$x^3 = x \Rightarrow x^2(x-1) = 0 \Rightarrow x = 0,1$$

The points of intersection are $x = 0$ and $x = 1$(Fig. 4.29). Note that in the range $0 \le x \le 1,\ x^3 \le x$. Therefore, the region of integration is:

$$0 \le x \le 1\,;\, x^3 \le y \le x$$

The area A is:

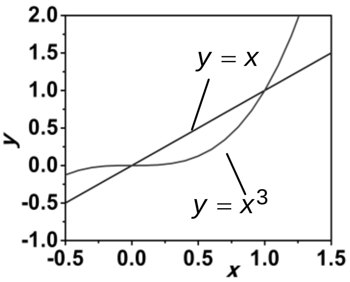


Fig. 4.29: **Region of integration is the area enclosed between the curves $y = x$ and $y = x^3$ in range** $0 \le x \le 1$.

$$A = \int_{x=0}^{1}\int_{y=x^3}^{x}(1)\,dy\,dx = \int_0^1 [y]_{x^3}^{x}\,dx = \int_0^1 \left[x - x^3\right]dx$$ [Integrating over y first]

$$= \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{4}\text{units}.$$ [Integrating over x]

2. Using the double integral, we can define the volume under the plane $z = 4 - y$ [see Example 4.1(ii)] as:

$$V = \iint_R (4 - y)\,dx\,dy \qquad \text{(i)}$$

where R is the region in the xy plane which is enclosed by the circle $x^2 + y^2 = 4$.

$$\int \sqrt{a^2 - x^2}\,dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

The region of integration R for Eq. (i) is

$$-2 \le x \le 2;\ -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2} \qquad \text{(ii)}$$

$$\therefore \quad V = \int_{x=-2}^{2}\left[\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4 - y)\,dy\right]dx$$

$$= \int_{-2}^{2} 8\sqrt{4 - x^2}\,dx$$

(after integrating over y).

On integrating over x this gives us (read the margin remark):

$$V = 16\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2} = 16\pi$$

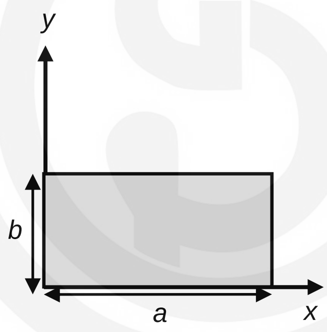


Fig. 4.30: **A rectangular lamina** $0 \le x \le a$, $0 \le y \le b$.

3. Refer to Fig. 4.30, which is Fig. 4.23 repeated here for convenience.

$$I_{xy} = \int_0^b\int_0^a \frac{m}{ab}\,xy\,dx\,dy$$

where m is the mass of the rectangle. Using Eq. (A2.12) we can write:

$$\therefore \quad I_{xy} = \frac{m}{ab}\left(\int_0^a x\,dx\right)\left(\int_0^b y\,dy\right)$$

Evaluating both the integrals separately we get:

$$\therefore \quad I_{xy} = \frac{m}{ab}\left[\frac{x^2}{2}\right]_0^a\left[\frac{y^2}{2}\right]_0^b = \frac{mab}{4}$$

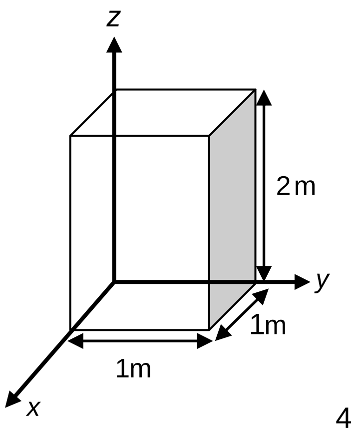


Fig. 4.31: **Diagram for solution of TQ 4.**

4. We determine the mass of the box m using Eq. (4.35) with $\gamma(x,y,z) = \left(9 - z^3\right)\text{kg m}^{-3}$ and Ω (see Fig. 4.31) as defined by the equations:

$0 \le x \le 1;\ 0 \le y \le 1;\ 0 \le z \le 2$ (i)

Then m is:

$$m = \iiint_{\Omega} \left(9 - z^3\right) dx\, dy dz = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{2} \left(9 - z^3\right) dx\, dy\, dz \quad \text{(ii)}$$

We can evaluate this integral as follows (read the margin remark):

$$m = 18 - [x]_0^1 [y]_0^1 \left[\frac{z^4}{4}\right]_0^2 = 14\,\text{kg}$$

Note that: $\int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{2} dx\, dy\, dz$ is the volume of the cube.

5. The flux of the vector field $\vec{\mathbf{F}}$ is:

$$\Phi = \oiint_S \vec{\mathbf{F}} . d\vec{\mathbf{S}} = \oiint_S \left(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} - 2z\,\hat{\mathbf{k}}\right) . d\vec{\mathbf{S}} \quad \text{(i)}$$

We evaluate the flux using the Divergence theorem (Eq. 4.40).

$$\therefore \quad \Phi = \oiint_S \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} - 2z\hat{\mathbf{k}}\right) . d\vec{\mathbf{S}} = \iiint_V \vec{\nabla} . \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} - 2z\hat{\mathbf{k}}\right) dV \quad \text{(ii)}$$

We evaluate the integral on the RHS of Eq. (ii) by first calculating:

$$\vec{\nabla} . \left(x\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} - 2z\,\hat{\mathbf{k}}\right) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(-2z) = 0 \quad \text{(iii)}$$

By replacing the divergence of the vector field from Eq. (iii) into Eq. (ii) we get:

$$\Phi = 0$$

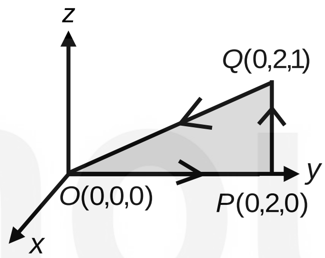


Fig. 4.32: **Path** *OPQ* **for solution of TQ 6.**

6. First we shall calculate $\oint_C \vec{\mathbf{A}} . d\vec{\mathbf{l}}$ where C is shown in Fig. 4.32. Here

$$\vec{\mathbf{A}} = z^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}},\ d\vec{\mathbf{l}} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$$

$$\therefore \quad \vec{\mathbf{A}} . d\vec{\mathbf{l}} = z^2 dy + yz\, dz$$

Now $\oint_{\mathbf{C}} \vec{\mathbf{A}} . d\vec{\mathbf{l}} = \int_{OP} \vec{\mathbf{A}} . d\vec{\mathbf{l}} + \int_{PQ} \vec{\mathbf{A}} . d\vec{\mathbf{l}} + \int_{QO} \vec{\mathbf{A}} . d\vec{\mathbf{l}}$

For the straight line *OP*, $x = 0$, $0 \le y \le 2$, $z = 0$. Hence $\int_{OP} \vec{\mathbf{A}} . d\vec{\mathbf{l}} = 0$.

