
UNIT 2 COMPOUND STATEMENTS AND THEIR TRUTH-VALUES

Contents

- 2.0 Objectives
- 2.1 Introduction
- 2.2 Simple and Compound Statements
- 2.3 Sentential Connectives
- 2.4 Compound Propositions and Their Truth-Values
- 2.5 Other Forms of Compound Proposition
- 2.6 Let Us Sum Up
- 2.7 Key Words
- 2.8 Further Readings and References
- 2.9 Answers to Check Your Progress

2.0 OBJECTIVES

After you grasp the contents of this unit you should be in a position to:

- analyse any compound proposition to determine its truth-value.
- realise that always symbolic representation of statements helps better understanding than verbal representation which is not only more complicated in structure but also ambiguous.
- understand that a compound proposition may be highly complicated as far as structure is concerned, but it does not affect the technique of determining the truth-value.
- determine the width of spectrum of compound proposition and simple form of compound from the complicated form of compound proposition.

2.1 INTRODUCTION

In this unit an attempt is being made to project the structure of and variety in proposition in a new perspective. Secondly, two shades of meaning of compound proposition will be distinguished in order to accommodate one type of statements, which looks like simple. A clear definition of truth-function is attempted by considering two parameters simultaneously.

2.2 SIMPLE AND COMPOUND STATEMENTS

In this unit, we consider two kinds of statements; simple and compound. This kind of distinction is similar to grammatical distinction. However, there is a sharp difference. A

compound statement in grammatical sense is independent of its components as far as its truth-value is concerned. However, in logical sense the truth or falsity of compound proposition depends upon the truth or falsity of its components. Simple proposition does not need any definition. It consists of only one sentence in grammatical sense. Compound statement, on the other hand, consists of two or more than two 'statements'. The last word should be carefully observed. It just says 'statements'. In other words, the components of a compound statement may be simple or themselves compound. Though the distinction per se is too a simple, statements may be deceptive. Consider the following examples:

- 1 Grass is green.
- 2 Einstein is a physicist and Lorenz was his professor.
- 3 Descartes is a philosopher and mathematician.

It is easy to conclude that the first statement is simple and the second statement is compound. However, we should not be hasty in judging the third proposition. It only seems to be a simple proposition. In reality, it is a compound statement. It can be analysed as follows: Descartes is a philosopher and Descartes is a mathematician. In the language of predicate logic compound proposition can be understood as follows; if there are two predicates then there are two propositions. And if there are three predicates, then there are three propositions and so on.

2.3 SENTENTIAL CONNECTIVES

A compound proposition can be generated in several ways. Classical logic says that a proposition is generated when subject and predicate terms are conjoined by copula. Likewise, modern logic says that a compound proposition is generated when two or more than two propositions are conjoined by what is known as sentential connective. There are five types of sentential connectives and therefore, there are five types of compound statements. 'And', 'if...then', 'or', 'not' and 'if and only if' (iff) are the connectives used to conjoin the statements. While providing descriptive account, connectives are shown, initially, in upper case letters for the sake of clarity only. Further, all letters printed in lower case below statements symbolise respective statements.

- I) AND: 'AND' is one type of sentential connective. When two propositions are connected by this connective, a compound proposition is generated. This type of compound proposition is known as 'CONJUNCTIVE' proposition or we simply say 'CONJUNCTION'. Consider first simple propositions:

Water flows down hill.

The sun is bright.

It is very easy to form a conjunctive proposition; just place 'AND' between 'water flows downhill' and 'the sun is bright'. We get the statement

Water flows downhill AND the sun is bright.

‘inclusive’ or and the second one is called ‘exclusive’ or. The process of obtaining disjunction is very simple. The connective ‘OR’ is placed between simple propositions. The resultant statement is a disjunctive one. Take these statements:

- | | | |
|----|--|---|
| 10 | Reason is the true friend of mankind. | p |
| 11 | Treason is the worst enemy of the state. | q |

With these two statements we obtain the required disjunctive statement:

12 ‘Reason is the true friend of mankind OR Treason is the worst enemy of the state’.

