# UNIT 4 DEFINITE INTEGRALS

## Structure



## 4.1 INTRODUCTION

We have seen in Unit 2 that one of the problems which motivated the concept of a derivative was a geometrical one – that of finding a tangent to a curve at a point. The concept of integration was also similarly motivated by a geometrical problem – that of finding the areas of plane regions enclosed by curves. Some recently discovered Egyptian manuscripts reveal that the formulas for finding the areas of triangles and rectangles were known even in 1800 BC. Using these formulas, one could also find the area of any figure bounded by straight line segments. But no method for finding the area of figures bounded by curves had evolved till much later.

In the third century BC, Archimedes was successful in rigorously proving the formula for the area of a circle. His solution contained the seeds of the present day integral calculus. But it was only later, in the seventeenth century, that Newton and Leibniz were able to generalize. Archimedes' method and also to establish the link between differential and integral calculus. The definition of the definite integral of a function, which we shall give in this unit, was first given by Riemann in 1854. We will also acquaint you with various application of integration.

### Objectives

After studying this unit, you should be able to

- define the definite integral of a given function as a limit of a sum,
- state the Fundamental Theorem of Calculus,
- use the Fundamental Theorem to calculate the definite integral of an integrable function,
- learn the different properties of definite integral, and
- use the definite integrals to evaluate areas of figures bounded by curves.

## 4.2 DEFINITE INTEGRALS

We have studied indefinite integrals in Unit 3. In this unit, we define a definite integral and see how it can be used to find the area under certain curves.

### 4.2.1 Definite Integral as the Limit of a Sum

Let $f$ be a continuous function defined on a classed interval $[a, b]$. Assume that all the values taken by the function are non-negative, i.e. the graph of the function is a curve above the $x$-axis.

**Figure 4.1**

Consider the area of Figure 4.1. Let us find the area of this region.

Let $AM$ and $BN$ be the ordinates for $x = a$ and $x = b$. Divide $MN$ into $n$ equal parts of length $h$ each and let $M_1 P_1, M_2 P_2, \ldots, M_{n-1} P_{n-1}$ be the ordinates at $M_1, M_2, \ldots, M_{n-1}$, then $nh = b - a$, i.e. $h = \frac{b-a}{n}$.

Also abscissae of the point $A, P_1, P_2, \ldots, P_{n-1}, B$ are $a, a+h, a+2h, \ldots, a+\overline{n-1}\,h, b$.

$$\therefore \quad MA = f(a)$$

$$M_1 P_1 = f(a+h)$$

$$M_2 P_2 = f(a+2h)$$

$$\vdots$$

$$M_{n-1} P_{n-1} = f(a + \overline{n-1}\,h)$$

$$MB = f(b)$$

We consider the left end points of these sub regions and construct rectangles 1, 2, 3, . . . , $n$ as shown in Figure 4.1.

Area of the first rectangle = $h\,f(a)$

Area of the second rectangle = $h\,f(a+h)$

Area of the third rectangle = $h\,f(a+2h)$

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Area of the $n$th rectangle = $h\ f(a + \overline{n-1}\,h)$

$\therefore$ Sum of these areas $= h\ f(a) + h\ f(a+h) + \ldots + h\ f(a + \overline{n-1}\, h)$

We note that this area is approximately equal to the area of the region AMNB. Further as the number of sub-divisions increases, the estimation becomes better. Let the subdivisions become very large, i.e. $n \to \infty$, then $h = \dfrac{b-a}{n} \to 0$, which in turn implies that the area of the region AMNB

$$= \operatorname*{Lt}_{h \to 0} h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a + \overline{n-1}\, h)] \qquad \ldots (4.1)$$

The expression on the R. H. S of Eq. (4.1) is called the definite integral of $f(x)$ from $a$ to $b$ and is denoted by $\int_a^b f(x)\,dx$, where $a$ is called the lower limit and $b$ is called the upper limit.

Thus, $\int_a^b f(x)\,dx = \operatorname*{Lt}_{h\to 0} [f(a) + f(a+2h) + \ldots + f(a + \overline{n-1}\, h)]$, where $nh = b - a$.

**Cor.**

$\int_a^b f(x)\,dx$ = the area of the region below the curve $y = f(x)$ above the $x$-axis and bounded by the ordinates $x = a$ and $x = b$.

