UNIT 16 MATRICES AND DETERMINANTS

Structure

16.1 Introduction

Objectives

16.2 Matrices

16.2.1 Equality of Matrices

16.2.2 Sum of Matrices

16.2.3 Scalar Multiplication

16.2.4 Transpose and Conjugate of a Matrix

16.2.5 Special Matrices

16.2.6 Partitioning of Matrices

16.2.7 Submatrices

16.3 Matrix Multiplication

16.4 Matrices and Linear Transformations

16.5 Rank of Matrix

16.6 Inverse of a Matrix

16.7 Determinants

16.7.1 Properties of Determinants

16.7.2 Laplace Expansion of Determinants

16.8 Determinants and Matrices

16.8.1 Cramer's Rule

16.9 Summary

16.10 Solutions/Answers

16.1 INTRODUCTION

In Unit 14, you have seen the role that linear transformations play in understanding the structure of the solutions set of a system of linear equations. The computational aspects of linear transformations can be understood by using matrices. We introduce matrices and various operations involving matrices in this unit. The notions of rank and inverse of a matrix are also introduced. After introducing the notion of determinant of a square matrix, some important properties of determinants are observed. These properties are useful in computing the determinants more efficiently. Finally, determinants are used to understand certain aspects of matrices.

Objectives

The main objectives of this unit are

* To introduce the notion of matrices and operations involving matrices.
* To understand the relation between matrices and linear transformations and thus introduce the concepts of rank and inverse of a matrix,
* To introduce determinants of order n and give some useful properties of determinants, and
* To understand certain aspects of matrices in terms of determinants.

16.2 MATRICES

We know that linear transformations are very intimately connected with systems of linear equations. The linear transformations are also related to certain objects called

Matrices. Thus matrices are also related with systems of linear equations. In this section, we introduce the notion of a matrix and then study some aspects of matrices which are extremely useful to understand several engineering systems.

A *matrix* of order m by n is an array of mn numbers (real or complex), arranged in m ordered rows (each row containing n numbers) and n ordered columns (each column containing m numbers). Note that the order, m by n, of a matrix determines the number of rows, m, and the number of columns, n, associated with the matrix. Let the ith row of a m by n matrix consist of the n numbers

$$a_{i1}\ a_{i2} \ldots a_{in},$$

for $i = 1$ to m. If we donote this matrix by A, then we usually write

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

or, more compactly as,

$$A = (a_{ij}),\ i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n.$$

Note that the first suffix i in a_{ij} refers to the row index and the second suffix j in a_{ij} refers to the column index. The number a_{ij} is called the *entry* or *element* in the ith row and j th column of the matrix A.

A matrix is called a *rectangular matrix* or a *square matrix* according as $m \neq n$ or $m = n$. A square matrix with n rows and n columns is also referred to as a n-square matrix or a square matrix of order n. A matrix of order m by 1 is called a *column matrix*. Similarly, a matrix of order 1 by n is called a *row matrix*. For example, the following are matrices of orders mentioned against each matrix:

$$\begin{bmatrix} 1 & 2 & 3+i \\ i & 0 & 5 \end{bmatrix} \quad : \text{ matrix of order 2 by 3.}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad : \text{ square matrix of order 3.}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad : \text{ column matrix of order 3 by 1.}$$

$$[1\ \ 2\ \ 1+2i] \quad : \text{ row matrix of order 1 by 3.}$$

Amongst all the m by n matrices, a matrix with every entry 0 is called a *null* (or *zero*) matrix. A null matrix is denoted by 0. Note that we have a null matrix of every order and each of these is denoted by the same symbol, viz., 0. The following are example of null matrices:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{null matrix of order 3 by 2.}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{null matrix of order 2 by 4.}$$

Amongst all the n square matrices, we also single out the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} = (a_{ij}),$$

i.e., $a_{ij} = 0$ for $i \neq j$ and $a_{ii} = 1$ for $i = 1$ to n and $j = 1$ to n. This matrix is called the *identity matrix* (or *Unit Matrix*) and is denoted by I . Again, note that we have an identity matrix I for every n and each of these is denoted by the same symbol, viz. I. For example, the identity matrix of order 3 is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We remark that the context in which a null matrix or identity matrix is used usually indicates the order of these matrices. Otherwise, the order of a null or identity matrix should be specified.

A square matrix $A = (a_{ij})$ is called a *diagonal matrix* if $a_{ij} = 0$ for $i \neq j$ and a_{ii} are arbitrary. The diagonal matrices play an important role. The following is an example of a diagonal matrix of order 4 :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2-i & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that some diagonal entries , a_{ii} , of a diagonal matrix may also be zero.

A square matrix $A = (a_{ij})$ is called an *upper triangular matrix* if $a_{ij} = 0$ for $i > j$ and a *lower triangular matrix* if $a_{ij} = 0$ for $i < j$. A matrix which is either upper trangular or lower triangular is called *triangular matrix*. Note that a diagonal matrix is both upper triangular and lower triangular. The following are examples of triangular matrices:

$$\begin{bmatrix} 1 & 2 & 3+i & 4 \\ 0 & -2 & 1 & 2 \\ 0 & 0 & 1+i & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix} \quad : \text{ upper triangular matrix of order 4}$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 2+i & i & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 5 & 6+i \end{bmatrix} \quad : \text{ lower triangular matrix of order 4.}$$

Either of these matrices is an example of a triangular matrix.

The mn entries of a m by n matrix can be regarded as a vector in R^{mn}. The arithmetic of R^{mn}, with which we are already familiar, can be used to introduce the arithmetic in the set of all m by n matrices. We repeat this arithmetic to get acquainted with the double suffix notation a_{ij} used in matrices.

16.2.1 Equality of Matrices

Let $A = (a_{ij})$ and $B = (b_{ij})$ be any two matrices of the same order. Then , we define

$$A = B \Leftrightarrow a_{ij} = b_{ij} \text{ for all } i \text{ and } j.$$

Thus, two matrices are *equal* if

(a) their orders are same, and

(b) the entries in the corresponding positions are same.

Observe that the following properties are satisfied by the above concept of equality of matrices:

(i) $A = B \Rightarrow B = A$

(ii) $A = B$ and $B = C \Rightarrow A = C$.

16.2.2 Sum of Matrices

The *sum* $A + B$ of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same order is defined by

$$A + B = (a_{ij} + b_{ij}).$$

Thus, we add the corresponding entries of two matrices to find the sum of the given matrices. For example, if

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3+i & 4 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} i & -1 & 1 \\ -6 & 3+2i & 0 \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} 1+i & 1 & 1 \\ -3+i & 7+2i & -1 \end{bmatrix}.$$

The following proprties of addition of matrices, as defined above, are easy to prove:

(iii) the sum of any two matrices of the same order is a matrix of the same order.

