UNIT 4 PROVING INVALIDITY

Contents

4.0 OBJECTIVES

In our study of logic this is the second unit dealing with the role played by the technique of proving the invalidity of arguments. Therefore the objective of this unit is a repetition of the earlier unit on proving invalidity. However, there is no harm in reiterating our earlier goal. Proving invalidity is significant not in negative sense, but in positive sense. The singular objective of this unit is very clear. If we know what is wrong, we will know what is right in the *right* sense of the word and we will avoid consciously the pitfalls of illogical ways of arguing. Otherwise, we may walk into the trap of fallacies. Thus by the end of this unit one should be in a position to establish the invalidity of seemingly valid arguments and also to identify the difference between proving tautology and proving invalidity

4.1 INTRODUCTION

Building up of a consistent proof system is an efficient mode to show that an argument is valid. Suppose that an argument happens to be invalid. Then it is not possible to construct proof of its validity using any rule applied so far. The reason is simple. There are rules to govern the *right* path, but there are no rules to govern the wrong paths. This maxim holds good in all walks of life. Surely, logic is no exception. Therefore when compared with the technique of proving validity, the technique of determining or proving invalidity turns out to be somewhat indirect. The difference is more pronounced when we compare this method with the methods of proving validity or tautology. We became familiar with one aspect of proving invalidity when in the previous unit we established the invalidity of arguments consisting of truth-functionally compound statements. We accomplished this task by assigning the truth-values to those simple statements, which constitute the structure of respective truth-functionally compound propositions, in such a way as to make their premises true and their conclusions false. We use exactly the same method to prove the invalidity of arguments involving quantifiers with an additional assumption which becomes clear shortly.

4.2 METHOD OF PROVING INVALIDITY -1

Before we embark upon our task we have to be sure of the worthiness of our starting point and also very clear about what that starting point amounts to. Otherwise, we may go astray. The foundation of quantification logic is characterized by the assumption that we are dealing with nonempty sets. This turns out to be the model for which the arguments which we are testing must conform. In other words, the presupposition means that there is *at least* one individual. Again, it must be pointed out, and there is a good deal of value in repeating what was pointed out earlier, that there is a difference between saying *at least* one and saying there is *only* one…. 'At least one' surely does not exclude (it does not mean that it includes) the possibility of two or more than two members whereas the latter prohibits. Our study of the technique of proving invalidity is based upon the assumption that there are two or more than two members in the universe of discourse though we begin with the assumption that there is exactly one member.

So far, we were concerned with the assumption which we have made. Now we should turn to the basic logical principle which has to be followed. In the previous unit on proving invalidity we dealt with truth-functionally compound propositions which constituted the arguments. Quantification logic deals with arguments consisting of general and singular statements to which familiar Rules cannot be applied directly. Hence there is need to try a different method. The logical principle involved here is that there is equivalence between general propositions which are non-compound and truth-functional singular propositions. These singular propositions can be said to be the units or constituents of non-compound general propositions. The equivalence between these kinds can be established very easily when we use the principles of truth-functional logic. There is a certain method of establishing the same. Suppose that there is only one member, viz., 'a' or two members, a and b in the nonempty universe of discourse. Then we do not have any reason to apply the rule UI. Therefore this technique is independent of the rules of quantification. This is an important aspect to remember. In such a case we get the following result.

1 $(x)\,(\Phi x) \equiv \Phi a$
2 $(\exists x)\,(\Phi x) \equiv \Phi a$

It is obvious that in a nonempty universe of discourse with exactly one member, the proposition with universal quantifier is logically equivalent to proposition with existential quantifier. Therefore the following equation holds good.

3 $(x)\,(\Phi x) \equiv \Phi\, a \equiv (\exists x)\,(\Phi x)$

However, the result alters when we consider two members, say a and b. Then proposition with universal quantifier takes this form.