**Remarks**

For simplicity of the above concept, we have taken non-negative values of $f(x)$. In fact it makes sense for negative values of $f(x)$ as well.

**Example 4.1**

Evaluate $\int_a^b x^2\,dx$ as the limit of a sum.

**Solution**

$\int_a^b x^2\,dx = \operatorname*{Lim}_{h\to 0} h\,[f(a) + f(a+h) + \ldots + f(a + \overline{n-1}\, h)]$, where $nh = b - a$

and $f(x) = x^2$.

i.e. $$\int_a^b x^2\,dx = \operatorname*{Lim}_{h\to 0} h\left[a^2 + (a+h)^2 + f(a+2h)^2 + \ldots + (a + \overline{n-1}\, h)^2\right]$$

$$= \operatorname*{Lim}_{h\to 0} h\left[(a^2 + a^2 + \ldots + a^2) + 2ah\,(1 + 2 + 3 + \ldots + \overline{n-1}) + (1^2 + 2^2 + 3^2 + \ldots + \overline{n-1}^2)\,h^2\right]$$

$$= \operatorname*{Lim}_{h\to 0} h\left[na^2 + 2ah\,\frac{(n-1)\,n}{2} + \frac{h^2\,(n-1)\,n\,[2(n-1)+1]}{6}\right]$$

$$= \operatorname*{Lim}_{h\to 0} h\left[a^2\,nh + a\,(nh)\,(nh-h) + \frac{1}{6}\,nh\,(nh-1)\,(2nh-h)\right]$$

$$= \operatorname*{Lim}_{h\to 0} h\left[a^2\,(b-a) + a\,(b-a)\,(b-a-h) + \frac{1}{6}\,(b-a)\,(b-a-h)\,(2\overline{b-a}-h)\right]$$

$$= a^2\,(b-a) + a\,(b-a)^2 + \frac{1}{6}\,(b-a)^2\,(b-a-h)\,(b-a)$$

$$= (b-a)\left[a^2 + a(b-a) + \frac{1}{3}(b-a)^2\right] = \frac{1}{3}(b^3 - a^3)$$

**Example 4.2**

Evaluate $\int_0^2 e^x\,dx$ as a limit of a sum.

**Solution**

$$b - a = 2 - 0 \therefore nh = 2,\ \text{i.e. } h = \frac{2}{n} \text{ and } f(x) = e^x$$

$$\int_0^2 e^x\,dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + \ldots + f(a + \overline{n-1}\,h)]$$

$$= \lim_{h\to 0} h\,[e^a + e^{a+h} + e^{a+2h} + \ldots + e^{\overline{n-1}\,h}]$$

$$= \lim_{h\to 0} h\,[e^0 + e^h + e^{2h} + \ldots + e^{\overline{n-1}\,h}] \text{ as } a = 0$$

$$= \lim_{h\to 0} h\left[\frac{e^{nh} - 1}{e^h - 1}\right] \text{ using the formula for the sum of a G. P.}$$

$$= \lim_{h\to 0} h\left[\frac{e^2 - 1}{e^h - 1}\right]$$

$$= \frac{e^2 - 1}{\lim\limits_{h\to 0} \frac{e^h - 1}{h}} = e^2 - 1.$$

**Example 4.3**

Evaluate $\int_0^\pi \sin x\,dx$ as the limit of a sum.

**Solution**

Here $f(x) = \sin x,\ a = 0,\ b = \pi$

$\therefore \quad h = \frac{b-a}{n} = \frac{\pi}{n}$, i.e. $nh = \pi$

$f(a) = f(0) = \sin 0 = 0$

$f(a+h) = f(h) = \sin h$

$f(a+2h) = f(2h) = \sin 2h$

. . . . . . . . . . . . . . . . . . . . . . .