(iv) $A + B = B + A$ for any two matrices A and B of the same order.

(v) $(A + B) + C = A + (B + C)$ for any three matrices A, B, and C of the same order.

(vi) $A + 0 = 0 + A = A$ for any matrix A, where 0 is a null matrix of the same order as that of A.

(vii) For any matrix A, we can associate a matrix B such that $A + B = B + A = 0$. In fact, if $A = (a_{ij})$, then $B = (-a_{ij})$. The matrix B thus obtained is usually denoted by $(-A)$ and is called the *negative* of A.

16.2.3 Scalar Multiplication

We define multiplication of a matrix $A = (a_{ij})$ by a scalar (number) α by the following rule:

$$\alpha A = A\alpha = (\alpha a_{ij}),$$

i.e., we multiply every entry of A by α to obtain αA. For example, if

$$A = \begin{bmatrix} 1 & 0 & -3 & 4 \\ i & 3+7i & 1 & -4 \end{bmatrix}$$

then

$$4A = \begin{bmatrix} 4 & 0 & -12 & 16 \\ 4i & 12+28i & 4 & -16 \end{bmatrix}$$

and

$$iA = \begin{bmatrix} i & 0 & -3i & 4i \\ -1 & -7+3i & i & -4i \end{bmatrix}.$$

The following properties of scalar multiplication follow from the definition:

(viii) αA is a matrix of the same order as that of A for every α.

(ix) $\alpha(A + B) = \alpha A + \alpha B$

(x) $(\alpha + \beta)A = \alpha A + \beta B$

(xi) $(\alpha\beta)A = \alpha(\beta A)$

(xii) $1A = AI = A$.

We observe that addition of matrices of order m by n and multiplication of these matrices by a scalar have identical properties with the corresponding arithmetic of R^{mn} (or C^{mn}).

E 1

Verify properties (v), (ix), (x) and (xi) with the following data:

$$A = \begin{bmatrix} 2 & 3-i & 4 & -i \\ 1 & 0 & 2i & 4+i \\ 3 & 4 & i & 1+i \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2+i & 3 & 2i \\ 0 & 1-i & i & 1 \\ 0 & 3 & 4i & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2-i & 3i & 1 \\ -2 & 2i & 3 & -5 \\ 4 & 2 & 0 & 6i \end{bmatrix}$$

and $\alpha = 2, \beta = -3i$.

16.2.4 Transpose and Conjugate of a Matrix

Let $A = (a_{ij})$ be a m by n matrix. A new matrix, of order n by m, called the *transpose* of A and denoted by A^T, can be associated with the matrix A as follows:

$$A^T = (a_{ji}), j = 1 \text{ to } n \text{ and } i = 1 \text{ to } m.$$

Thus the ith row of A is the ith column of A^T and the jth column of A is the jth row of A^T. To obtain the transpose of a matrix, we have to interchange its row and columns. For example, if

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 5 \end{bmatrix}.$$

If a_{ij} are complex numbers, we can also define the *conjugate*, denoted by $\overline{A}$, of the matrix $A = (a_{ij})$ as follows:

$$\overline{A} = (\overline{a}_{ij})$$

i.e., we replace every a_{ij} by its conjugate $\overline{a}_{ij}$ to obtain $\overline{A}$. For example, if

$$A = \begin{bmatrix} 1+i & 2 \\ 3-i & 4+2i \end{bmatrix} \text{ then } \overline{A} = \begin{bmatrix} 1-i & 2 \\ 3+i & 4-2i \end{bmatrix}.$$

We observe the following properties which follow from the definition itself:

(xiii) $(A^T)^T = A$

(xiv) $(A+B)^T = A^T + B^T$

(xv) $\overline{(\overline{A})} = A$

(xvi) $\overline{(A+B)} = \overline{A} + \overline{B}$.

(xvii) $(\alpha A)^T = \alpha A^T$

(xviii) $\overline{(\alpha A)} = \overline{\alpha}\,\overline{A}$

E 2

Verify the properties (xiii) to (xviii) using the data given in E1.

16.2.5 Special Matrices

Using the notions of conjugate and transpose of a matrix, we now define some special types of matrices. These matrices have some properties which are not valid for arbitrary matrices.

A matrix A is called a *symmetric matrix* if

$$A = A^T.$$

Thus, $A = (a_{ij})$ is a symmetric matrix if,

For example, the matrix

$$\begin{bmatrix} -1 & 2 & 4 \\ 2 & 0 & 3 \\ 4 & 3 & 5 \end{bmatrix}$$

is a symmetric matrix.

A matrix $A = (a_{ij})$ is called *skew-symmetric* if

$a_{ij} = -a_{ji}$ for all i and j,

i.e., $A = -A^T$.

For example, the matrix

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

is a skew–symmetric matrix.

Note that $a_{ii} = 0$ for all i in a skew-symmetric matrix because

$a_{ii} = -a_{ii}$.

A matrix $A = (a_{ij})$ is called a *Hermitian matrix* if

$A = \overline{A}^T$,

i.e., $a_{ij} = \overline{a}_{ji}$ for all i and j.,

For example, the matrix

$$\begin{bmatrix} 1 & 2+i & 3-2i \\ 2-i & 0 & 2i \\ 3+2i & -2i & 4 \end{bmatrix}$$

is a Hermitian matrix.

Note that a_{ii} is real for all i in a Hermitian matrix, because $a_{ii} = \overline{a_{ii}}$.

A matrix $A = (a_{ij})$ is called *skew-Hermitian* if

$A = -\overline{A}^T$

i.e, $a_{ij} = -\overline{a_{ji}}$ for all i and j.

For example, the matrix

$$\begin{bmatrix} 2i & 4+3i & 5i \\ -4+3i & 0 & 2+i \\ 5i & -2+i & i \end{bmatrix}$$

is a skew- Hermitian matrix.

Note that a_{ii} is purely imaginary or zero for all i in a skew-Hermitian matrix, because $a_{ii} = -\overline{a_{ii}}$. We observe that we can talk of symmetric , skew-symmetric Hermitian matrix only for square matrix.

We, also observe that a symmetric (skew-symmetric) matrix with real entries is a Hermitian (skew-Hermitian) matrix.