For the straight line *PQ*, $x = 0$, $y = 2$, $0 \le z \le 1$. Hence $dy = 0$ and

$$\int_{PQ} \vec{\mathbf{A}} . d\vec{\mathbf{l}} = \int_0^1 2z\, dz = 1$$

And for the straight line *QO*, $x = 0$, $y = 2z$, $1 \le z \le 0$. Also $dy = 2dz$ (see margin remark) and

$$\int_{QO} \vec{\mathbf{A}} . d\vec{\mathbf{l}} = \int_{QO} z^2 dy + \int_{QO} yz dz = \int_{QO} 2z^2 dz + \int_{QO} (2z)\, z\, (dz) = \int_1^0 4z^2\, dz = -\frac{4}{3}$$

OQ is a straight line in the *yz* plane and its equation is $y - 2z = 0$

$$\therefore \quad \oint_C \vec{\mathbf{A}} . d\vec{\mathbf{l}} = 0 + 1 - \frac{4}{3} = -\frac{1}{3}$$

Next we evaluate the integral using Stokes' theorem.

$$\vec{\nabla}\times\vec{\mathbf{A}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & z^2 & yz \end{vmatrix} = -z\hat{\mathbf{i}}$$

Since the path C is traversed anticlockwise, we have $d\vec{\mathbf{S}} = dS\,\hat{\mathbf{i}}$.

Moreover, as S lies on the yz plane, $d\vec{\mathbf{S}} = dydz\,\hat{\mathbf{i}}$

$$\therefore\ \left(\vec{\nabla}\times\vec{\mathbf{A}}\right).d\vec{\mathbf{S}} = \left(-z\hat{\mathbf{i}}\right).\left(dydz\,\hat{\mathbf{i}}\right) = -z\,dydz$$

S is defined by the equations $0 \le y \le 2; 0 \le z \le y/2$

$$\iint_S \left(\vec{\nabla}\times\vec{\mathbf{A}}\right).d\vec{\mathbf{S}} = -\iint_S z\,dy\,dz = -\int_0^2\left(\int_0^{y/2} z\,dz\right)dy$$

$$= -\int_0^2 \frac{y^2}{8}dy = -\frac{1}{8}\left|\frac{y^3}{3}\right|_0^2 = -\frac{1}{3}$$

$$\therefore\ \iint_S \left(\vec{\nabla}\times\vec{\mathbf{A}}\right).d\vec{\mathbf{S}} = \int_C \vec{\mathbf{A}}.d\vec{\mathbf{l}}$$

which is Stokes' theorem.

7. Let S be the surface bounded by the closed curve C. We first note that the given line integral can be written as $\oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}}$ where

$$\vec{\mathbf{F}} = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$$

Applying Stokes' theorem we can write:

$$\oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \iint_S \left(\vec{\nabla}\times\vec{\mathbf{F}}\right).d\vec{\mathbf{S}} = \iint_S \left[\vec{\nabla}\times\left(yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}\right)\right].d\vec{\mathbf{S}}$$

We next find $\vec{\nabla}\times(yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}})$ which is:

$$\vec{\nabla}\times\vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = 0$$

Since $\vec{\nabla}\times\vec{\mathbf{F}} = \vec{\mathbf{0}}$, from Stokes' theorem we get that $\oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = 0$ for any closed contour C.

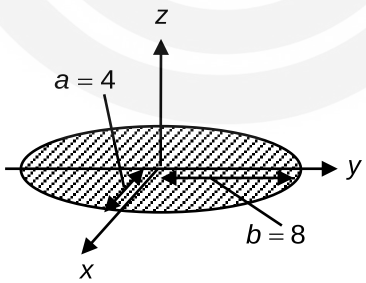


Fig. 4.33: **The shaded region is the surface of integration** *S.*

8. The surface of integration is the shaded region shown in Fig. 4.33 which is an ellipse in the xy plane defined by the equation:

$$\frac{x^2}{16} + \frac{y^2}{64} = 1;\quad z = 0$$

The parameters (semi-major and semi-minor axes) of the ellipse are a = 4 and b = 8. C is the curve enclosing the region. According to Stokes' theorem:

$$I = \oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \iint_S \left(\vec{\nabla}\times\vec{\mathbf{F}}\right).d\vec{\mathbf{S}} \qquad \text{(i)}$$

We first calculate:

$$\vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2x & z^2 \end{vmatrix} = 2\hat{\mathbf{k}}$$

On the xy plane $d\vec{\mathbf{S}} = dx\, dy\, \hat{\mathbf{k}}$.

$$\therefore I = \iint_S (\vec{\nabla} \times \vec{\mathbf{F}}).d\vec{\mathbf{S}} = \iint_S (2\hat{\mathbf{k}}).(dx\, dy\, \hat{\mathbf{k}}) = 2\iint_S dx\, dy$$

By the area property of the double integral, the integral is just:

$$I = 2 \text{ (Area of the Ellipse)}$$

The area of the ellipse is πab so with $a = 4$ and $b = 8$ we get:

$$I = 2[\pi(4 \times 8)] = 64\pi$$

9. Using the divergence theorem, the flux of the vector field $\oiint_S \vec{\mathbf{F}}.d\vec{\mathbf{S}}$ where S is the surface of the sphere of radius two units, is the volume integral $\iiint_V (\vec{\nabla}.\vec{\mathbf{F}})dV$, where V is the volume enclosed by the sphere. We first evaluate $\vec{\nabla}.\vec{\mathbf{F}}$:

$$\vec{\nabla}.\vec{\mathbf{F}} = \frac{\partial(z)}{\partial x} + \frac{\partial(2y)}{\partial y} + \frac{\partial(-x^3)}{\partial z} = 2 \qquad \text{(i)}$$

Using the result of Eq. (i) and the divergence theorem we can write the flux of the vector field $\vec{\mathbf{A}}$ as (see margin remark):

$$\oiint_S \vec{\mathbf{F}}.d\vec{\mathbf{S}} = \iiint_V (\vec{\nabla}.\vec{\mathbf{F}})dV = 2\iiint_V dV = 2\left[\frac{4}{3}\pi(2)^3\right] = \frac{64\pi}{3} \qquad \text{(ii)}$$

> We have used the volume property of the triple integral to write $\iiint_V dV$ as the volume of a sphere of radius 2 units.