$f(a + \overline{n-1}\,h) = \sin \overline{n-1}\,h$

$$\therefore \quad \int_0^\pi \sin x\,dx = \mathop{\mathrm{Lt}}_{h\to 0} h\left[0 + \sin h + \sin 2h + \ldots + \sin(\overline{n-1}\,h)\right]$$

$$= \mathop{\mathrm{Lt}}_{h\to 0} \frac{h}{2\sin\frac{h}{2}}\left[2\sin h \sin\frac{h}{2} + 2\sin 2h \sin\frac{h}{2} + \ldots + 2\sin(nh - h)\sin\frac{h}{2}\right]$$

$$= \underset{h \to 0}{\text{Lt}} \frac{1}{\frac{\sin \frac{h}{2}}{\frac{h}{2}}} \left[ \left( \cos \frac{h}{2} - \cos \frac{3h}{2} \right) + \left( \cos \frac{3h}{2} - \cos \frac{5h}{2} \right) + \ldots + \right.$$

$$\left. \left[ \cos \left( nh - \frac{3h}{2} \right) - \cos \left( nh - \frac{h}{2} \right) \right] \right]$$

$$= \underset{h \to 0}{\text{Lt}} \frac{1}{\frac{\sin \frac{h}{2}}{\frac{h}{2}}} \left[ \cos \frac{h}{2} - \cos \left( nh - \frac{h}{2} \right) \right]$$

$$= 1 \,.\, \underset{h \to 0}{\text{Lt}} \left[ \cos \frac{h}{2} - \cos \left( \pi - \frac{h}{2} \right) \right] \text{ as } nh = \pi \text{ and } \underset{h \to 0}{\text{Lt}} \frac{\sin \frac{h}{2}}{\frac{h}{2}} = 1$$

$$= \underset{h \to 0}{\text{Lt}} \left[ \cos \frac{h}{2} + \cos \frac{h}{2} \right] = 1 + 1 = 2 \,.$$

**SAQ 1**

Evaluate the following definite integrals as a limit of a sum :

(i) $\int_a^b e^x \, dx$

(ii) $\int_a^b \cos x \, dx$

(iii) $\int_1^2 (x^2 - 1) \, dx$

## 4.3 FUNDAMENTAL THEOREM OF CALCULUS

### 4.3.1 Area Function

We have defined $\int_a^b f(x) \, dx$ as the area of the region bounded by the curve $y = f(x)$, $x$-axis and the ordinates $x = a$ and $x = b$. Let $x \in [a, b]$.

Then $\int_a^x f(x) \, dx$ represents the area of the shaded region in Figure 4.2.

(Here it is assumed that $f(x) > 0$ for $x \in [a, b]$.

**Figure 4.2**

The area of this shaded region depends on $x$, i.e. in other words is a function of $x$. We denote it by $A(x)$

$$\therefore \qquad A(x) = \int_a^x f(x)\, dx$$

We will show that $A'(x) = f(x)$.

### 4.3.2 First Fundamental Theorem of Integral Calculus

Let the area function be defined by $A(x) = \int_a^x f(x)\, dx$ for all $x \geq a$,

where the function $f$ is continuous on $[a, b]$. Then $A'(x) = f(x)$ for all $x \in [a, b]$.

(We assume it without proof).

### 4.3.3 Second Fundamental Theorem of Integral Calculus

Let $f$ be a continuous function defined on an interval $[a, b]$ and $F$ is an antiderivative of $f$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

(We assume it without proof).

**Remarks**

(i) $\int_a^b f(x)\, dx$ = (value of an antiderivative at the upper limit $b$) – (value of the same antiderivative at the lower limit $a$).

(ii) This theorem is very useful as it gives us a method of calculating a definite integral more easily without calculating the limit of a sum.

For convenience $F(b) - F(a)$ is denoted by $F(x)|_a^b$.

(iii) If we consider $F(x) + c$ to be an antiderivative value of $f(x)$ instead of $F(x)$, then

$$\int_a^b f(x)\, dx = [F(x) + c]_a^b = (F(b) + c) - (F(a) - c)$$

$$= F(b) - F(a)$$

Hence, there is no need to keep the integration constant $c$.