Let A be any square matrix, then clearly

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T).$$

Further, we observe that

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

and

$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T).$$

Thus, any square matrix A can be written as the sum of a symmetric matrix, $\frac{1}{2}(A + A^T)$, and a skew- symmetric matrix, $\frac{1}{2}(A - A^T)$. Further, such a decomposition of A is unique in the following sense:

If $A = B + C$,

where B is symmetric and C is skew-symmetric, then we must have

$$B = \frac{1}{2}(A + A^T) \text{ and } C = \frac{1}{2}(A - A^T).$$

Similarly, we also have

$$A = \frac{1}{2}(A + \overline{A}^T) + \frac{1}{2}(A - \overline{A}^T),$$

where $\frac{1}{2}(A + \overline{A}^T)$ is Hermitian and $\frac{1}{2}(A - \overline{A}^T)$ is skew-Hermitian. Such a decomposition of a square matrix as the sum of a Hermitian matrix and a skew-Hermitian matrix is also unique.

E 3

Show that the sum of any two Hermitian matrices is a Hermitian matrix.

E 4

Let A be a Hermitian matrix. What can you say about αA when

(a) α is a real number

(b) α is a complex number (not real).

E 5

Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & 4 \\ 6 & 4 & 5 \end{bmatrix}$$

Determine B and C such that $A = B + C$ where B is symmetric and C is skew-symmetric.

E 6

Show that $A + A^T$ is symmetric for any square matrix A.

16.2.6 Partitioning of Matrices

We have assumed that the entries of a matrix are numbers. This restriction is imposed because we use the arithmetic of numbers to introduce the arithmetic in the set of all m by n matrices. Since we are now familiar with the arithmetic of matrices, we can permit matrices themselves as elements of a matrix. For example, we can write

$$A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix},$$

where the elements P, Q, R and S are matrices of appropriate orders. Note that P and Q (*and* R and S) must have the same number of rows. Similarly, P and R (and Q and S) must have the same number of columns. Otherwise, A is not a matrix. It may be noted that the orders of P, Q, R and S, satisfying the restrictions just stated, can be varied. This process of viewing a matrix with elements as matrices is called *partitioning of matrices*. In practice, the matrices P, Q, R, S refer to sub-systems of the system represented by the matrix A.

Suppose that two matrices A and B are *similarly partitioned* as

$$A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \text{ and } B = \begin{bmatrix} L & M \\ N & T \end{bmatrix},$$

i.e., the orders of corresponding elements (matrices now) of A and B are same. Then, clearly we have

$$A + B = \begin{bmatrix} P+L & Q+M \\ R+N & S+T \end{bmatrix},$$

and

$$\alpha A = \begin{bmatrix} \alpha P & \alpha Q \\ \alpha R & \alpha S \end{bmatrix}$$

$$A^T = \begin{bmatrix} P^T & R^T \\ Q^T & S^T \end{bmatrix},$$

and

$$\bar{A} = \begin{bmatrix} \bar{P} & \bar{Q} \\ \bar{R} & \bar{S} \end{bmatrix}.$$

Thus, the arithmetic of the set of all m by n matrices can be used in similarly partitioned matrices.

Let

$$A = (a_{ij}), i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n,$$

be a matrix of order m by n. The following partitionings of A, in terms of its rows and columns, is found extremely useful:

$$A = \begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ P_m \end{bmatrix} = \begin{bmatrix} Q_1 Q_2 \cdots Q_n \end{bmatrix},$$

where

$$P_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}, \; i = 1 \text{ to } m,$$

and

$$Q_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdots \\ a_{mj} \end{bmatrix}, \quad j = 1 \text{ to } n.,$$

Thus P_i is a 1 by n row matrix, representing the ith row of A. Similarly, Q_j is a m by 1 column matrix representing the jth column of B. We can also regard P_i as a vector in n-dimensional space and Q_j as a vector in m-dimensional space.

E 7

Let

$$A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

(a) Is A symmetric when P, Q, R, and S are symmetric?

(b) Are P, Q, R, and S symmetric when A is symmetric?

16.2.7 Submatrices

Let $A = (a_{ij})$ be a matrix. We can form several matrices by using the elements a_{ij} of A such that the relative positions of a_{ij} are not changed. For instance, we can delete some rows and columns of A and concentrate on the remaining elements of A. The matrix thus

obtained is called *submatrix* of A. For example, let

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 0 & 5 & 6 \\ 2 & 3 & 1 & 7 \end{bmatrix}.$$

Then, the following are some of the submatrices of A:

$[1\ 0\ 5\ 6]$ obtained by deleting the first and third rows of A

$\begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$ obtained by deleting the first, second and fourth columns of A

$\begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}$ obtained by deleting the third row of A and second and fourth columns of resulting matrix.

$\begin{bmatrix} 1 & 0 & 6 \\ 2 & 3 & 7 \end{bmatrix}$ obtained by deleting the first row of A and third column of *resulting matrix.*

16.3 MATRIX MULTIPLICATION

As remarked earlier, the operations of addition and scalar multiplicatiopn of matrices of order m by n is identical with the corresponding operations in R^{mn} (or C^{mn}). In this section, we introduce the important concept of matrix multiplication which, indeed, makes the study of matrices distinctive from that of R^{mn}.

Let $A = (a_{ij})$ be a matrix of order m by n and $B = (b_{jk})$ be a matrix of order n by p so that $i = 1$ to m, $j = 1$ to n and $k = 1$ to p. Note that n is common in the orders of A and B, i.e., the number of columns of A equals the number of rows of B. If A and B have the orders as specified, then we say that A *is conformable or compatible with* B for multiplication. Further, the *product* of A with B, denoted by AB, is defined as follows:

$$AB = (c_{ik}),\ i = 1 \text{ to } m, k = 1 \text{ to } p,$$

where

$$c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + \ldots + a_{in} b_{nk}.$$

Thus each of the n nunmbers in ith row of A, viz ,a_{ij} is multiplied with the corresponding number in the k th column of B, viz ,b_{jk} and these n products are added to obtain c_{ik}, the element in the ith row and kth column of the matrix AB. Thus, we can write

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

Note that the matrix AB, as defined above, is of order m by p ,where m is the number of rows of A and p is the number of columns of B.

As an example, let

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 1 & 0 \\ 5 & 2 & 1 \end{bmatrix}.$$

Since the number of columns of A equals the number of rows of B, we can multiply A with B. AB is of order 2 by 3. Thus , we have

$$AB = \begin{bmatrix} 1\times4+2\times5 & 1\times1+2\times2 & 1\times0+2\times1 \\ 4\times4+5\times5 & 4\times1+5\times2 & 4\times0+5\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 5 & 2 \\ 41 & 14 & 5 \end{bmatrix}.$$

It may be noted that AB may be defined whereas BA may not be defined, because B may not be conformable with A for multiplication. However, if A and B are square matrices of the same order then both AB adn BA are defined.