4. $(x)\,(\Phi x) \equiv [\Phi a \wedge \Phi b]$

5. $(\exists x)\,(\Phi x) \equiv [\Phi a \vee \Phi b]$

It is not possible to say that 1 ≡ 2 without further evidence. 4 is true only when Φ a is true and also Φ b is true. However, for 5 to be true it is sufficient if at least one of the two components on

R.H. S. is true. We shall generalize our conclusion. If there are ‘n’ members, then the respective equations become

$$6.\ (x)\ (\Phi x) \equiv [\Phi a \wedge \Phi b \wedge \Phi c \wedge \ldots\ \Phi n]$$

$$7.\ (\exists x)\ (\Phi x) \equiv [\Phi a \vee \Phi b \vee \Phi c \ldots\ \Phi n]$$

It is obvious that every equation is a biconditional proposition. Therefore both sides of the equation must have the same truth-value. Since we are dealing with non-empty universe of discourse, 0 cannot be the value assigned to any side. From this stage onwards we restrict ourselves to the method of assigning truth-values.

Now our task is very much simplified. An argument involving quantifiers must possess such a structure that after assigning 0 to the conclusion it must be possible to assign 1 to every premise. We must remember that the conjunction of an indefinite number of true premises with just one false premise produces a false conjunction. Therefore in an invalid argument it is imperative that all premises are true while the conclusion is false. This is the first step in testing the invalidity of arguments. In the next step we prove the invalidity of a given argument by displaying or describing a model in which the given argument is logically equivalent to an invalid truth-functional argument. This is achieved by translating the given argument involving quantifiers to a logically equivalent argument involving only singular propositions and truth-functional compounds of them, and then using the method of assigning truth-values to prove that the argument is invalid. This is the actual method of proving the invalidity of arguments. We shall examine a few arguments to become familiar with this technique. Consider this argument:

1.

All sharks are dangerous.
All lions are dangerous.
Therefore all sharks are lions.

We symbolize this argument as follows.

$$(x)\ (Sx \Rightarrow Dx)$$
$$(x)\ (Lx \Rightarrow Dx)$$

$$\therefore (x)\ (Sx \Rightarrow Lx)$$

In the case of a model containing exactly one individual, say, (a) the given argument is logically equivalent to:

$$(Sa \Rightarrow Da)$$
$$(La \Rightarrow Da)$$
$$\therefore (Sa \Rightarrow La)$$

Assign truth-values to the propositions beginning with the conclusion according to the norm. While assigning the 0 value to the conclusion we must remember that it should be assigned to the sentential connective.

(Sa => Da)
1 1 1

(La => Da)
0 1

∴ (Sa => La)

1 0 0

Since the premises are true when the conclusion is false, the argument is invalid. In the second premise there is no reason to assign the value to the consequent since the premise is known to be true beforehand.

Now suppose that there are two members, say a and b, in the universe of discourse. Then the argument takes this form.

(Sa => Da) Λ (Sb => Db)

(La => Da) Λ (Lb => Db)

∴ (Sa => La) Λ (Sa => La)

Now assign the truth-values as we did in the previous case.

(Sa => Da) Λ (Sb => Db)
1 1 1 1 1 1 1
(La => Da) Λ (Lb => Db)
0 1 1 1 1 1 1
∴ (Sa => La) Λ (Sb => Lb)
1 0 0 0 1 1 1

The argument turns out to be invalid according to this fixation of values. The student is advised to obtain the result when the second component of the conclusion takes 0.

One more example is considered.

2.

All snakes are animals.

All elephants are animals.
Therefore all snakes are elephants.

We symbolize the argument as follows.

(Sa => Aa)
(Ea => Aa)

∴ (Sa => Ea)

When the truth-value is assigned to the logically equivalent form of this argument with exactly one member, the result is as follows.

(Sa => Ua)
1 1 1
(Ea => Ua)
0 1 1
∴ (Sa => Ea)

1 0 0

There is no need to consider a non-empty universe of discourse with two members. The method of testing is the same as the one for the first example.

Consider an example where two premises are universal and conclusion is particular.

3.

All flowers are attractive.

All attractive things are temporary.

Therefore some flowers are temporary.

Assuming that in non-empty universe of discourse there is only one member, we shall assign the truth-values. Let us assign 0 to Fa.