**Example 4.4**

Evaluate $\int_0^4 x^{\frac{3}{2}}\, dx$

**Solution**

$$\int x^{\frac{3}{2}}\, dx = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} = \frac{2}{5}\, x^{\frac{5}{2}}$$

$$\therefore \quad \int_0^4 x^{\frac{3}{2}}\, dx = \frac{2}{5}\, x^{\frac{5}{2}} \Big|_0^4 = \frac{2}{5}\left(4^{\frac{5}{2}} - 0\right)$$

$$= \frac{2}{5} \,.\, 2^5 = \frac{64}{5}$$

**Example 4.5**

Evaluate $\int_0^1 \frac{\tan^{-1} x}{1+x^2}\, dx$

**Solution**

$$\int \frac{\tan^{-1} x}{1+x^2}\, dx = \frac{(\tan^{-1} x)^2}{2}$$

$$\therefore \quad \int_0^1 \frac{\tan^{-1} x}{1+x^2}\, dx = \frac{1}{2}\,(\tan^{-1} x)^2 \Big|_0^1$$

$$= \frac{1}{2}\left[(\tan^{-1} 1)^2 - (\tan^{-1} 0)^2\right] = \frac{1}{2}\left[\left(\frac{\pi}{4}\right)^2 - 0\right]$$

$$= \frac{1}{2} \cdot \frac{\pi^2}{16} = \frac{\pi^2}{32}$$

**Example 4.6**

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx$

**Solution**

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx = \int_0^{\frac{\pi}{2}} \frac{2 \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{2 \tan x \sec^2 x}{\tan^4 x + 1}\, dx$$

Now $$= \int \frac{2 \tan x \sec^2 x}{\tan^4 x + 1}\, dx = \tan^{-1} \tan^2 x$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin^4 x + \cos^4 x}\, dx = \tan^{-1} \tan^2 x \Big|_0^{\frac{\pi}{2}}$$

$$= \tan^{-1} \tan^2 \frac{\pi}{2} - \tan^{-1} \tan^2 0$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

## 4.3.4 Evaluation of a Definite Integral by Substitution

When we use the method of substitution for evaluating an integral $\int_b^a f(x)\, dx$, we follow the following steps :

**Step 1**

Substitute $x = g(y)$.

**Step 2**

Integrate the new integrand with respect to $y$.

**Step 3**

Resubstitute the value of $y$ in terms of $x$ in the answer.

**Step 4**

Find the value of the answer in Step 3 at the given limits and find the difference.

In order to quicken this method we can proceed as follows :

After performing Step 2, there is no need for Step 3. Instead the integral will be kept in the new variable $y$ and the limit of the integral will be accordingly changed.

**Example 4.7**

Evaluate $\int_{-1}^{1} 5x^4 \sqrt{x^5 + 1}\, dx$.

**Solution**

Let $t = x^5 + 1$, then $dt = 5x^4\, dx$

When $x = 1, t = 1^5 + 1 = 2$ and when $x = -1$, $t = (-1)^5 + 1 = -1 + 1 = 0$.

$$\therefore \quad \int_{-1}^{1} 5x^4 \sqrt{x^5 + 1}\, dx = \int_0^2 \sqrt{t}\, dt$$

$$= \frac{2}{3} t^{\frac{3}{2}} \Big|_0^2$$

$$= \frac{2}{3}\left(2^{\frac{3}{2}} - 0\right) = \frac{4\sqrt{2}}{3}.$$

**SAQ 2**

(a) Evaluate

(i) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx$

(ii) $\int_0^2 \frac{6x+3}{x^2+4} \, dx$

(iii) $\int_0^{\pi} \left( \sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$

(b) Evaluate

(i) $\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \cos x \, dx$

(ii) $\int_0^{\pi} \frac{dx}{5 + 4\cos x}$

(iii) $\int_0^{\frac{\pi}{2}} \frac{dx}{2\cos x + 4\sin x}$

## 4.4 PROPERTIES OF DEFINITE INTEGRALS

We consider below some important properties of the definite integral. These will be useful in evaluating the definite integrals more easily.

**Property 1**

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

**Property 2**

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx \text{ for } a < c < b$$

**Property 3**

$$\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$$

**Property 4**

$$\int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx \text{ if } f(2a-x) = f(x)$$

$$= 0 \qquad \text{if } f(2a - x) = -f(x)$$

**Property 5**

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \quad \text{if } f \text{ is an even function.}$$

$$= 0 \qquad \text{if } f \text{ is an odd function.}$$

We give proof of these properties.

**Property 1**

Let $F$ be an antiderivative of $f$.