The following properties of matrix multiplication are easy to prove:

(xix) $A(BC) = (AB)C$

(xx) $A(B+C) = AB+AC$

(xxi) $(A+B)C = AC+BC$

(xxii) $(\alpha A)B = \alpha(AB) = A(\alpha B)$

(xxiii) $(AB)^T = B^T A^T$

(xxiv) $\overline{AB} = \overline{A}\,\overline{B}$

(xxv) $AI = IA = A$.

Here we have assumed that the orders of the matrices A, B and C are such that the various matrices involved in the above properties are well defined. For example in (xx), if A is of order m by n then the orders of both B and C must be n by p for some p.

Most of the properties that we have listed about matrix operations (addition, scalar multiplication, transportation, conjugation, multiplication) are also valid in numbers. However, there are some important differences also. We bring out the important differences in the following examples.

Example 1:

We show that $AB \neq BA$, in general.

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ -1 & 1 \\ 2 & 3 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 4 & 11 \\ 1 & 4 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & -2 \\ 2 & 7 & 9 \end{bmatrix}.$$

In general, let A be of order m by n and B be of order n by m. Then AB is of order m by m and BA is of order n by n. Thus if $m \neq n$, then AB and BA are of different orders. Therefore, AB can never equal BA if A and B are not square matrices.

Even if $m = n$, AB is not always equal to BA. For example, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then,

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that $AB \neq BA$.

Example 2 :

We observe that $AB = 0$ does not mean that either $A = 0$ or $B = 0$.

Consider the matrices A *and* B of example 1. We have $BA = 0$ but $A \neq 0$ and $B \neq 0$.

It may also be noted that $BA = 0$ but $AB \neq 0$.

Example 3 :

Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then,

$$B - C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Example 1 shows that

$$A(B-C) = 0,$$

i.e., $AB - AC = 0$, i.e., $AB = AC$.

Thus, $AB = AC$ but $B \neq C$.

Example 4 :

Let

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Then,

$$A^2 = AA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B^2 = BB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, $A^2 = B^2$ but $A \neq B$.

Example 5 :

Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

then

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus, $A^2 + B^2 = 0$ but neither $A = 0$ nor $B = 0$.

The above examples show that we have to be careful while working with various matrix operations. The usual rules of the arithmetic of numbers are not valid in the case of matrices.

The following examples illustrate that the valid properties of matrix operations can be used effectively to prove some assertions about matrices.

Examples 6 :

Let A be a symmetric matrix. Then, we must have

$$A = A^T.$$

Now,

$$(A^2)^T = (AA)^T = A^T A^T = AA = A^2.$$

Since $(A^2)^T = A^2$, we conclude that A^2 is a symmetric matrix.

Note that computing A^2 and then observing the symmetry property of A^2 by using the symmetry property of A is much more difficult.

Example 7 :

In this example, we observe that any system of linear equations can be compactly written as a *matrix equation.*

Consider the following system of m linear equations in n variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad\quad \vdots &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

Using the arithmetic of matrices, this system can be represented by the matrix equation

$$AX = b,$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Let us partition A in terms of its columns, viz.,

$$A = \begin{bmatrix} Q_1 Q_2 \dots Q_n \end{bmatrix},$$

where Q_j is the j th column of A .Then, the matrix equation $AX = b$ can also be written as

$$Q_1 x_1 + Q_2 x_2 + \dots + Q_n x_n = b.$$

You may now try the following exercises:

E 8

Let $A(\theta)$ denote the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Prove that $A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2)$.

E 9

Let A be a 2 by 3 matrix such that
$[\ 1\ \ 1\ ]A = [\ 0\ \ 1\ \ 1\ ]$ and $[\ 0\ \ 1\ ]A = [\ 1\ \ 0\ \ 1\ ]$.
Determine $[\ 2\ \ 3\ ]A$.

E 10

Find $2A^4 - A^2 + A - 5I$, where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

E11

Show that $A^3 - 6A^2 - 15A - 8I = 0$, where

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

E 12

Let A and B be square matrices of order n such that $AB = BA$. Prove that

$$(A+B)^2 = A^2 + 2AB + B^2$$

and $(A+B)(A-B) = A^2 - B^2$.

E 13

Let $A = (a_{ij})$ be a square matrix of order n such that $a_{ij} = 0$ for $i \geq j$, Prove that $A^n = 0$.

E 14

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Show that both A and B are symmetric, but AB is not symmetric.

E 15

Let A and B be symmetric matrices such that $AB = BA$. Is it necessary that AB is symmetric ?

E 16

Let A and B be skew-symmetric matrices such that $AB = BA$. What can you say about AB ?

E 17

If A is a Hermitian matrix, show that $\overline{B}^T AB$ is Hermitian for every matrix B.

E 18

Let A be a symmetric matrix such that $A^2 = 0$. Is it necessary that $A = 0$?

E 19

Let A be a n by n matrix such that $AB = BA$ for all n by n matrices B. Show that $A = \alpha I$ for some α.

E 20

Let $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ and A, B be matrices such that $C = AB - BA$.

Prove that $c_{11} + c_{22} = 0$.

E 21

Let A and B be symmetric matrices such that AB is a diagonal matrix.

Prove that $AB = BA$.

E 22

If A and B are Hermitian matrices, show that $-i\,(AB - BA)$ is Hermitian.

E 23

Let A and B be square matrices of order 2. Show that $(AB - BA)^2 = \alpha I$ for some scalar α.

16.4 MATRICES AND LINEAR TRANSFORMATION

Let A be any m by n matrix with real entries and $X \in R^n$. We can regard X as a column vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

i.e., as a n by 1 matrix. Then AX is a m by 1 matrix, i.e., a column vector in R^m. Thus, using any matrix A, we can associate a unique vector AX in R^m with every vector $X \in R^n$, i.e., a mapping from R^n to R^m can be defined by using a m by n matrix. The properties of matrix multiplication show that

$$A(X+Y) = AX+AY \qquad \text{(property (xx))}$$

and $$A(\alpha X) = \alpha AX \qquad \text{(property (xxi))}$$

Thus, the mapping $X \to AX$ is a linear transformation.

Conversely, let $T : R^n \to R^m$ be any linear transformatiom and $\{E_1, E_2, \ldots E_n\}$ denote the standard basis of R^n.