10. We have to evaluate $\iint_S \vec{\mathbf{A}}.d\vec{\mathbf{S}}$, where S is the surface of the cube. Using the divergence theorem we can write

$$\iint_S \vec{\mathbf{A}}.d\vec{\mathbf{S}} = \int_0^1\int_0^1\int_0^1 (\vec{\nabla}.\mathbf{A})\, dx\, dy\, dz$$

$$= \int_0^1\int_0^1\int_0^1 10y\ dx\, dy\, dz + 4\int_0^1\int_0^1\int_0^1 dx\, dy\, dz$$

$$= 10\int_0^1 dx \int_0^1 y\, dy \int_0^1 dz + 4\int_0^1\int_0^1\int_0^1 dx\, dy\, dz$$

$$= 10\,[x]_0^1 \left[\frac{y^2}{2}\right]_0^1 [z]_0^1 + 4.1 = 9 \text{ units.}$$

APPENDIX A1

BASIC CONCEPTS OF VECTOR ALGEBRA

In this Appendix, we state the basic results of vector algebra that you need to know while studying this course. You have studied these concepts in Units 1 and 2 of the course BPHCT-131 entitled Mechanics.

Recall that in its geometric representation, the vector is described by a directed line segment. In its algebraic representation the vector is described by its components in a specific coordinate system like the Cartesian coordinate system. The properties of vectors do not depend on the chosen representation.

A vector is represented geometrically or graphically by a **directed line segment** or an arrow, that is, a straight line with an arrowhead. The length of the arrow represents the **magnitude** of the vector quantity, which is a positive scalar quantity and the **arrowhead points along the direction of the vector**.

In the Cartesian coordinate system, a vector $\vec{\mathbf{a}}$ in **two-dimensional space** with tail at the point (x_1, y_1) and head at the point (x_2, y_2) can be represented algebraically in terms of its *x* and *y* components as

$$a = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} \qquad \text{(A1.1a)}$$

where

$$a_x = x_2 - x_1 = a\cos\theta, \quad a_y = y_2 - y_1 = a\sin\theta \qquad \text{(A1.1b)}$$

The **magnitude** of the vector is given by $\left|\vec{\mathbf{a}}\right| = a = \sqrt{a_x^2 + a_y^2}$ (A1.1c)

and its **direction** is given by the angle θ that the vector makes with the positive *x*-axis: $\theta = \tan^{-1}\left(\dfrac{a_y}{a_x}\right)$ (A1.1d)

A vector $\vec{\mathbf{a}}$ in **three-dimensional space** with tail at the point (x_1, y_1, z_1) and head at the point (x_2, y_2, z_2) can be represented algebraically in terms of its *x*, *y* and *z* components as

$$\vec{\mathbf{a}} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} + a_z\,\hat{\mathbf{k}} \qquad \text{(A1.2a)}$$

where $a_x = x_2 - x_1, \quad a_y = y_2 - y_1, \quad a_z = z_2 - z_1$ (A1.2b)

The **magnitude** of $\vec{\mathbf{a}}$ is given by $a = \sqrt{(a_x^2 + a_y^2 + a_z^2)}$ (A1.2c)

The **direction** of the vector $\vec{\mathbf{a}}$ is given by the direction cosines cos α, cos β and cos γ where α, β and γ are the angles that the vector $\vec{\mathbf{a}}$ makes with the *x*, *y* and z-axes. Thus,

$$a_x = (\vec{\mathbf{a}}.\hat{\mathbf{i}}) = a\cos\alpha \qquad \text{(A1.2d)}$$

$$a_y = (\vec{\mathbf{a}}.\hat{\mathbf{j}}) = a\cos\beta \qquad \text{(A1.2e)}$$

$$a_z = (\vec{\mathbf{a}}.\hat{\mathbf{k}}) = a\cos\gamma \qquad \text{(A1.2f)}$$

We now state the properties of vectors.

Equality of Vectors

Two free vectors are equal if they have the same magnitude and direction, regardless of the position of the tail of the vector or their respective components are equal:

$$\vec{\mathbf{a}} = \vec{\mathbf{b}} \text{ iff } a_x = b_x,$$

$$a_y = b_y,$$

$$a_z = b_z$$

If a vector $\vec{\mathbf{b}}$ has the same magnitude but the opposite direction as any other vector $\vec{\mathbf{a}}$, then we can write

$$\vec{\mathbf{b}} = -\vec{\mathbf{a}}$$

Unit Vector

A vector of length or magnitude 1 is called a **unit vector**. By convention, unit vectors are taken to be dimensionless. A unit vector is used to denote a direction in space. Any vector $\vec{\mathbf{a}}$ can be represented as the product of its magnitude (a) and a unit vector along its direction denoted by $\hat{\mathbf{a}}$. Then we have:

$$\vec{\mathbf{a}} = a\,\hat{\mathbf{a}} \quad \text{or} \quad \hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{a} = \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} \qquad \text{(A1.3)}$$

Addition and Subtraction of Vectors

Triangle Law of Vector Addition: If two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ to be added are represented in magnitude and direction by the two sides of a triangle taken in order (which means that the tail of $\vec{\mathbf{b}}$ is at the head of the vector $\vec{\mathbf{a}}$), then their sum or resultant is given in magnitude and direction by the third side of the triangle taken in the opposite order, that is from the tail of the first vector to the head of the second vector (Fig. A1.1).

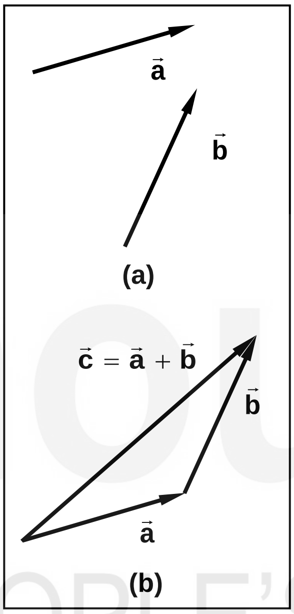


Fig. A1.1: **The triangle law of vector addition.**

Parallelogram Law of Vector Addition: If the two vectors to be added are represented in magnitude and direction by the adjacent sides of a parallelogram, then their resultant is given in magnitude and direction by the diagonal of the parallelogram drawn through the point of intersection of the two given vectors (see Fig. A1.2).

The expressions for the magnitude and direction of the resultant $\vec{\mathbf{c}}$ for two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ having the angle θ between them are given as follows:

$$c = \sqrt{b^2 + 2ab\cos\theta + a^2} \qquad \text{(A1.4a)}$$

$$\alpha = \tan^{-1}\left[\frac{a\sin\theta}{b + a\cos\theta}\right] \qquad \text{(A1.4b)}$$

Here a, b and c are the magnitudes of the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$, respectively, and the angle α between the vectors $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ gives the direction of the vector $\vec{\mathbf{c}}$ (see Fig. A1.2).