Then $\int_a^b f(x)\,dx = F(b) - F(a) = -[F(a) - F(b)] = -\int_b^a f(x)\,dx$

**Property 3**

Let $t = a - x$

Then $dt = -dx$

When $x = 0$, $t = a$ and when $x = a$, $t = 0$

$$\therefore \quad \int_0^a f(x)\,dx = -\int_a^0 f(a - t)\,dt$$

$$= +\int_0^a f(a - t)\,dt \text{ by Property 1}$$

$$= +\int_0^a f(a - x)\,dx \text{ by changing the variable } t \text{ to } x.$$

**Property 4**

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \text{ by Property 2}$$

Put $t = 2a - x$ in the second integral

Then $\int_a^{2a} f(x)\,dx = -\int_a^0 f(2a - t)\,dt$

$$= \int_0^a f(2a - t)\,dt = \int_0^a f(2a - x)\,dx$$

$$\therefore \quad \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a - x)\,dx$$

$$= 2\int_0^a f(x)\,dx \text{ or } 0$$

according as $f(2a - x) = f(x)$

or $f(2a - x) = -f(x)$

Property 2 and Property 5 are left as exercises.

**Example 4.8**

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

**Solution**

Let $$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \qquad \ldots (1)$$

by Property 3

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \qquad \ldots (2)$$

Adding Eqs. (1) and (2), we have

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$= \int_0^{\frac{\pi}{2}} dx = x \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$\therefore \qquad I = \frac{\pi}{4}$

**Example 4.9**

Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 x \, dx$

**Solution**

$\cos^2 x$ is an even function.

$\therefore$ by Property 5

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 x \, dx = 2 \int_0^{\frac{\pi}{4}} \cos^2 x \, dx$$

$$= 2 \int_0^{\frac{\pi}{4}} \frac{1 + \cos 2x}{2} dx$$

$$= \frac{2}{2} \left| x + \frac{\sin 2x}{2} \right|_0^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{4} + \frac{1}{2} \sin \frac{\pi}{2}\right) - \left(0 + \frac{1}{2} \sin 0\right) = \frac{\pi}{4} + \frac{1}{2}$$

**Example 4.10**

Evaluate $\int_0^{\frac{\pi}{2}} \log \sin x \, dx$

**Solution**

Let $I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx$

Then $I = \int_0^{\frac{\pi}{2}} \log \sin \left(\frac{\pi}{2} - x\right) dx$

$$= \int_0^{\frac{\pi}{2}} \log \cos x \, dx$$

$\therefore \quad 2I = \int_0^{\frac{\pi}{2}} \{\log \sin x + \log \cos x\} \, dx$

$$= \int_0^{\frac{\pi}{2}} (\log \sin x \cos x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} \log \left(\frac{2 \sin x \cos x}{2}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \log \frac{\sin 2x}{2} \, dx$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \log 2 \, . \, x \Big|_0^{\frac{\pi}{2}}$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \frac{\pi}{2} \log 2 \qquad \ldots (1)$$

But $\int_0^{\frac{\pi}{2}} \log \sin 2x \, dx$

Putting $2x = t$ we have $\int_0^{\frac{\pi}{2}} \log \sin 2x \, dx = \frac{1}{2} \int_0^{\pi} \log \sin t \, dt$

$$= \frac{1}{2} \int_0^{\pi} \log \sin x \, dx$$

$$= \frac{1}{2} \, . \, 2 \int_0^{\frac{\pi}{2}} \log \sin x \, dx \text{ as } \sin(\pi - x) = \sin x$$

$$= I \qquad \ldots (2)$$

From Eqs. (1) and (2), we have

$\therefore \quad 2I = I - \frac{\pi}{2} \log 2$

i.e. $\quad I = -\frac{\pi}{2} \log 2$

**SAQ 3**

Evaluate

(i) $\int_0^{\frac{\pi}{4}} \log(1 + \tan x)\, dx$

(ii) $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx$

(iii) $\int_0^{\frac{\pi}{2}} \frac{x\, dx}{\sin x + \cos x}$

(iv) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x\, dx$

(v) $\int_0^1 \frac{\log(1 + x)}{1 + x^2}\, dx$

## 4.5 APPLICATIONS

We have seen that the area below (or above) the curve $y = f(x)$, above (or below) the $x$-axis and between the ordinates $x = a$ and $x = b$ is represented by the definite integral

$$\int_a^b f(x)\, dx = \int_a^b y\, dx$$

Likewise the area enclosed between the graph of the curve $x = F(y)$, $y$-axis and the lines $y = c$, $y = d$ is given by

$$\int_c^d F(y)\, dy = \int_c^d x\, dy$$

**Example 4.11**

Draw a rough sketch of the curve $y = \sqrt{3x + 4}$ and find the area under the curve, above the $x$-axis and between $x = 0$, $x = 4$.