Consider the matrix

$$A = [T(E_1)\,T(E_2) \ldots T(E_n)],$$

where $T(E_j)$ is the jth column of A. Then,

$$T(X) = x_1 T(E_1) + x_2 T(E_2) + \ldots + x_n T(E_n),$$

because

$$X = x_1 E_1 + x_2 E_2 + \ldots + x_n E_n.$$

Further,

$$AX = [T(E_1)\ T(E_2) \ldots T(E_n)] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 T(E_1) + x_2 T(E_2) + \ldots + x_n T(E_n).$$

Note that we have used the matrix multiplication when A is partitioned in terms of its columns. Thus, we find that

$$T(X) = AX \text{ for all } X \text{ in } R^n.$$

Thus every linear transformation can be viewed as the mapping $X \to AX$, where A is as defined above. Note that the basis $\{E_1, E_2, \ldots, E_n\}$ has played an important role in associating the matrix A with the linear transformation T. In fact ,by changing the basis of R^n, we can indeed associate various matrices with the same transformation. However, we do not pursue this topic at this stage. Note that a linear transformation from C^n to C^m can be associated, proceeding as above, with a given m by n matrix with complex entries.

The association of linear transformations with matrices, as discussed here, can be used to interpret the matrix multiplication rule. Let A be a m by n matrix and B be a n by p matrix. Then, we know that AB is a m by p matrix. Let us denote the corresponding linear transformation also by A, B and AB. How is the linear transformation AB related to the linear transformations A and B ? We note that

$$B : R^p \to R^n$$

$$A : R^n \to R^m$$

and $$AB : R^p \to R^m.$$

We know that the transformation AB associates the vector ABX in R^m with the vector X in R^p. Using property (xix) of matrix multiplication, we have

$$(AB)X = A(BX),$$

i.e., to find $(AB)X$, we first find BX in R^n by using the transformation B and then find $A(BX)$ in R^m by using the transformation A. Thus the transformation AB is the resultant of the transformations A and B (first B and then A). The matrix multiplication rule, that we have adopted, ensures that the resultant transformation AB should correspond to the product AB of the matrices A and B. We, however, omit the details at this stage.

16.5 RANK OF A MATRIX

We know that a m by n matrix can be used to define a linear transformation from R^n to R^m. Further, every linear transformation has a well defined rank. We define the rank of the associated linear transformation as the rank of the matrix. In this section ,we show how the rank of any matrix can be computed efficiently.

Let $A = (a_{ij})$ be a m by n matrix. Let us partition A in terms of its columns, i.e,

$$A = [\, Q_1 Q_2 \cdots Q_n \,],$$

where

$$Q_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix} \text{ for } j = 1 \text{ to } n.$$

We know that the linear transformation T associated with the matrix A is defined by

$$T(E_j) = Q_j, \text{ for } j = 1 \text{ to } n,$$

where $\{E_1, E_2, \ldots E_n\}$ is the standard basis of R^n. We also know that the rank of T is the dimension of the subspace $L\{Q_1, Q_2, \ldots Q_n\}$ of R^m. Thus, the rank of A is the *maximum number of linearly independent columns of A,* which is the dimension of $L\{Q_1, Q_2, \ldots Q_n\}$.

Let us now partition A in terms of its rows, i.e.,

$$A = \begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ P_m \end{bmatrix},$$

where

$$P_i = [\, a_{i1}\, a_{i2} \ldots a_{in} \,] \text{ for } i = 1 \text{ to } m$$

We know that the linear transformation T associated with the matrix A can also be defened by

$$T(X) = \begin{bmatrix} P_1 X \\ P_2 X \\ \cdot \\ \cdot \\ \cdot \\ P_m X \end{bmatrix},$$

where $X \in R^n$ is considered as a column vector. The nullity of T is the dimension of the subspace $\{X \mid T(X) = 0\}$ of R^n. Now, $T(X) = 0$ is equivalent to

$$P_i X = 0 \quad , \text{ for } i = 1 \text{ to } m,$$

i.e., every X satisfying $T(X) = 0$ is orthogonal to the subspace $L\{P_1, P_2, \ldots P_m\}$ of R^n. Thus $\nu(T)$, the nullity of T, is the dimension of the orthogonal complement of the subspace $L\{P_1, P_2, \ldots, P_m\}$ of R^n. Therefore,

$$\nu(T) = n - \text{dimension of } L\{P_1, P_2, \ldots, P_m\}.$$

Details of the proof may be left at this stage.

Since we know that

$$\nu(T) + \text{dimension of } L\{Q_1, Q_2, \ldots, Q_n\} = n,$$

we conclude that

$$\text{dimension of } L\{P_1, P_2, \ldots, P_m\} = \text{dimension of } L\{Q_1, Q_2, \ldots, Q_n\}.$$

Thus we have proved the surprising result that the maximum number of linearly independent columns of any matrix A always equals the maximum number of linearly independent rows of A. This common number (i.e. the maximum number of linearly independent rows or column of a matrix) is called the *rank of matrix*.. We now show how the rank of a matrix can be computed efficiently.

Let A be any m by n matrix considered in terms of its rows. Let us consider the following operations:

(a) interchange any two rows of A

(b) add any multiple of some row of A to another row of A.

Let B denote the matrix obtained on using any of the above operations on A. Use the definition of the rank of a matrix to verify that the rank of A equals the rank of B. Clearly, these operations can be used sequentially, in any order, without changing the rank of the matrix. As in the Gauss Elimination Method, we can use these operations to obtain a matrix $C = (c_{ij})$ in which $c_{ij} = 0$ for $i > j$. Then the rank of A equals the number of non-zero rows of C. We illustrate the computation of the rank of a matrix in example 9.

Example 9 :

Let us compute the rank of the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 & 3 \\ 2 & 1 & 4 & 5 & 6 \\ 4 & 1 & 2 & 9 & 12 \\ 0 & 0 & 1 & 3 & 4 \end{bmatrix}.$$

Multiplying the first row by –2 and adding the second row, we get the matrix

$$\begin{bmatrix} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & 6 & 1 & 0 \\ 4 & 1 & 2 & 9 & 12 \\ 0 & 0 & 1 & 3 & 4 \end{bmatrix}.$$

Multiplying the first row of this matrix by – 4 and adding to its third row, we get the matrix

$$\begin{bmatrix} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 1 & 3 & 4 \end{bmatrix}.$$

Subtracting the second row of this matrix from its third row, and then interchanging the third and fourth rows, we get the matrix

$$C = \begin{bmatrix} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = (c_{ij}).$$

Now, we observe that $c_{ij} = 0$ for $i > j$. Since there are 3 non-zero rows of the matrix C, we conclude that the rank of the matrix A is 3.

E 24

Compute the rank of the matrix

$$\begin{bmatrix} 1 & 2 & -3 & 4 & 1 \\ 0 & 2 & 1 & 1 & 3 \\ 1 & 4 & 2 & 5 & 8 \\ 1 & 4 & -2 & 5 & 4 \end{bmatrix}.$$

E 25

Let $X \neq 0$ be a n by 1 column matrix and $Y \neq 0$ be 1 by n row matrix. Show that the rank of $A = XY$ is one. Also prove that $A^2 = \alpha A$, where $\alpha = (X, Y)$.