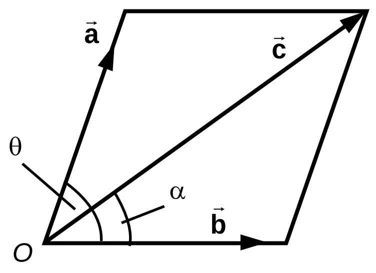


Fig A1.2: **The parallelogram law of vector addition.**

In terms of components in the Cartesian system, the sum

$$\vec{\mathbf{c}} = (c_x \hat{\mathbf{i}} + c_y \hat{\mathbf{j}} + c_z \hat{\mathbf{k}})$$

of the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is given as

$$\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}} \Rightarrow (a_x + b_x)\,\hat{\mathbf{i}} + (a_y + b_y)\,\hat{\mathbf{j}} + (a_z + b_z)\,\hat{\mathbf{k}}$$

It is possible to add any number of vectors by the repeated application of the triangle law of vector addition. We also use the polygon law of vector addition.

Polygon Law of Vector Addition: If a number of vectors are represented in magnitude and direction, by the sides of a polygon, taken in order, then the resultant vector is represented in magnitude and direction by the closing side of the polygon taken in the opposite order, that is from the tail of the first vector to the head of the last vector (see Fig. A1.3).

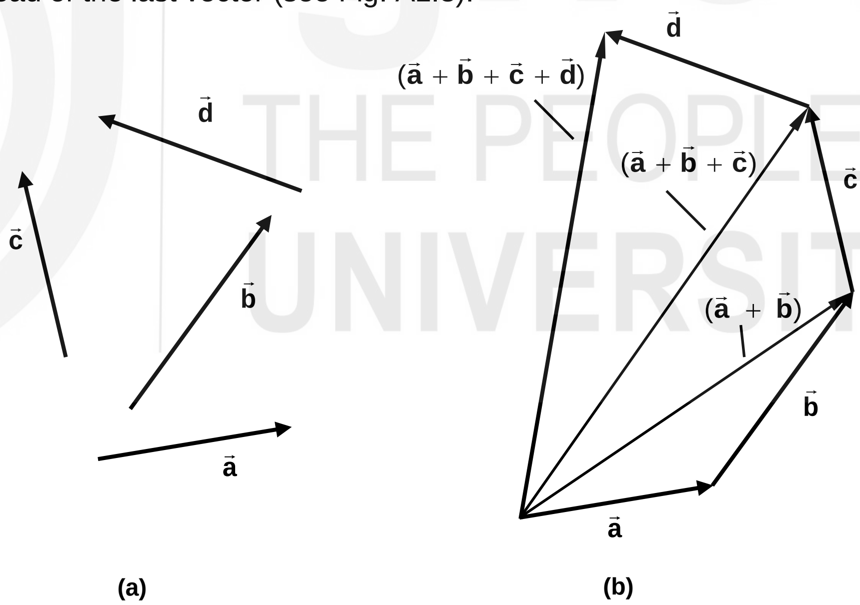


Fig. A1.3: **Polygon law of vector addition applied for determining the resultant $(\vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}} + \vec{\mathbf{d}})$ of four vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ and $\vec{\mathbf{d}}$.**

Vector addition is commutative and associative:

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}} + \vec{\mathbf{a}} \quad \text{and} \quad (\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}} = \vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) \qquad \text{(A1.5)}$$

Subtraction of a vector $\vec{\mathbf{b}}$ from a vector $\vec{\mathbf{a}}$ denoted by $\vec{\mathbf{a}} - \vec{\mathbf{b}}$ is just the sum of the vectors $\vec{\mathbf{a}}$ and $(-\vec{\mathbf{b}})$:

$$\vec{\mathbf{a}} - \vec{\mathbf{b}} = \vec{\mathbf{a}} + (-\vec{\mathbf{b}}) \qquad \text{(A1.6)}$$

Multiplication of a vector by a scalar

A vector $\vec{\mathbf{a}}$ when multiplied by a scalar quantity m, is equal to the vector $m\vec{\mathbf{a}}$, in the same direction as $\vec{\mathbf{a}}$ and having the magnitude $|m||\vec{\mathbf{a}}|$. The following is true for the multiplication of a vector by a scalar:

$m(n)\vec{\mathbf{a}} = (m)n\vec{\mathbf{a}} = mn\vec{\mathbf{a}}$ **Associative Law** (A1.7a)

$(m+n)\vec{\mathbf{a}} = m\vec{\mathbf{a}} + n\vec{\mathbf{a}}$ and $m(\vec{\mathbf{a}} + \vec{\mathbf{b}}) = m\vec{\mathbf{a}} + m\vec{\mathbf{b}}$ **Distributive Laws** (A1.7b)

If $m = 0$, then $m\vec{\mathbf{a}}$ is a **null** or **zero** vector, which has zero magnitude but no definite direction.

Components of a vector in a given direction

A vector can be resolved into its **component vectors** along any arbitrary direction. The components of a vector $\vec{\mathbf{a}}$ parallel and perpendicular to any other vector $\vec{\mathbf{b}}$ which makes an angle θ with the vector $\vec{\mathbf{a}}$ are given as (see also Fig. A1.4):

The component of $\vec{\mathbf{a}}$ parallel to $\vec{\mathbf{b}} = \dfrac{\vec{\mathbf{a}}.\vec{\mathbf{b}}}{|\vec{\mathbf{b}}|} = |\vec{\mathbf{a}}|\cos\theta$ (A1.8a)

The component of $|\vec{\mathbf{a}}|$ perpendicular to $\vec{\mathbf{b}} = |\vec{\mathbf{a}}|\sin\theta$ (A1.8b)

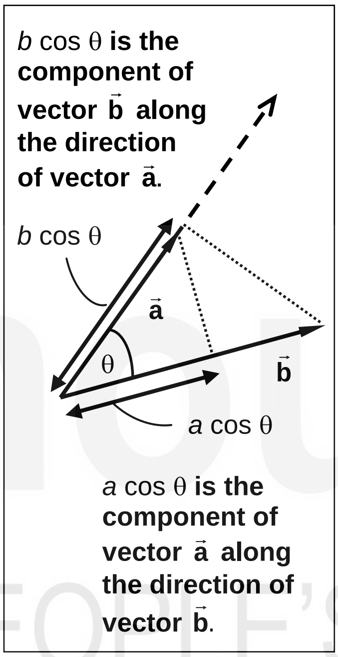


Fig. A1.4: **Projection or the component of a vector along the direction of another vector**.