When it is symbolized, it becomes p or q. The connective ‘OR’ is symbolized by using the symbol ‘ \vee ’. This symbol is called Wedge. p or q becomes $p \vee q$. This particular statement is an example for ‘inclusive’ OR. It is called inclusive because the statement also includes third possibility. Accordingly, it can be further extended in the following manner:

13 ‘Reason is the true friend of mankind or treason is the worst enemy of the state’ or both.

The last word ‘both’ is the extended part of original compound statement. This is third possibility, which cannot be logically ruled out. If third possibility is admissible in any disjunctive proposition, then ‘OR’ becomes inclusive. There are cases when third possibility is not admissible. Consider these two statements:

14 ‘Rich people are generous or greedy.’

It does not admit further extension. It does not make sense to say that

15 ‘Rich people are generous or greedy or both generous and greedy.’

Since the extended part is inadmissible in this example ‘OR’ is regarded as exclusive or. When disjunction consists of exclusive or, the proposition is symbolized as

$$p \vee q$$

At this juncture a clarification is necessary. When is ‘OR’ inclusive and when is it exclusive? There is no law of logic as such which stipulates the conditions under which ‘OR’ becomes inclusive and conditions under which ‘OR’ becomes exclusive. We have to depend upon the ‘meaning’ of certain terms employed in the construction of statements. Consider propositions 10 and 11. We admit that these two statements do not exclude each other based on what these statements ‘really’ mean. However the same is not the case with propositions 14. The terms ‘greedy’ and ‘generous’ mean so differently that they both ‘cannot’ be the attributes of the very same class or individual. In other words, if rich people are greedy surely some other class of people can be generous and vice versa. Hence meaning alone can be our guide in determining whether ‘or’ is inclusive or exclusive.

Generally, disjunction is expressed in terms of 'EITHER ... OR'. There is no harm in omitting the former. Both usages are admissible.

IV) NOT: In modern logic, when the connective NOT is appended to the given propositions, it becomes a compound proposition. However, grammar does not allow it. Therefore we have to treat this as a special case within the structure of modern logic. We obtain 'NEGATION' when NOT is used. This is another kind of compound proposition in strictly logical sense because the use of this word alters the truth-value of the given proposition. The connective NOT is appended to the given propositions in several ways. Negation may begin with expressions like "It is NOT the case that... .." or "it is NOT true that... .." Consider this example:

16 The sun rises in the east. - p

Now this statement is negated and expressed in three different ways.

17 It is NOT the case that the sun rises in the east. - NOT p

17a It is NOT true that the sun rises in the east. - NOT p

17b The sun does NOT rise in the east. - NOT p

It must be noted that all these three statements exactly mean the same and all of them negate the statement 16. Now we will symbolize the statement, using symbol for negation, '¬'

16 p

17 ¬ p

'Not' was symbolized earlier in a different way. The symbol '∼' was used earlier to denote negation. This is read curl or tilde. Russell and others used this symbol.

V IF AND ONLY IF: When this connective is used we obtain 'BICONDITION'. We will insert this connective between two statements to obtain 'BICONDITIONAL' proposition. Consider these two examples:

18 Mr. A is a bachelor. - p

19 Mr. A is an unmarried male. - q

Now connect 18 and 19 using the given connective.

Mr. A is a bachelor IF AND ONLY IF Mr. A is an unmarried male.

This connective is symbolized in this manner '⟷'. BICONDITIONAL proposition is represented as follows; p ⟷ q. Negation (¬) and biconditional are (p ⟷ q) special kinds of compound proposition. This will become clear in the next section.

Check Your Progress I

Note: a) Use the space provided for your answer.

b) Check your answers with those provided at the end of the unit.

1) Distinguish 'compound' in grammatical sense from 'compound' in logical sense.

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.....

2) Bring out the difference and similarity with respect to copula and sentential connective.

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.....