E 26

If A and B are square matrices of order n, prove that rank of $AB \leq$ Min [rank of A, rank of B].

E 27

Let A be a 3 by 4 matrix such that

$$AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

What can you say about the rank of A ?

E 28

Prove that the rank of A is the same as the rank of A^T.

16.6 INVERSE OF A MATRIX

Let A be any n by n matrix and T be the linear transformation from R^n to R^n associated with the matrix A. We know that T is invertible if and only if it is one- one onto. Assuming that the linear transformation T associated with the given matrix A is invertible, we can compute the linear transformation T^{-1}.

Let $B = [\, T^{-1}(E_1)T^{-1}(E_2) \ldots T^{-1}(E_n)\,]$,

i.e., B is the matrix associated with T^{-1}. Using the definition of T^{-1}, it can be verified that the matrix B satisfies the equation

$$AB = BA = I.$$

The above discussion helps us to define the inverse of a square matrix. We define a square matrix A to be *invertible* (or *non-singular*) if there exists a matrix B such that

$$AB = BA = I$$

Further, the matrix B, if it exists, is called the inverse of the matrix A and is usually denoted by A^{-1}.

We first observe that the inverse of a matrix is unique, provided that it exists. To prove it, let us assume that B and C are two inverses of A. Then, we must have

$$AB = BA = I$$

and $$AC = CA = I.$$

Pre- multiplying $AB = I$ with C, we get

$$C(AB.) = CI = C$$

$$\Rightarrow (CA)B = C$$

$$\Rightarrow IB = C \qquad (\text{use } CA = I)$$

$$\Rightarrow B = C.$$

The above arguments show that the inverse of a matrix, if it exists, must be unique. To compute the inverse of a matrix, we are required to solve the following n systems of linear equations

$$AX = E_j \text{ for } j = 1 \text{ to } n.$$

We know that a unique solution exists for every j, if the matrix A is non-singular. Let X_j denote this unique solution. Then, we must have

$$A^{-1} = [X_1 X_2 .. X_n\,].$$

Since the coefficient matrix A is same for every j, the Gauss Elimination Method can be used to solve these n systems simultaneously. The Gauss Elimination Method also shows that the rank of a n by n non-singular matrix is n. We illustrate the procedure to compute the inverse of a matrix in example 10.

Example 10.

Let us compute the inverse of the following matrix

$$A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}.$$

In detached coefficient form, the 3 systems of linear equations $AX = E_j$ for $j = 1$ to 3, look as follows:

			E_1	E_2	E_3
−1	2	−3	1	0	0
2	1	0	0	1	0
4	−2	5	0	0	1

We now use the Gauss operations, in an appropriate sequence, with the objective of reducing the matrix A to the identity matrix I. Remember that each operation is also to be performed on the right hand side matrix I. Then the resulting matrix in the position of $E_1\, E_2\, E_3$ is the matrix A^{-1}.

Let us use the notation

$$R_i \leftarrow \alpha R_i + \beta R_k$$

for the operation : replace the ith row by α times the i th row added to β times the kth row. Using this notation, the various steps of the Gauss Elimination Method gives the following equivalent systems:

$R_1 \leftarrow -R_1$

1	−2	3	−1	0	0
2	1	0	0	1	0
4	−2	5	0	0	1

$R_2 \leftarrow R_2 - 2R_1$

1	−2	3	−1	0	0
0	5	−6	2	1	0
4	−2	5	0	0	1

$R_3 \leftarrow R_3 - 4R_1$

1	−2	3	−1	0	0
0	5	−6	2	1	0
0	6	−7	4	0	1

$R_2 \leftarrow \frac{1}{5}R_2$

1	−2	3	−1	0	0
0	1	$\frac{-6}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	0
0	6	−7	4	0	1

$R_3 \leftarrow R_3 - 6R_2$

1	−2	3	−1	0	0
0	1	$-\frac{6}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	0
0	0	$\frac{1}{5}$	$\frac{8}{5}$	$\frac{-6}{5}$	1

We can use the back substitution method at this stage to compute A^{-1}. This also can be computed simultaneously by further reductions as follows:

$R_3 \leftarrow 5R_3$

$$\begin{matrix} 1 & -2 & 3 \\ 0 & 1 & \frac{6}{5} \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} -1 & 0 & 0 \\ \frac{2}{5} & \frac{1}{5} & 0 \\ 8 & -6 & 5 \end{matrix}$$

$R_2 \leftarrow R_2 + \frac{6}{5}R_3$ and $R_1 \leftarrow R_1 - 3R_3$

$$\begin{matrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} -25 & 18 & -15 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{matrix}$$

$R_1 \leftarrow R_1 + 2R_2$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{matrix}$$

Now we observe that the position originallly occupied by matrix A has been reduced to I. Hence,

$$A^{-1} = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}.$$

It may be noted that is reducing A to form I, we have written E as post factor and used only row operation. We can , also calculate A^{-1} by writing E as pre-factor and using only column operations and when A becomes I, reduced E becomes A^{-1}.

E 29 Compute the inverse of the following matrices

(a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$.

E 30

Let A and B be n by n matrices such that AB is singular and B is non-singular. Prove that A must be singular.

E 31

Let A be a 2 by 2 matrix such that $[\,1\ 0\,]A = [\,-2\ 0\,]$ and $[\,0\ 1\,]A = [\,-1\ \ 1\,]$. Is A non-singular?

E 32

Let A be a non-singular matrix. Prove that the rank of AB equals the rank of B.

E 33

If A and B are non-singular, prove that $(AB)^{-1} = B^{-1}A^{-1}$.

E 34

If A is singular, prove that $AB = I$ cannot be satisfied by any matrix B.

E 35

Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Prove that $A^{-1} = A^T$.

16.7 DETERMINANTS

You are familiar with determinants of order two and three. In this section, we define determinants of higher orders and give some important properties without proof. These properties can be used to compute the determinants efficiently. We avoid rigour in order to be practical.

With each square matrix A, we associate a number $D(A)$ called the determinant of A, calculated with the help of its entries. It may be noted that, in our discussion, notationwise $D(A) = |A| = det(A)$.

Recall that when $n = 1,2,3$, the determinant $D(A) = |A|$ is defined as follows:

$$D(A) = |a| = a \text{ if } A = (a) \text{ is a } 1 \times 1 \text{ matrix,}$$

$$D(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

$$\text{if } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ is a } 2 \times 2 \text{ matrix.}$$

$$D(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

(expansion along first column)

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$ (expansion along the first row),

where $A = (a_{ij})$. is a 3×3 matrix.