Scalar product

The **scalar product** of two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ called "a dot b" and denoted by $\vec{\mathbf{a}}.\vec{\mathbf{b}}$ is a scalar quantity defined as

$$\vec{\mathbf{a}}.\vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\cos\theta = ab\cos\theta \quad \text{(A1.9a)}$$

In component form $\vec{\mathbf{a}}.\vec{\mathbf{b}} = a_x b_x + a_y b_y + a_z b_z$ (A1.9b)

Vector product

The **vector product** of two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ called "a cross b" and denoted by $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is a vector quantity defined as

$$\vec{\mathbf{c}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}} = ab\sin\theta\,\vec{\mathbf{c}} \text{ with magnitude } c = ab\sin\theta \quad \text{(A1.10a)}$$

The direction of $\vec{\mathbf{c}}$ is determined by the right hand rule. The vector $\vec{\mathbf{c}}$ is perpendicular to the plane containing the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

In the component form,

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\hat{\mathbf{i}} + (a_z b_x - a_x b_z)\hat{\mathbf{j}} + (a_x b_y - a_y b_x)\hat{\mathbf{k}}$$

(A1.10b)

APPENDIX A2

DOUBLE INTEGRALS

In Unit 3, you have learnt how to integrate vector functions of single variables. It is possible to extend the idea of a definite integral to calculate double and triple integrals which are integrals of functions of two and three variable respectively. Double and triple integrals have many applications in physics. For example, we use these integrals to determine the volume of an object bound by an arbitrary surface, its mass, its centre of mass or its moment of inertia. In this appendix, we explain in brief how to evaluate a double integral, which is an integral of a function of two variables.

A2.1 DOUBLE INTEGRALS

We first develop the geometrical concept of the double integral. Before we do this, however, you should revise the concept of the definite integral of a function.

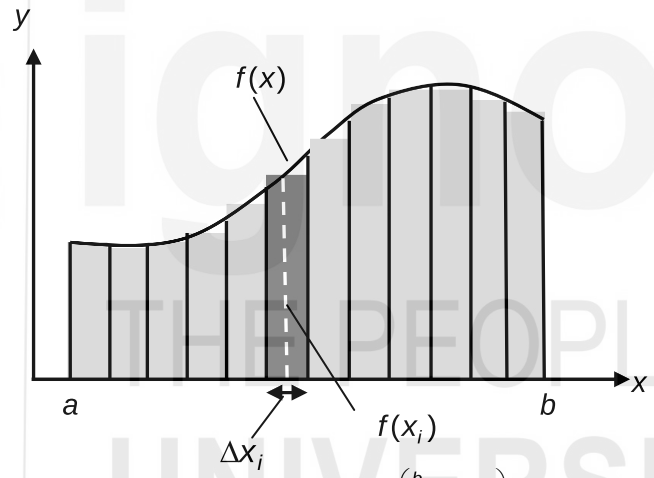


Fig. A2.1: **Definite integral of the function** $f(x)$: $\left(\int_a^b f(x)dx\right)$ **as area under a curve.**

We define the definite integral of a function *f* (*x*) over the interval [*a*, *b*] on the *x*-axis, denoted by $\int_a^b f(x)dx$ as the limit of a sum. We start by dividing the interval [*a*, *b*] into *n* sub-intervals, the *i*th sub-interval having a width Δx_i, as shown in Fig. A2.1. The sum $\sum f(x_i)\Delta x_i$ is the total area of the *n* rectangles we see in the figure. Then the sum of the areas of the rectangles is approximately the area under the curve. It is also clear that if we increase the number of sub-intervals, i.e., increase the value of *n*, the rectangles become narrower, and the total area of the rectangles comes closer and closer to the area under the curve. The exact area is then given by the limit of the sum as *n* goes to infinity:

$$\text{Area under the curve } f(x) = \lim_{n\to\infty} \sum_{i=1}^{i=n} f(x_i)\Delta x_i \qquad \text{(A2.1a)}$$

The expression on the right hand side of Eq. (A2.1a) is called the definite integral of *f*(*x*) from *a* to *b* and denoted as follows:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{i=n} f(x_i)\Delta x_i \qquad \text{(A2.1b)}$$

This definition of the definite integral holds even if *f* (*x*) has both positive and negative values in the interval [*a*, *b*]. The integral exists if the function *f* is continuous on [*a*, *b*] or has only a finite number of jump discontinuities.

Let us now explain the concept of **double Integral** of the function *f* (*x*,*y*) over a bounded region *R* on the *xy*-plane denoted by

$$\iint_R f(x,y)\,dx\,dy$$

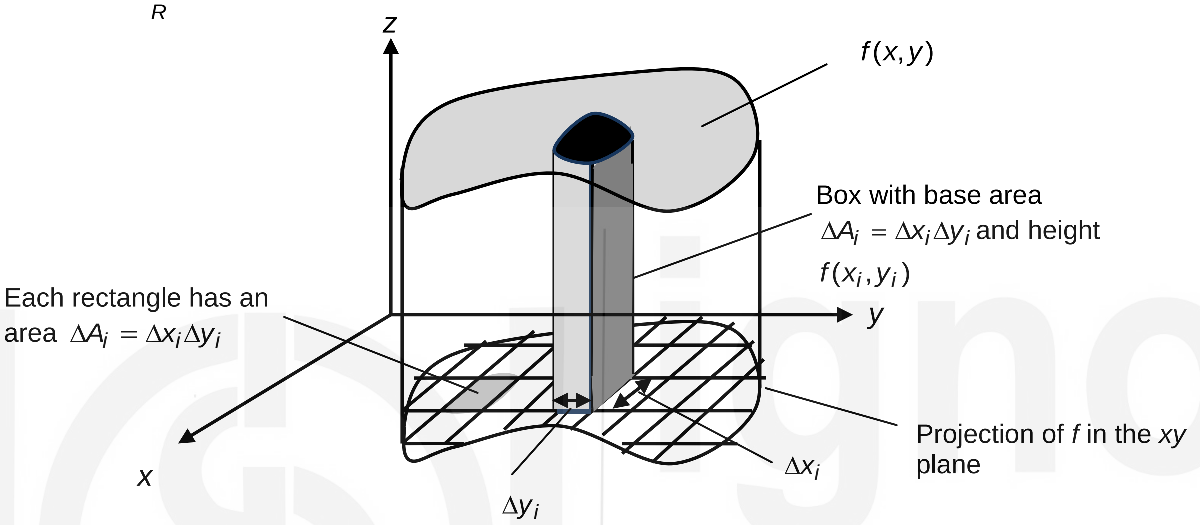


Fig. A2.2: **Double Integral** $\iint_R f(x,y)dxdy$ **as the volume under a surface** $f(x,y)$ **and above the region** *R* **in the** *xy* **plane.**

The definition of the double integral is similar to that of a definite integral. We divide the region *R* in the *xy* plane into *n* tiny rectangles by drawing lines parallel to the *x* and *y* axes. Each rectangle has an area ΔA_i (see Fig. A2.2).