2.4 COMPOUND PROPOSITIONS AND THEIR TRUTH-VALUES

Classical logic stipulates that any proposition is either true or false. The truth-value of a true proposition is TRUE and the truth-value of a false proposition is FALSE. Truth-value refers to the designating of a statement either as true or false. Likewise, any compound proposition is either true or false. There is a technique of determining the truth-value of compound proposition. In effect the truth-value of a compound proposition is a function of the truth-value of its constituent or component statements. Barring a few cases, which are exceptions, in all other cases this particular specification applies to compound proposition. Therefore it is very important to distinguish these two kinds of compound proposition. It is distinguished as follows: **'A compound proposition is said to be truth-functionally compound if and only if its truth-value is a function of the truth-value of its components'**. In other words, truth-function is a compound statement whose truth-value is completely determined by the truth-values of its components. Logic which deals with truth-functional compound statements is called truth-functional logic: this is the part that we are presently studying.

The construction of truth-table (which is the list that shows the various values a truth-function may assume) is a technique adopted in order to determine the truth-value of compound propositions. It is interesting to learn that even when the propositions remain the same, different types of compound propositions exhibit different truth-values because sentential connectives change from one compound to another compound. This clearly shows that the sentential connective plays a crucial role in determining the truth-value of a compound proposition. *Therefore the truth-value of a compound proposition is determined by the truth-values of components and also the sentential connective used.* In order to drive home this point, let us retain the same set of statements, which form parts of compound proposition, but at the same time obtain different results in terms of truth-values by using different sentential connectives.

21 The stars are self-luminous. - p

22 Glass is fragile. - q

Let us construct truth-tables to determine the truth-values of compound propositions (As usual '1' stands for 'True' and '0' stands for 'False'). Generally, no justification for determination of truth-value is called for. They are to be treated as the truth-conditions of respective compound propositions.

I) IMPLICATION:

An implicative proposition is false only under one circumstance, i.e., *when the antecedent is true and the consequent is false*. It means that false conclusion does not follow from true premise and under all other circumstances it is true. In the case of implication antecedent is the premise and consequent is the conclusion. Let us illustrate it in the form of a table.

Table 1:

	p	q	p => q
1	1	1	1
2	1	0	0
3	0	1	1
4	0	0	1

From this table one aspect becomes clear; a false premise implies any conclusion (whether true or false). It also means that a true conclusion follows from any premise. This is admissible because there is no necessary relation between the premise and the conclusion as pointed out earlier. (See 1.4) Implication as understood in logic is very different from common man's perception. This is exactly what Russell meant when he introduced the term 'material implication'.

Let us consider implication in verbal form. The statement 'If the stars are self-luminous, then glass is fragile' is false only when it is true that the stars are self-luminous and it is not the case that glass is fragile; and under all other circumstances it is true. This entire expression is hidden in Table 1. It is anybody's guess that Table 1 is more intelligible and understood with less effort than verbal form.

II) CONJUNCTION:

A Conjunction is true if and only if both the conjunctions are true; otherwise, it is false. Therefore, the truth-table for conjunction is as follows:

Table: 2

	p	q	p \wedge q
1	1	1	1
2	1	0	0
3	0	1	0
4	0	0	0

Conjunction corresponds to a familiar algebraic rule. When two positive numbers are added we will get sum. However, when a negative number is added to a positive number, we are only subtracting. And addition of two negative numbers also amounts to subtraction only. $(-4 + (-4) = -8$; and $-8 < -4$). Let us restate conjunction in verbal form:

- i) The stars are self-luminous: 1
- ii) Glass is fragile: 1

Conjunction:

1	The Stars are self-luminous and glass is fragile.	1
2	The Stars are self-luminous and glass is not fragile:	0
3	The Stars are not self-luminous and glass is fragile:	0
4	The Stars are not self luminous and glass not fragile:	0

III) DISJUNCTION:

A disjunction is true when at least one of the disjuncts is true. The condition of its truth-value can also be stated in this manner. A distinction is false if and only if both the disjuncts are false. Stated in this form, disjunction is just the inversion of conjunction. The truth-value for disjunction is as follows.