We observe that determinants of order 3 can be defined in terms of determinants of order 2. Similarly determinants of order 4 can be defined in terms of determinants of order 3 and by induction determinants of any order n can be defined in terms of determinants of order n–1. Although there are difficulties in giving rigorous proofs of results, if we define determinants by induction, but from practical point of view, for our purposes, this approach in reasonably good. We shall discuss this later as Laplace expansion of determinants.

Rightnow, we define the determinant of an $n \times n$ matrix $A = (a_{ij})$ as follows:

$$D(A) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$= a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13} - a_{14} M_{14} + \ldots + (-1)^{n+1} \; a_{1n} M_{1n}$$

(expansion along first row)

or $$a_{11} M_{11} - a_{21} M_{21} + a_{31} M_{31} - a_{41} M_{41} + \ldots + (-1)^{n+1} \; a_{n1} M_{n1}$$

(expansion along first column)

where M_{ij}, called (i, j)th minor, is the determinant of $(n-1) \times (n-1)$ matrix obtained by deleting ith row and jth column of the $n \times n$ matrix. It is known that the two values thus obtained will be equal. More details will be given when we discuss Laplace expansion of determinants.

We, now, emphasize on the efficient computation of determinants and applications. Computation of determinants of higher order is a time consuming process although it is routine. We give below the main properties of determinants without, proof. These properties are used to compute the determinants efficiently.

16.7.1 Properties of Determinants

Let us partition the matrix A in terms of its rows,

i.e., $$A = \begin{bmatrix} R_1 \\ R_2 \\ . \\ . \\ . \\ R_n \end{bmatrix},$$

so that $R_i = [\, a_{i1} \; a_{i2} \ldots a_{in} \,]$ is the ith row of A. We shall denote the determinant of A by $D(R_1, R_2, \ldots, R_n)$ or $D(A)$. We, now, give some properties of determinants which can be used to compute determinants more efficiently.

Property 1.

$D(R_1, \ldots tR_i, \ldots, R_n) = tD(R_1, \ldots, R_i, \ldots, R_n)$, i.e., multiplying every element of any one row of a determinant by any number is equivalent to multiplying the determinant by the number.

Property 2.

If $R_i = R'_i + R''_i$ for some fixed i, then

$D(R_1, \ldots, R_i, \ldots, R_n) = D(R_1, \ldots, R_i', \ldots, R_n) + D(R_1, \ldots, R_i'', \ldots, R_n)$.

Property 3.

If $R_i = R_j$ for some fixed i,and j, then $D(R_1, \ldots, R_i, R_{i+1}, \ldots, R_n) = 0$.

Property 4.

For any fixed i, $D(R_1, \ldots, R_{i+1}, R_i, \ldots, R_n) = -D(R_1, \ldots, R_i, R_{i+1}, \ldots, R_n)$, i.e., interchanging any two consecutive rows of a determinant is equivalent to multiplying the determinant by –1.

Property 5.

For any i and j,

$$D(R_1, \ldots, R_i, \ldots, R_j, \ldots, R_n) = -D(R_1, \ldots, R_j, \ldots, R_i, \ldots, R_n)$$

i.e. interchanging any two rows of a determinant is equivalent to multiplying the determinant by –1.

Property 6.

If any two rows of A are identical, then D (A) = 0

Property 7.

If a determinant B is obtained by adding a scalar multiple of any row of a determinant A to any other row of the determinant A, then $D(A) = D(B)$.

Property 8.

$D(A) = D(A^T)$ for any determinant A

Property 9.

If $A = (a_{ij})$ is a triangular matrix, then $D(A) = a_{11}\, a_{22} \ldots a_{nn}$.

Property 10.

$D(AB) = D(A)\, D(B)$ for any pair of $n \times n$ matrices A and B.

Remark :

Since $D(A) = D(A^T)$, properties 1 to 7 are valid if the similar operations are performed on columns of A rather than on rows of A.

Let $A = (a_{ij})$ be any matrix. We can repeatedly add suitable multiples of some row (or column) to some other row (or column) till the matrix becomes triangular. Property 7 shows that the determinant of the resulting triangular matrix is $D(A)$. Then property 9 can be used to obtain $D(A)$. Thus we observe that these properties of determinants can be used to compute $D(A)$ efficiently. Compare these computations with those required to compute $D(A)$ by using the definition of a determinant. These computations are illustrated in example 11.

Example 11 :

Use the properties of determinants to evaluate

$$\begin{vmatrix} 1 & 1 & 1 & -1 \\ 1 & 3 & 3 & 3 \\ 2 & 3 & 4 & -3 \\ 1 & 2 & 1 & 2 \end{vmatrix}.$$

We know that replacing R_j by $R_j + t\,R_i$ does not change the value of the determinant . Thus

$$\begin{vmatrix} 1 & 1 & 1 & -1 \\ 1 & 3 & 3 & 3 \\ 2 & 3 & 4 & -3 \\ 1 & 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 3 \end{vmatrix} \qquad \begin{aligned} R_2 &\leftarrow R_2 - R_1 \\ R_3 &\leftarrow R_3 - 2R_1, \\ R_4 &\leftarrow R_4 - R_1. \end{aligned}$$

$$= \begin{vmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -1 & 1 \end{vmatrix} \qquad \begin{aligned} R_3 &\leftarrow R_3 - \frac{1}{2}R_2 \\ R_4 &\leftarrow R_4 - \frac{1}{2}R_2 \end{aligned}$$

$$= \begin{vmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & -2 \end{vmatrix} \qquad R_4 \leftarrow R_4 + R_3$$

$$= -4 \qquad \text{using property 9.}$$

16.7.2 Laplace Expansion of Determinants

We use the definition of a determinant to show that a determinant of order n is related, in a natural way, to n determinants of order $n-1$. Thus, a determinant of order n can be evaluated by evaluating n determinants of order $n-1$.

Let $A = (a_{ij})$ be a square matrix of order n. Let A_{ij} denote the matrix obtained by deleting the *ith row and* jth column of the matrix A. Clearly, A_{ij} is a square matrix of order $n-1$ for fixed i and j. The determinant of A_{ij} namely $D(A_{ij})$, is called the *minor of* a_{ij}. *Further* $(-1)^{i+j} D(A_{ij})$ *is called the cofactor of* a_{ij}. *We shall use the notation*

$$M_{ij} = D(A_{ij}) \text{ for minor of } a_{ij} = (i, j)\text{-th minor}$$

and $C_{ij} = (-1)^{i+j} M_{ij}$ for cofactor of a_{ij} = (i, j) -th cofactor.