(Fa => Aa)
0 1 0
(Aa => Ta)
0 1
∴ (Fa Λ Ta)
0 0

With this set of values the argument turns out to be invalid no matter what value is assigned to Ta in the conclusion and in the premise. Of course, if Aa takes 1, then Ta must take 1. We must remember that nowhere did we use any rule of quantification.

Let us consider a non-empty universe of discourse consisting of three individuals. This is being discussed only with the intention of reinforcing what we have learnt so far. Once we arrive at desired result, it is possible to safely assume that the result holds good for n number of members.

Suppose that there are only three men in the model of men, viz. a, b and c. In such a case the proposition 'A' can be represented in the following manner.

1. $(x)\ (\Phi x) \equiv (\Phi a \wedge \Phi b \wedge \Phi c)$
   The LHS is true if and only if Φa is true, Φb is true and Φc is true. If any one of them is false, then the LHS is false. Similarly, the proposition 'E' becomes

2. $(x)\ (\neg \Phi x) \equiv (\neg \Phi a \wedge \neg \Phi b \wedge \neg \Phi c)$

If a, b and c are the only men in the model of men, then as in the previous case, in the present case also the LHS is true. This is because a conjunction is true (RHS) if and only if everyone of the three components is true. If any one of them is false then LHS also is false.

While the propositions with universal quantifiers are translated to the conjunction mode, those with existential quantifiers are translated to the disjunction mode. If we persist with the same model, then

3. $(\exists x)\ (\Phi x) \equiv (\Phi a \vee \Phi b \vee \Phi c)$

4. $(\exists x)\ (\neg \Phi x) \equiv (\neg \Phi a \vee \neg \Phi b \vee \neg \Phi c)$

These four equations are in the form of bicondtional proposition and a biconditional proposition is true only when both sides of the equation have the same truth-value. From these four equations, it is clear that the truth-status of propositions with quantifiers is determined by the truth-conditions of corresponding compound proposition. This relation is in perfect consonance with the definition of universal and existential quantifiers. We always assume that the propositions are true as long as we are dealing with the set of premises. Therefore both sides of the equations representing the premises must necessarily be true.

On logical grounds, there is a qualitative difference between a model containing only one individual and another model containing two or more than two individuals. For the sake of convenience let us call the first model monadic and the second one polyadic or n-adic model for n number of members. For example, if there are two individuals, then the model is dyadic and if there are three members, then triadic and so on. There is a qualitative difference because in a monadic model an invalid argument may correspond to a valid truth-functional argument whereas the very same argument in any other model may correspond to an invalid truth-functional argument. Let us consider an argument which is invalid from traditional angle.

4.

All politicians are lawyers.

All judges are lawyers.
∴ All judges are politicians.

1. p1: (x) {Px => Lx}
2. p2: (x) {Jx => Lx}
   ∴ (x) {Jx => Px}

Since there is only member, this argument is logically equivalent to

3. p1: {Pa => La}
4. p2: {Ja => La}
   ∴ Ja => Pa

In a monadic model (x) (Φx) ≡ Φa ≡ (∃x) (Φx)

∴The argument is logically equivalent to

5. Pa Λ La
6. Ja Λ La

∴ Ja Λ Pa

If we assign the value 0 to any one of the components of the conclusion, then not only the conclusion is false but also one of the premises becomes false. However, according to definition, the premises must be true. It is logically impossible to derive a false conclusion from true premises. Therefore the argument is valid. However, the same argument is invalid in a dyadic model.

We shall symbolize the previous argument.

1. p1: (x) {Px => Lx}
2. p2: (x) {Jx => Lx}
   ∴(x) {Jx => Px}

Since we are considering a dyadic model the symbolic presentation is logically equivalent to

3. (Pa => La) Λ (Pb => Lb)
4. (Ja => La) Λ (Jb => Lb)

∴ (Ja => Pa) Λ (Jb => Pb)

Assign 0 to Pa and 1 to the rest. The result can be computed as follows:

5. (Pa => La) Λ (Pb => Lb)
   0 1 1 1 1 1 1
6. (Ja => La) Λ (Jb => Lb)
   1 1 1 1 1 1 1

∴ (Ja => Pa) Λ (Jb => Pb)
   1 0 0 0 1 1 1

The conjunction of the truth-values in 5 and 6 yields true premises whereas the conclusion is false. Hence the argument is invalid. This result can be generalized to include other polyadic models with 3 or more than 3 members. Whatever holds good to a dyadic model in this case also holds good to any other polyadic model. In order to become familiar with this method let us work with some more problems.