We number the rectangles within *R* from *i* = 1 to *i* = *n* and choose a point $(x_i,\ y_i)$ in each rectangle. Now consider the sum:

$$S_n = \sum_{i=1}^{n} f(x_i,\ y_i)\Delta A_i = \sum_{i=1}^{n} f(x_i,\ y_i)\Delta x_i \Delta y_i \qquad \text{(A2.2)}$$

We can evaluate this sum for increasing values of *n* such that the maximum diagonal of the rectangles goes to zero as the number of rectangles goes to infinity. If $f(x,y)$ is a continuous function in *R*, these sums (also called the Reimann sums) converge to a limiting value which does not depend on either the values of (x_i, y_i) or the choice of subdivision. This limit is the double integral of the function $f(x,y)$ over the region *R*.

The double integral of a function *f*(*x, y*), which is defined for all (*x, y*) in a closed, bounded region *R* in the *xy* plane, is written as the limit of a sum as follows:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i = \iint_R f(x,y)dA = \iint_R f(x,y)dxdy \qquad \text{(A2.3)}$$

Just as the area under the curve *f* (*x*) in Fig. A2.1b is the area under the curve above the *x*-axis.

For $f(x,y) \geq 0$, the double integral gives us the volume of the solid that lies below the surface $f(x,y)$ and above the region *R* in the *xy* plane (read the margin remark).

PROPERTIES OF THE DOUBLE INTEGRALS

For two functions $f(x,y)$ and $g(x,y)$, which are defined and continuous in a region *R*:

1. $$\iint_R c\, f(x,y)\,dxdy = c \iint_R f(x,y)\,dxdy \qquad \text{(A2.4)}$$

   where *c* is a constant.

2. **Linearity**

   $$\iint_R [\alpha f(x,y) + \beta g(x,y)]\,dxdy = \alpha \iint_R f(x,y)\,dxdy + \beta \iint_R f(x,y)\,dxdy \qquad \text{(A2.5)}$$

   where α and β are constants.

3. **Additivity**

   If the region of integration *R* can be broken up into a finite number of non overlapping regions R_1 , R_2 ……..R_n, (Fig. A2.3), then we can write:

   $$\iint_R f(x,y)\,dxdy = \iint_{R_1} f(x,y)\,dxdy + \iint_{R_2} f(x,y)\,dxdy + ... + \iint_{R_n} f(x,y)\,dxdy \qquad \text{(A2.6)}$$

4. **Area Property**

   If the function being integrated is *f(x,y)* = 1, then

   $$\iint_R [1]\,dxdy = \text{Area of the region } R \qquad \text{(A2.7)}$$

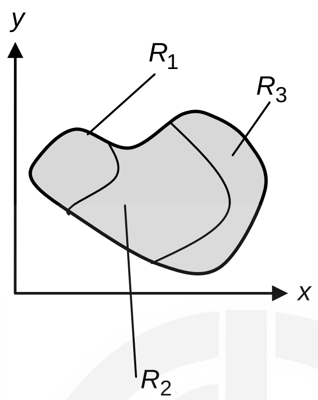


Fig. A2.3: **The region *R* is broken up into three overlapping regions R_1, R_2 and R_3.**

We briefly explain how to evaluate double integrals

A2.2 EVALUATION OF DOUBLE INTEGRALS

To evaluate a double integral $\iint_R f(x,y)\,dxdy$ over a region *R*, we have to carry out two successive integrations over the variables *x* and *y*. How is this done? You will see that there are actually two ways of doing this. Let us see what these are.

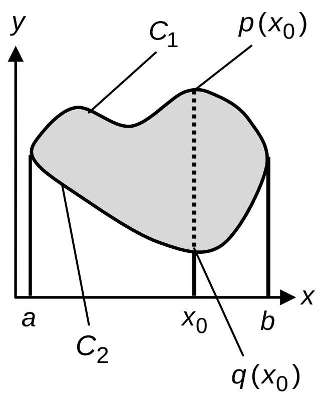


Fig. A2.4: **Evaluating the double integral as an iterated integral (Eq. A2.9).**

First let us define the region of integration *R* shown in Fig. A2.4 as:

$$a \leq x \leq b;\; q(x) \leq y \leq p(x) \qquad \text{(A2.8)}$$

As you can see from Fig. A2.4, the values of the *x* coordinate at the two extremities of the region are *x* = *a* and *x* = *b*. Now in between these two values of *x*, the region *R* is bound by the two curves C_1 and C_2. These two curves are given by the equations *y* = *p*(*x*) and *y* = *q*(*x*), respectively. What does this tell us?

It tells us that for each value of *x* in the interval [*a*, *b*], the value of *y* ranges between *q*(*x*) and *p*(*x*) which are the points on the lower and upper curves bounding the region *R*. So for any value of *x*, for example $x = x_0$ in [a, b], the values of *y* range from $q(x_0)$ to $p(x_0)$. Now if we were to first integrate the function *f*(*x, y*) over the variable *y*, holding *x* as a constant, the limits in *y* would be from *y* = *q*(*x*) to *y* = *p*(*x*). The result would be a function of only *x*. Next we integrate this function of *x* with respect to *x* from *x* = *a* to *x* = *b*. Thus, we cover the entire region of *R* while integrating over the two variables.

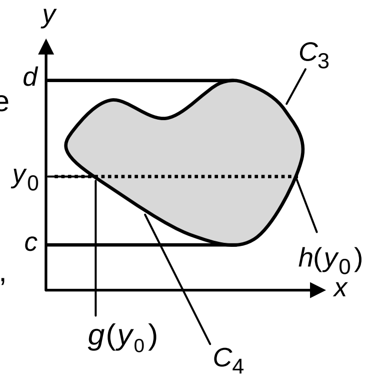


Fig. A2.5: **Evaluating the double integral as an iterated integral (Eq. A2.11).**

Therefore:

$$\iint_R f(x,y)dxdy = \int_{x=a}^{b}\left[\int_{y=q(x)}^{p(x)} f(x,y)dy\right]dx \qquad \text{(A2.9)}$$

The quantity in the brackets, which is evaluated first is the integral of *f(x, y)* over *y* alone, with the limits as specified. The result of this integral is a function of *x* alone which is then integrated over *x*, over the limits shown.