Table: 3

	p	q	p \vee q
1	1	1	1
2	1	0	1
3	0	1	1
4	0	0	0

At a later stage we will have an opportunity to understand the significance of the way in which the truth-value conditions of disjunction and conjunction differ. For the time being, let us consider the verbal form of disjunction.

- i) The Stars are self-luminous. 1
- ii) Glass is fragile. 1

Disjunction:

1	The stars are self-luminous or glass is fragile.	1
2	The stars are self-luminous or glass is not fragile.	1
3	The stars are not self luminous or glass is fragile.	1
4	The stars are not self luminous or glass is not fragile.	0

IV) NEGATION:

The simplest form of truth-functionally compound proposition is negation. In this case we have only two rows because there is only one proposition whereas in all other cases there are four rows because there are two propositions.

Table: 4

	p	$\neg p$
1	1	0
2	0	1

If p stands for ‘The stars are self-luminous’, $\neg p$ stands for ‘The stars are not self luminous’. Therefore if ‘it is true that the stars are self-luminous’, then it is not true that the stars are not self-luminous’. And if it is not the case that the stars are self-luminous, then it is true that the stars are not self-luminous. Again, it is obvious that the verbal form is more complex than the truth-table. Since negation connects one proposition only, it is called unary whereas all other connectives are called binary since they connect two propositions.

V) BI-CONDITION:

A biconditional proposition is true only when both the components have the same truth-value. Otherwise, it is false. The truth-value of biconditional proposition is as follows:

Table: 5

	p	q	$p \Leftrightarrow q$
1	1	1	1
2	1	0	0
3	0	1	0
4	0	0	1

Now let us consider verbal form for bicondition. ‘The stars are self-luminous if and only if glass is fragile’ is true when ‘it is the case that the stars are self-luminous’ and also ‘it

is the case that glass is fragile' or when 'it is not the case that the stars are self-luminous' and also it is not the case that glass is fragile'. Under remaining circumstances, it is false. In such cases the verbal form is as follows:

- 1 The stars are not self-luminous if and only if glass is fragile.
- 2 The stars are self-luminous if and only if glass is not fragile.

Again, let it be made clear that whether we say it is not the case that 'the stars are self-luminous' or we say that 'the stars are not self-luminous, there is no difference in intended meaning.

Negation and bicondition are unique for different reasons. Negation is unique because, though in grammatical sense, the statement 'the stars are not self-luminous' is a simple statement, modern logic regards it as a compound statement only because its truth-value depends upon the inclusion or exclusion of the connective 'not'. So what determines the compound nature of a proposition is not really the number of statements, but it is the truth-functional quality of proposition. In this connection it is worthwhile to refer to exceptions mentioned in the beginning of this section. While all truth-functional statements are compound, all compound statements are not truth-functional. In other words, in exceptional cases, the truth-value of components does not determine the truth-value of 'apparent' compound propositions. Consider these propositions, which, obviously, have this form.

23. If there is rise in the temperature, then there is rise in mercury level.

24. If India has to win the cricket match, then the gods must be crazy.

(23) and (24) differ in structure, which we generally, do not notice easily. In order to clearly understand the difference, let us break (23) and (24) to get their respective components.

23a There is rise in temperature.

23b There is rise in mercury level.

24a India has to win.

24b The gods must be crazy.

(23a) and (23b) are true or false together. But the same cannot be said about (24a) and (24b). They are, really, neither true nor false together. Therefore though (24) is a compound sentence, it is not truth-functionally compound. Therefore what is grammatically a compound statement may not be truth-functionally compound and vice-versa.

Biconditional proposition is unique for another reason. Implication does not allow simple transposition of antecedent and consequent whereas biconditional proposition allows only simple transposition of components. Consider $p \Rightarrow q$ and $q \Rightarrow p$ respectively with the help of truth-table.

Table: 6

	p	q	p \Rightarrow q	q \Rightarrow p
1	1	1	1	1
2	1	0	0	1
3	0	1	1	0
4	0	0	1	1

From rows (2) and (3) it becomes clear that $(p \Rightarrow q) \neq (q \Rightarrow p)$. This is because the truth of implication does not allow simple transposition. However, the case of biconditional proposition is different. We should remember that many disputes can be settled with the help of truth-table.