5. (x) (Dx => ¬ Ex)
(x) (Ex => Fx)
∴ (x) (Fx => ¬ Dx)

Let us restrict this argument to a dyadic model. If this argument is invalid in this model, then it is invalid in all other polyadic models. The logically equivalent form of 3 is as follows.

1. (Da => ¬ Ea) Λ (Db => ¬ Eb)
2. (Ea => Fa) Λ (Eb => Fb)
∴(Fa => ¬ Da) Λ (Fb => ¬ Db)

Assign 0 to ¬ Da. Accordance to the law of contradiction Da = 1. Similarly, ¬ Db is assigned 0. Therefore Db = 1. Assign 1 to ¬ Ea. Ea takes 0. Assign 1 to ¬ Eb. Eb takes 0. Assign 1 to Fa and Fb. The result can be computed as follows.

3. (Da => ¬ Ea) Λ (Db => ¬ Eb)
1 1 1 1 1 1 1
4. (Ea => Fa) Λ (Eb => Fb)
0 1 1 1 0 1 1
∴ (Fa => ¬ Da) Λ (Fb => ¬ Db)
1 0 0 0 1 0 0

In this argument also the conjunction of the truth-values in 5 and 6 yields true premises whereas the conclusion is false. Hence the argument is invalid. This result can be generalized to include other polyadic models with 3 or more than 3 members. Whatever holds good to a dyadic model in this case also holds good to any other polyadic model.

6.
1 (∃x) (Jx Λ Kx)
2 (∃x) (Kx Λ Lx)
∴(∃x) (Lx Λ Jx)

We shall consider this argument also in a dyadic model. This is logically equivalent to
3 (Ja Λ Ka) v (Jb Λ Kb)
4 (Ka Λ La) v (Kb Λ Lb)
∴(La Λ Ja) v (Lb Λ Jb)

There is a difference between this argument and the previous arguments. In this argument the premises and conclusion are disjunctive unlike the previous arguments which have conjunctive statements. The difference is due to quantifiers. In case of universal quantifiers conjunction is the connective whereas in case of existential quantifiers disjunction is the connective.
Assign the truth-values as follows; 0 to La and Jb and 1 to the rest. The result is computed as follows.

5 (Ja Λ Ka) v (Jb Λ Kb)

1 1 1 1 0 0 1

6 (Ka Λ La) v (Kb Λ Lb)
1 0 0 1 1 1 1
∴ (La Λ Ja) v (Lb Λ Jb)

0 0 1 0 1 0 0

In this argument also the conjunction of the truth-values in 5 and 6 yields true premises whereas the conclusion is false. Hence the argument is invalid. This result can be generalized to include other polyadic models with 3 or more than 3 members. Whatever holds good to a dyadic model in this case also holds good to any other polyadic model.

4.3 METHOD OF PROVING INVALIDITY -2

We shall examine those immediate inferences which are regarded as valid by traditional logic, but invalid according to modern logic. Let us begin with conversion per limitation for a dyadic model.

7.

All women are pious.

Therefore some pious persons are women.

Symbolize the argument and assign truth-values:

(Wa => Pa) Λ (Wb => Pb)

1 1 1 1 0 1 1

∴ (Pa Λ Wa) Λ (Pb => Wb)

1 1 1 0 1 0 0

Consider O proposition.

8.

Some soldiers are not graduates.
Therefore some graduates are not soldiers.

Again, we consider this argument for a dyadic model. Then the rest follows.

(Sa Λ ¬ Ga) Λ (Sb Λ ¬ Gb)
1 1 1 1 1 1 1

$\therefore$ (Ga Λ ¬ Sa) Λ (Gb Λ ¬ Sb)
0 0 0 0 0 0 0

Consider opposition of relations. Let us begin with superaltern.