**Solution**

$$y = \sqrt{3x + 4}$$

$\therefore$ Its domain consists of those $x$ for which $3x + 4 \geq 0$, i.e. $x \geq -\frac{4}{3}$.

We construct the table of values as under

| **x** | $-\frac{4}{3}$ | $-1$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|
| **y** | 0 | 1 | 2 | $\sqrt{7}$ | $\sqrt{10}$ | $\sqrt{13}$ | 4 |

A portion of the rough sketch of curve is shown in Figure 4.3.

Required area is the shaded area $= \int_0^4 f(x)\, dx$.

$$= \int_0^4 \sqrt{3x+4}\, dx$$

$$= \frac{(3x+4)^{\frac{3}{2}}}{\frac{3}{2} \cdot 3} \Big|_0^4 = \frac{2}{9}\left(16^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)$$

$$= \frac{2}{9}\left[(4^2)^{\frac{3}{2}} - (2^2)^{\frac{3}{2}}\right]$$

$$= \frac{2}{9}[4^3 - 2^3] = \frac{2}{9}[64 - 8] = \frac{112}{9} \text{ sq. units}$$

**Figure 4.3**

## Example 4.12

Make a rough sketch of the graph of the function $y = 3 \sin x,\ 0 \leq x \leq \pi$ and determine the area enclosed by the curve and the $x$-axis.

**Solution**

We construct the table of values as under

| **x** | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|---|---|
| **y** | 0 | $\frac{3}{2}$ | $\frac{3\sqrt{3}}{2}$ | 3 | $\frac{3\sqrt{3}}{2}$ | $\frac{3}{2}$ | 0 |

A rough sketch of the curve is shown in Figure 4.4.

**Figure 4.4**

$$\text{Required Area} = \int_0^{\pi} f(x)\, dx$$

$$= \int_0^{\pi} 3 \sin x \, dx$$

$$= [3(-\cos x)] \Big|_0^{\pi} = -3\,[\cos \pi - \cos 0^o]$$

$$= -3(-1-1) = 6 \text{ sq. units}$$

**Note :** Since the curve is symmetrical about the line $x = \frac{\pi}{2}$.

$\therefore$ Required Area = 2 Area OAM

$$= 2 \int_0^{\frac{\pi}{2}} f(x)\, dx = 2 \int_0^{\frac{\pi}{2}} 3 \sin x \, dx$$

$$= -6 \cos x \Big|_0^{\frac{\pi}{2}} = -6 \left( \cos \frac{\pi}{2} - \cos 0^o \right)$$

$$= -6(0-1) = 6 \text{ sq. units}$$

## Remark

In case of symmetrical closed area, find the area of the smaller part and multiply the result by the number of symmetrical parts.

## Example 4.13

Find the area enclosed between the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{x}{a} + \frac{y}{b} = 1$ which lies in the first quadrant.

**Solution**

The given ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots (1)$$

and the line is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \ldots (2)$$

Line (2) meets the curve (1) in $A$ $(a, 0)$ and $B$ $(0, b)$. The required area is shown in Figure 4.5.

**Figure 4.5**

For the ellipse

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$

i.e. $$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

i.e. for the first quadrant

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

Shaded Area = Area OATB – Area of the triangle OAB

Area of the triangle $\text{OAB} = \frac{1}{2}\text{OA . OB} = \frac{1}{2}\,ab$

Area OATB = Area bounded by the ellipse, $x$-axis in the first quadrant.

$$= 2\int_0^a y\,dx = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]\Bigg|_0^a$$

$$= \frac{b}{2a}\,[0 + a^2 \sin^{-1} 1 - (0 + a^2 \sin^{-1} 0)]$$

$$= \frac{b}{2a}\left[a^2\,\frac{\pi}{2} - a^2\,.\,0\right] = \frac{\pi\,ab}{4}$$

Required area $= \frac{\pi\,ab}{4} - \frac{1}{2}\,ab = \frac{(\pi - 2)\,ab}{4}$ sq. units.