We could have chosen to carry out this integration in another way. Refer to Fig. A2.5. We can write down the limits on *x* and *y* for the same region *R* in a different way as we describe below:

$$c \le y \le d;\ g(y) \le x \le h(y) \qquad \text{(A2.10)}$$

Now for any value of *y* in the interval [*c*, *d*] the value of *x* is decided by the function *h*(*y*) (curve C_3) and *g*(*y*) (curve C_4), which respectively now defined the upper and lower boundaries of *R*. Now we can integrate the function *f*(*x, y*) over the variable *x*, holding *y* as a constant, the limits of the integral would be from *x* = *g*(*y*) to *x* = *h*(*y*) and the result would be a function of only *y*. We then integrate this function of *y* over *x* from *y* = *c* to *y* = *d*. So we get an alternative expression for the evaluation of the double integral:

$$\iint_R f(x,y)dxdy = \int_{y=c}^{d}\left[\int_{x=g(y)}^{h(y)} f(x,y)dx\right]dy \qquad \text{(A2.11)}$$

Both these iterated integrals defined in Eqs. (A2.9) and (A2.11) are equal if *p*(*x*), *q*(*x*), *g*(*y*), *h*(*y*) are continuous functions, for the limits defined in Eqs. (A2.8) and (A2.10). This is the consequence of a theorem in multivariable calculus called the Fubini's theorem, which is beyond the scope of this course.

As before, the integral within the brackets is carried out first. Both Eqs. (A2.9) and (A2.11) are equivalent methods of determining the double integral. In Eq. (A2.9), the integral over the variable *y* is carried out first. In Eq. (A2.11), the integral over *x* is carried out first.

Suppose *R* cannot be represented by the inequalities shown in Eq. (A2.8) or Eq. (A2.10), but can be subdivided into many parts that can be represented by inequalities, then we evaluate the double integral over each part and sum up to get the result as the double integral over *R*.

The integrals of the form $\int_{y=c}^{d}\left[\int_{x=g(y)}^{h(y)} f(x,y)dx\right]dy$ and $\int_{x=a}^{b}\left[\int_{y=q(x)}^{p(x)} f(x,y)dy\right]dx$ are called **iterated** (repeated) **integrals** because they are evaluated first by integrating with respect to one variable, either *x* or *y*, as the case may be and then integrating the result with respect to the second variable. Multiple integrals are usually integrated as iterated integrals. We shall use the same technique to evaluate triple integrals as you will see in Unit 4. Let us summarise these results.

Recap

EVALUATION OF A DOUBLE INTEGRAL

Suppose that $f(x, y)$ is a continuous function on the region R. If R is described by the inequalities $a \le x \le b, q(x) \le y \le p(x),$ then

$$\iint_R f(x,y)dxdy = \int_{x=a}^{b}\left[\int_{y=q(x)}^{p(x)} f(x,y)dy\right]dx \qquad \text{(A2.9)}$$

If R is described by the inequalities $c \le y \le d, g(y) \le x \le h(y)$, then

$$\iint_R f(x,y)dxdy = \int_{y=c}^{d}\left[\int_{x=g(y)}^{h(y)} f(x,y)dx\right]dy \qquad \text{(A2.11)}$$

In some textbooks that the iterated integrals are sometimes written without the bracket as follows:

$$\int_a^b\left[\int_c^d f(x,y)dy\right]dx = \int_a^b\int_c^d f(x,y)dydx \qquad \text{(A2.12a)}$$

and

$$\int_c^d\left[\int_a^b f(x,y)dx\right]dy = \int_c^d\int_a^b f(x,y)dxdy \qquad \text{(A2.12b)}$$

By convention, the limits of integration on the variable over which the integration is carried out first, appears on the inner integral sign.

Note: The order of integration in Eqs. (A2.12a and b) is different. It is as shown in the left hand side of each of these equations.

In Eq. (A2.12a), we write *dy dx* in the integrand. This means that we first integrate with respect to y over the interval $[c,d]$ and then with respect to x over $[a,b]$. In Eq. (A2.12b), we write *dx dy* in the integrand. So, the integration is first with respect to x and then with respect to y.

A Special Case

An important special case is when we evaluate double integrals for which the following is true:

i) the region R is a rectangle defined by the limits, say $a \le x \le b, c \le y \le d$ (Fig. A2.6).

ii) The function $f(x, y) = h(x)\, g(y)$, that is $f(x, y)$ is a product of two functions, one of which is a function of only x, $h(x)$ and the other a function of only y, that is $g(y)$.

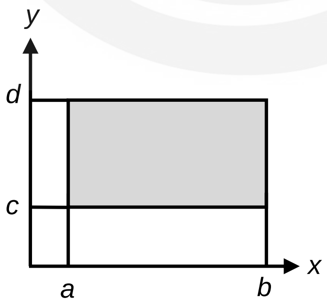


Fig. A2.6: **A rectangular region of integration** $a \le x \le b, c \le y \le d$.

Then the double integral can be evaluated as:

$$\iint_R f(x,y)dxdy = \left[\int_{x=a}^{b} h(x)dx\right]\left[\int_{y=c}^{d} g(y)dy\right] \qquad \text{(A2.13)}$$

As you can see, here we can integrate with respect to each variable separately. We now evaluate some double integrals to illustrate these methods.

EXAMPLE A2.1: DOUBLE INTEGRAL OVER A RECTANGULAR REGION

Evaluate the integral $\iint_R \sin x \cos y \, dx \, dy$ where R is a square on the xy plane defined by $0 \le x \le \pi/2, 0 \le y \le \pi/2$.

SOLUTION ■ Using (Eq. A2.13), we solve the integral as:

$$\iint_R \sin x \cos y dx dy = \left[\int_{x=0}^{\pi/2} \sin x \, dx\right]\left[\int_{y=0}^{\pi/2} \cos y \, dy\right] = [-\cos x]_0^{\frac{\pi}{2}} \times [\sin y]_0^{\frac{\pi}{2}} = 1$$

Note that if the region R is rectangular, but $f(x,y)$ cannot be written as the product of two functions i.e., $f(x,y) \ne h(x)g(y)$, we shall have to carry out an iterated integral.

Let us now work out an example of an iterated integral.

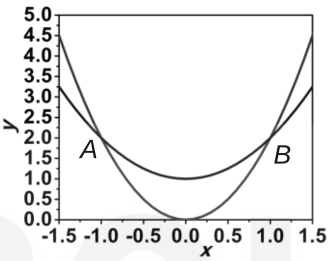


Fig. A2.7: **Region of integration for Example A2.2. The two curves intersect at** $x = -1$ **and** $x = 1$.