9.

All graduates are university educated.
Therefore some graduates are university educated.

we shall examine this argument for a dyadic model following the familiar norms.

(Ga => Ua) Λ (Gb => Ub)
0 1 1 1 1 1 1
$\therefore$ (Ga Λ Ua) Λ (Gb Λ Ub)
0 0 1 0 1 1 1

The argument is invalid for a dyadic model. Therefore it is invalid for polyadic model.
There is no need to work out the status of subaltern relation. It is sufficient if the statements of 9 are reversed. This is left as an exercise for the student.

Contrary is the relation to be considered now.

10.
All swimmers are medalists.
$\therefore$No swimmers are medalists.

Symbolize the statements and assign the truth-values as usual.

Sa => Ma
1 1 1
$\therefore$ Sa => ¬ Ma
1 0 0

Subcontrary relation is left out for the student as an exercise.

This method of proving invalidity in this manner is, perhaps, decisive since the invalidity of those arguments, which are held by traditional logic as valid, becomes clear from the last two examples.

Check Your Progress I

Note: Use the space provided for you answers.

1 Give examples to prove that subcontrary and subaltern relations are invalid.

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2 Distinguish between monadic and dyadic models.

4.4 EXERCISES

Using the method of assigning truth-values prove that the following arguments are invalid.

1.

(x) {Ax => (Bx ∨ Cx)}
(x) {Dx => (Ex ∨ Fx)}
(x) {¬ Bx => (Fx ∨ Gx)}
(∃x) {(Fx => Dx) Λ (¬ Ex => ¬ Dx)}
¬ Gx

∴ (x){Ax => (Dx ∨ Fx)}

2.

{Aa => (Ba ∨ Ca)}
{Da => (Ea ∨ Fa)}
{Ba => (Fa ∨ Ga)}
{(Fa => Da) Λ (¬ Ea => ¬ Da)}
¬ Ga

∴ {Aa => (Da ∨ Fa)}

3.

Aa => (Ba ∨ Ca)
Da => (Ea∨ Fa)
¬ Ba => (Fa ∨ Ga)
(Fa => Da) Λ (¬ Ea => Da)
¬ Ga

∴Aa => (Da ∨ Fa)

4.

(x){Px => (Qx ∨ Rx)}
(x){Sx => (Tx ∨ Ux)}
(x) {¬ Qx => (Ux ∨ Vx)}
(∃x){[(Ux => Sx) Λ (¬ Tx => ¬ Sx)}
¬ Vx

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∴ (x) {Px => (Sx ∨ Ux)}

Check Your Progress II

Note: Use the space provided for your answer.

1. Using the technique of proving invalidity show that the following arguments are invalid:

a)
Ix ∨ (Kx Λ Jx)
¬ (Ix ∨ Jx) ∨ (Lx ≡ ¬ Mx)
¬ (Lx => ¬ Mx) ∨ (¬ Nx Λ Mx)
(Nx => Ox) Λ (Ox => Mx)
¬ (¬ Jx ∨ Kx) ∨ Ox /∴Oa

b)

(Bx Λ Cx) ∨ Ax
(Ax ∨ Cx) => (Lx ≡ ¬ Mx)
(Lx ⊃ ¬ Mx) => (¬ Nx Λ Mx)
(Nx => Zx) Λ (Zx ⊃ Mx)
(¬ Cx ∨ Bx) => Ox /∴Za ∨ Kx

4.5 LET US SUM UP

In this unit we have learnt about the significance of proving invalidity. The method followed is assigning of truth-values to the statements. We have also learnt that the conclusion must be always assigned the value 0 while all premises must invariably take the value1. Some inferences held as valid in traditional logic are shown conclusively to be invalid with the help of this method. The importance of nonempty universe of discourse is demonstrated.

4.6 KEY WORDS

Nonempty universe of discourse or model: A universe of discourse with at least one member.

Monadic model: A model with exactly one member.

Polyadic or **n-adic model:** A model with an indefinite number of (n) members.

4.7 FURTHER READINGS AND REFERENCES

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