**Example 4.14**

Find the area of the region bounded by the parabola $y = x^2 + 2$ and the lines $y = x$, $x = 0$, $x = 3$.

**Solution**

$y = x$ is the equation of a straight line lying below the parabola and the line $x = 3$ meets the parabola at (3, 11). The line $y = x$ meets the line $y = x$ at (3, 3). The region whose area is required is shaded and shown in Figure 4.6.

**Figure 4.6**

Required Area = Area bounded by the parabola, $x$-axis and the ordinates $x = 0$, $x = 3$ − (Area bounded by the line $y = x$, $x$-axis and the ordinates $x = 0$, $x = 3$).

$$= \int_0^3 (x^2 + 2)\, dx - \int_0^3 x\, dx$$

$$= \left( \frac{x^3}{3} + 2x \right) \Big|_0^3 - \frac{x^2}{2} \Big|_0^3$$

$$= 9 + 6 - \frac{9}{2} = \frac{21}{2} \text{ sq. units}$$

**Note :** Area bounded by the line $y = x$, $x$-axis and the ordinates at $x = 0$, and $x = 3$ is also the area of the triangle $\text{OAM} = \frac{1}{2}\, \text{OM}.\text{AM} = \frac{1}{2} \cdot 3.3 = \frac{9}{2}$.

## SAQ 4

Find the area of the regions

(i) bounded by $y^2 = 9x$, $x = 2$ and $x = 4$ and the $x$-axis in the first quadrant.

(ii) bounded by $x^2 = y - 3$, $y = 4$, $y = 6$ and the $y$-axis in the first quadrant.

(iii) bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

(iv) bounded by the circle $x^2 + y^2 = 4$, the line $x = \sqrt{3}\, y$, $x$-axis lying in the first quadrant.

(v) bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

(vi) enclosed between the circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 1$.

## 4.6 SUMMARY

In this unit, we have covered the following points.

- If $f$ is continuous on $[a, b]$, then $\int_a^b f(x)\,dx$ represents the area of the region bounded by the curve $y = f(x)$, $x$-axis and the ordinates $x = a$, $x = b$.
- $\int_a^b f(x)\,dx = \lim\limits_{h \to 0} h \sum\limits_{i=1}^{n} f[a + (i-1)h]$, where $h = \dfrac{b-a}{n}$.
- Fundamental theorem of calculus
  - (i) If $f$ is continuous on $[a, b]$ then for $x \in [a, b]$ if $A(x) \int_a^x f(x)\,dx$ then $A'(x) = f(x)$ for all $x \in [a, b]$.
  - (ii) If $f$ is continuous function on $[a, b]$ and $F$ is an antiderivative of $f$ then $\int_a^b f(x)\,dx = F(b) - F(a)$.
- Area bounded by a curve $y = f(x)$, $x$-axis and the lines $x = a$, $x = b$ is $\int_a^b f(x)\,dx = \int_a^b y\,dx$.
- Area bounded by a curve $x = g(y)$, $y$-axis and the lines $y = c$, $y = d$ is $\int_c^d g(y)\,dy = \int_c^d x\,dy$.

## 4.7 ANSWERS TO SAQs

**SAQ 1**

(i) $c^b - e^a$

(ii) $\sin b - \sin a$

(iii) $\dfrac{4}{3}$

**SAQ 2**

(a) (i) 2

(ii) $3 \log 2 + \dfrac{3\pi}{8}$

(iii) 0

(b) (i) $\frac{2}{3}$

(ii) $\frac{\pi}{3}$

(iii) $\frac{1}{\sqrt{5}} \log\left(\frac{3+\sqrt{5}}{2}\right)$

**SAQ 3**

(i) $\frac{\pi}{8} \log 2$

(ii) $\frac{\pi}{4}$

(iii) $\frac{\pi}{2\sqrt{2}} \log (1 + \sqrt{2})$

(iv) $\frac{3\pi}{8}$

(v) $\frac{\pi}{8} \log 2$

**SAQ 4**

(i) $16 - 4\sqrt{2}$

(ii) $\frac{2}{3} (3\sqrt{3} - 1)$

(iii) $12 \pi$

(iv) $\frac{\pi}{3}$

(v) $\frac{9}{8}$

(vi) $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$