Table: 7

	p	q	p \Leftrightarrow q	q \Leftrightarrow p
1	1	1	1	1
2	1	0	0	0
3	0	1	0	0
4	0	0	1	1

From tables (6) and (7) it is clear that what allows or does not allow simple transposition is the truth-condition only. This particular characteristic can be brought out clearly only when bicondition is contrasted with implication.

The role played by sentential connectives in determining the truth-value of compound propositions vis-a-vis the truth-value of the components themselves is better understood when we compare the truth-table of all compound propositions. However, negation is not required for this purpose, since it does not have components.

Table: 8

	p	q	p \Rightarrow q	p \vee q	p \wedge q	p \Leftrightarrow q
1	1	1	1	1	1	1
2	1	0	0	1	0	0
3	0	1	1	1	0	0
4	0	0	1	0	0	1

Assume that in all columns p is replaced by proposition 21 and q is replaced by proposition 22. It is impossible that the truth-value of the proposition components differ from one situation to another. The position is like this; even when the same set of propositions with determinate truth-values form the components of various compounds propositions, the truth-value of one compound proposition differs from the truth-value of any other compound propositions. Before we arrive at this conclusion, we must compare

the truth-value of component propositions in all possible circumstance. Even if in one circumstance there is variation in the truth-value, our stand is vindicated. For example, in the table 8, the last two columns possess different truth-values only in the fourth row. Therefore it is clear that in spite of the fact the same set of propositions form components of different compound propositions, the truth-value varies from column to column because besides components, the sentential connective also determines the truth-value of given compound proposition. So the truth-value of a compound proposition is ‘uniquely’ determined by the truth-value of its components only with respect to that particular compound. However, if we have to explain variation from one column to another, then we also have to consider the role played by sentential connectives. The difference can be aptly summarized in this way; ‘vertical variation in truth-value of a compound proposition is a function of the truth-value of components only, whereas horizontal variation is a function of sentential connective’ only.

2.5 OTHER FORMS OF COMPOUND PROPOSITION

In the beginning of this unit, it was mentioned that the components of a compound propositions themselves can be compound propositions. We will consider a compound proposition with only three propositions because then we will have eight rows and if there are four propositions we will have sixteen rows. It is because, since any component takes two truth-values (i.e., either true or false), addition of a component would double the number of rows: thus for one component, only two rows as in the case of negation; for two, four rows, as we have seen in other truth table; for three, eight rows; for four, sixteen; for five, thirty two rows, and so on. However, with three simple propositions several compound propositions can be constructed. Therefore it will adequately serve our purpose. The variables and statements are as follows:

- 25 Alcoholism is a vice. p
- 26 Courage is a virtue. q
- 27 Yoga heals diseases. r

Various compound propositions can be constructed out of these propositions. Some of them are considered.

- 28 $(p \Rightarrow q) \wedge (\neg q \vee r)$
- 29 $(p \Rightarrow q) \vee (p \wedge q)$
- 30 $(q \vee r) \Rightarrow p$
- 31 $(q \Rightarrow r) \vee (p \wedge r)$

It should not be difficult to substitute statements of p, q and r. It is left as an exercise to the students to do the same. There is something more important to clarify.

Apart from the fact that the components of propositions 28 to 31 are themselves compound, there are parentheses also. The significance and necessity of parentheses can

be easily understood, when compared with simple arithmetic. Compare these two expressions:

- i). $(5+7)10 = 300$
- ii). $5+7 \times 10 = 75$

(1) is false. It is not even possible to say whether ii) is false or not. Knowing whether a certain expression is true or false is not very significant. But arriving at a determinate expression is significant. This is what exactly parentheses achieve when used appropriately. If they are not used, then it will be a mistake in mathematics, language and logic.