EXAMPLE A2.2: DOUBLE INTEGRAL

Evaluate the integral $\iint_R (x + 4y) \, dx \, dy$ where R is the region bounded by the curves $y = 2x^2$ and $y = 1 + x^2$.

SOLUTION ■ In Fig. A2.7, we plot the two curves $y = 2x^2$ and $y = x^2 + 1$ which define R. Now, as you can see from the figure, the two curves intersect at the points A and B. At the points of intersection of the two curves, we have $2x^2 = x^2 + 1$. Solving for x we have:

$$2x^2 = x^2 + 1 \Rightarrow x^2 = 1 \quad \Rightarrow x = -1, 1$$

So the points of intersection are $x = 1$ and $x = -1$. This marks the limits of x for the region of integration. As you can see, for each value of x in the range $-1 \le x \le 1$, the value of y will vary in the range $2x^2 \le y \le 1 + x^2$. Now let us use Eq. (A2.9) to evaluate the integral with $q(x) = 2x^2$ and $p(x) = x^2 + 1$. Then we write:

$$\iint_R (x + 4y) \, dx \, dy = \int_{x=-1}^{1} \left[\int_{y=2x^2}^{1+x^2} (x + 4y) dy\right] dx \qquad \text{(i)}$$

You can see that we have used $a = -1, b = 1$. Now we first carry out the integration within the bracket, integrating over y and taking x as a constant.

$$\int_{y=2x^2}^{x^2+1} (x + 4y) dy = \left[xy + 2y^2\Big|_{2x^2}^{1+x^2}\right] = x(1 + x^2 - 2x^2) + 2(1 + x^2)^2 - 2(2x^2)^2$$

$$= 2 + x + 4x^2 - x^3 - 6x^4 \qquad \text{(ii)}$$

Substituting the quantity in the bracket in Eq. (i) by the expression in Eq. (ii) we get,

$$\iint_R (x+4y)\,dx\,dy = \int_{x=-1}^{1} \left[2+x+4x^2-x^3-6x^4\right]dx$$

$$= \left[2x+\frac{x^2}{2}+4\frac{x^3}{3}-\frac{x^4}{4}-6\frac{x^5}{5}\right]_{-1}^{1} = \frac{64}{15} \quad \text{(iii)}$$

SAQ - Double integrals over a rectangular region

1. Evaluate the following integrals:

   a) $\int_0^2\int_0^2 xye^{(x+y)}dydx$

   b) $\int_{-1}^{0}\int_0^2 y\sin\frac{\pi x}{4}dxdy$

2. Evaluate $\iint_R (x^4-2y)dx\,dy$, where R is the region defined by the equations $-1 \le x \le 1$ and $-x^2 \le y \le x^2$.

A2.3 SOLUTIONS AND ANSWERS

Self-Assessment Questions

When we integrate over *y*, *x* is a constant.

1. a) Using Eq. (A2.13), we write the integral:

$$I = \int_{x=0}^{2}\int_{y=0}^{2} x\,y\,e^{(x+y)}\,dy\,dx = \left[\int_0^2 x\,e^{x}dx\right]\left[\int_0^2 y\,e^{y}dy\right]$$

$$= \left[x e^x - e^x\right]_0^2 \times \left[y e^y - e^y\right]_0^2 = \left(e^2+1\right)^2$$

b) Using Eq. (A2.13), we write

$$I = \int_{y=-1}^{0}\int_{x=0}^{2} y\sin\left(\frac{\pi x}{4}\right)dx\,dy = \left[\int_{-1}^{0} y\,dy\right]\left[\int_0^2 \sin\left(\frac{\pi x}{4}\right)dx\right]$$

$$= \left[\frac{y^2}{2}\right]_{-1}^{0}\left[-\left(\frac{4}{\pi}\right)\cos\left(\frac{\pi x}{4}\right)\right]_0^2 = \frac{2}{\pi}$$

2. We use Eq. (A2.9) to write:

$$I = \int\int_R \left(x^4-2y\right)dx\,dy = \int_{x=-1}^{1}\left[\int_{y=-x^2}^{x^2}\left(x^4-2y\right)dy\right]dx$$

Carrying out the integral over *y* first and applying the limits of integration, we get:

$$I = \int_{-1}^{1}\left[x^4y-y^2\right]_{-x^2}^{x^2} dx = \int_{-1}^{1}\left[2x^6\right]dx$$

We now integrate over *x* to get:

$$I = \left[\frac{2x^7}{7}\right]_{-1}^{1} = \frac{4}{7}$$

TABLES OF DERIVATIVES AND INTEGRALS

Table A1.1: Derivatives of simple functions

S. No.	df/dx	S. No.	df/dx
1.	$\frac{d}{dx}(c) = 0$, c constant	10.	$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$, $x \neq 0$
2.	$\frac{d}{dx}(x) = 1$	11.	$\frac{d}{dx}(x^{-n}) = -\frac{n}{x^{n+1}}$
3.	$\frac{d}{dx}(x^n) = nx^{n-1}$	12.	$\frac{d}{dx}[g(x) + h(x)] = \left[\frac{d}{dx}g(x)\right] + \left[\frac{d}{dx}h(x)\right]$
4.	$\frac{d}{dx}(\sin x) = \cos x$	13.	$\frac{d}{dx}[f(x)\,g(x)] = \left[\frac{df}{dx}\right]g + f\left[\frac{dg}{dx}\right]$
5.	$\frac{d}{dx}(\cos x) = -\sin x$	14.	$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$, $g \neq 0$
6.	$\frac{d}{dx}(\tan x) = \sec^2 x$	15.	$\frac{d}{dx}(e^x) = e^x$
7.	$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$	16.	$\frac{d}{dx}\ln x = \frac{1}{x}$
8.	$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$	17.	$\frac{d}{dx}(c^x) = c^x \ln c$, $c > 0$
9.	$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$	18.	$\frac{d}{dx}\log_c x = \frac{1}{x \ln c}$ $c \neq 1$, $c > 0$

Table A1.2: Integrals of simple functions

S. No.	Integral	S. No.	Integral
1.	$\int a\, dx = ax + c$, a and c constants	5.	$\int \sin x\, dx = -\cos x + c$, c constant
2.	$\int x^n\, dx = \dfrac{x^{n+1}}{n+1} + c$, c constant	6.	$\int \cos x\, dx = \sin x + c$, c constant
3.	$\int \dfrac{1}{x}\, dx = \ln\lvert x\rvert + c$, c constant	7.	$\int \tan x\, dx = \ln\lvert\sec x\rvert + c$, c constant
4.	$\int e^x\, dx = e^x + c$, c constant	8.	$\int e^{ax}\, dx = ae^{ax} + c$, a and c constants