Let us consider statement 28 which has four connectives and therefore there are four compound propositions. Though one truth-table is sufficient for our purpose, in order to gain better understanding, we shall split the table:

Table: 9

	p	q	p ⇒ q
1	1	1	1
2	1	0	0
3	0	1	1
4	0	0	1

Table: 10

	r	¬q	¬q ∨ r
1	1	0	1
2	0	0	0
3	1	1	1
4	0	1	1

($r \vee \neg q$ is the same as $\neg q \vee r$.) To take next step let us assume that ($p \Rightarrow q$) is one component and $\neg q \vee r$ is another component. Let us transpose columns 3 of Table 9 and Table 10 to Table 11 to compute the result.

Table: 11

p	q	¬q	r	p ⇒ q	¬q ∨ r	(p ⇒ q) ∧ (¬q ∨ r)
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1	1	1	0	1	1	1	1
2	1	1	0	0	1	0	0
3	1	0	1	1	0	1	0
4	1	0	1	0	0	1	0
5	0	1	0	1	1	1	1
6	0	1	0	0	1	0	0
7	0	0	1	1	1	1	1
8	0	0	1	0	1	1	1

Before closing this section one has to learn the method of constructing truth-tables; it is a very interesting part of the study of symbolic logic. Truth-tables are constructed for truth-functions having statement variables that are customarily counted from the middle part of the alphabet like p, q, r, s, ... Accordingly, 'Bacon is a writer' is a statement in English; it can be symbolized as 'B'; it can be represented in a variable form as simply 'p'. Before beginning the work of constructing the truth-table we have fix the specific form of the given statement, determine the columns under which the truth-values are to be arranged and limit the number of rows in accordance with number of variables in the specific form of the statement. Let us work with a compound statement: $(A \Rightarrow B) \wedge (\neg B \vee C)$. Its specific form is $(p \Rightarrow q) \wedge (\neg q \vee r)$ Its truth-table is just above (no. 11).

[The students are advised to construct truth-tables for the remaining combinations, which are relatively simple. In all cases the number of rows is 8. Since practice makes man perfect, the students are advised to substitute statements for variables in all cases.]

Check Your Progress II

Note: a) Use the space provided for your answer.
b) Check your answers with those provided at the end of the unit.

1) Define truth-functional logic.

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2) Distinguish between implication and bicondition.

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2.6 LET US SUM UP

Modern logic distinguishes two kinds of statements. All truth-functional propositions are compound. 'Grammatical' compound is different from 'logical' compound. Truth-functional compound is a function of sentential connective and truth-values of components. Negation is the simplest (simplest in grammatical sense) form of compound. There are five types of compound propositions, each distinguished by its own set of truth-values. The truth-values of one compound differ from that of the others at least on one

occasion. Difference between implication and bicondition are notable. Components of compound proposition can themselves be compound. To have at least one compound within a compound, we need at least three propositions.

2.7 KEY WORDS

Ambiguity: When a word or a statement carries more than one legitimate meaning it is said to be ambiguous.

Turmoil: Turmoil is a state or condition of extreme confusion, agitation, or commotion.

Main Connective: The connective that determines the basic form of a statement is called main connective. For example, $(A \Rightarrow B) \wedge (\neg B \vee C)$ is a conjunction whose left hand conjunct is an implication and whose right hand conjunct is a disjunction.

2.8 FURTHER READINGS AND REFERENCES

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2.9 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress I

1. A compound statement in grammatical sense is independent of its components as far as its truth-value is concerned. However, in logical sense the truth or falsity of compound proposition depends upon the truth or falsity of its components.

2. Both copula and sentential connective perform the function of linking two distinct units; copula links two terms whereas sentential connective links two statements which may be true or false. The number of sentential connectives is always one less than that of statements. The same connective may occur more than once in the given compound proposition. While copula does not determine the truth of combination, the latter determines the same.

Check Your Progress II

1 Logic which deals with truth-functional compound statements is called truth-functional logic.

2 Implication is false only when the antecedent is true and consequent is false and under all other instances it is true. Bicondition is true only when both the components have the same truth-value, i.e., both components must be true or false together.