We have defined

$$D(A) = a_{11} C_{11} + a_{12} C_{12} + \ldots + a_{1n} C_{1n}$$

$$= a_{11} C_{11} + a_{21} C_{21} + \ldots + a_{n1} C_{n1}.$$

It is well known that $D(A)$ can be obtained by expanding along any row or any column and all values thus obtained will be equal. We shall not give the proof of this fact and assume that all expansions lead to same number $D(A)$. Thus we have the following:

$$D(A) = a_{i1} C_{i1} + a_{i2} C_{i2} + \ldots + a_{in} C_{in} \text{ (expansion along } i\text{th row)}$$

Similarly,

$$D(A) = a_{1j} C_{1j} + a_{2j} C_{2j} + \ldots + a_{nj} C_{nj} \text{ (expansion along } j\text{th column)}$$

Let us now consider the expression

$$a_{k1} C_{i1} + a_{k2} C_{i2} + \ldots + a_{kn} C_{in} \text{ where } k \neq i$$

Clearly, this is the value of the determinant of a matrix in which $R_i = R_k$ and other rows are rows of A. Property 6 shows that this value is zero. Thus,

$$a_{k1} C_{i1} + a_{k2} C_{i2} + \ldots + a_{kn} C_{in} = 0, \text{ if } k \neq i$$

Similarly,

$$a_{1k} C_{1j} + a_{2k} C_{2j} + \ldots + a_{nk} C_{nj} = 0, \text{ if } k \neq j$$

The expression given above for $D(A)$ in terms of C_{ij} is known as *Laplace expansion of a determinant*. Clearly any row or column can be used in this expansion. The computation of a determinant, using Laplace expansion, is illustrated in example 12.

Example12 :

Compute the following determinant by using Laplace expansion

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{vmatrix}$$

Using Laplace expansion with $i = 1$, we get

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{vmatrix} = 1 \begin{vmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{vmatrix} - 2 \begin{vmatrix} 2 & 4 & 5 \\ 3 & 5 & 6 \\ 4 & 6 & 7 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 & 5 \\ 3 & 4 & 6 \\ 4 & 5 & 7 \end{vmatrix}$$

$$- 4 \begin{vmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= \left\{ 3 \begin{vmatrix} 5 & 6 \\ 6 & 7 \end{vmatrix} - 4 \begin{vmatrix} 4 & 6 \\ 5 & 7 \end{vmatrix} + 5 \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} \right\}$$

$$- 2 \left\{ 2 \begin{vmatrix} 5 & 6 \\ 6 & 7 \end{vmatrix} - 4 \begin{vmatrix} 3 & 6 \\ 4 & 7 \end{vmatrix} + 5 \begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix} \right\}$$

$$+ 3 \left\{ 2 \begin{vmatrix} 4 & 6 \\ 5 & 7 \end{vmatrix} - 3 \begin{vmatrix} 3 & 6 \\ 4 & 7 \end{vmatrix} + 5 \begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} \right\}$$

$$- 4 \left\{ 2 \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - 3 \begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix} + 4 \begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} \right\}$$

$$= 3(35-36) - 4(28-30) + 5(24-25) - 4(35-36) +$$

$$8(21-24) - 10(18-20) + 6(28-30) - 9(21-24) +$$

$$15(15-16) - 8(24-25) + 12(18-20) - 16(15-16)$$

$$= -3+8-5+4-24+20-12+27-15+8-24+16 = 0.$$

E 36

Use the properties of determinants to prove the following:

(a) $$\begin{vmatrix} a+b & 1 & c \\ b+c & 1 & a \\ a+c & 1 & b \end{vmatrix} = 0$$

(b) $$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

(c) $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{vmatrix}.$$

(d) $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ab-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}.$$

E 37

Let $D_n = |d_{ij}|$ be a determinant of order n is which $d_{ii} = a$, $d_{i,i-1} = b$ for all i and $d_{ij} = C$ elsewhere . Prove that

$$D_n = a D_{n-1} - bc D_{n-2} \text{ for } n \geq 3.$$

If $a = 2$ and $b = c = 1$, show that $D_n = n+1$.

E 38

Show that $D(\overline{A}) = \overline{D(A)}$.

E 39

If A is Hermitian matrix, show that $D(A)$ is real .

E40

How does the value of a determinant change if each element a_{ij} of the determinant is multiplied by p^{i-j} where p is fixed .

E 41

If every element of a determinant of order n is $+1$ or -1 , prove that the determinant is divisible by 2^{n-1}.

16.8 DETERMINANTS AND MATRICES

In this section, we study the role of determinants in the study of some aspects of matrices.

Let $A = [A_1 A_2 \ldots A_n]$ be a singular matrix of order n , where A_j is the j th column of A. Then $\{A_1, A_2, \ldots, A_n\}$ is linearly dependent. Thus we can find scalars $x_1, x_2, \ldots, x_n$ such

that

$$x_1 A_1 + x_2 A_2 + ..., + x_n A_n = 0$$

and atleast one $x_j \neq 0$. Without any loss of generality, let $x_1 \neq 0$. Then, we get

$$x_1 A_1 = -x_2 A_2 - x_3 A_3 - ...- x_n A_n,$$

$$\text{i.e.,} A_1 = y_2 A_2 + y_3 A_3 + ...+ y_n A_n$$

where $y_j = -x_j/x_1$ for $j = 2$ to n. Now,

$$D(A_1, A_2, ..., A_n) = D(y_2 A_2 + ... + y_n A_n, A_2, A_3, ..., A_n)$$
$$= y_2 D(A_2, A_2, ..., A_n) + y_3 D(A_3, A_2, ..., A_n)$$
$$+ ... + y_n D(A_n, A_2, ..., A_n) = 0,$$

since in each determinant, on the right hand side, two columns are identical. Thus, we have proved that

$$\{A_1, A_2, ..., A_n\} \text{ is linearly dependent} \Rightarrow D(A_1, A_2, ..., A_n) = 0.$$

Now, let A and B be any two sqare matrices of the same order. It can be proved that

$$D(AB) = D(A)\, D(B)$$

i.e., the determinant of the product of two matrices equals the product of the determinants of the matrices.

Now , let A be a non-singular matrix. Then, there exists a matrix B such that

$$AB = I.$$

Hence,

$$D(AB) = D(I) = 1$$

$$\text{i.e. } D(A)D(B) = 1.$$

This shows that $D(A) \neq 0$ and $D(B) = D(A)^{-1}$. Hence we have shown that a matrix A is non-singular implies $D(A) \neq 0$.

Further , $D(A^{-1}) = D(B) = D(A)^{-1}$. We now show that $D(A) \neq 0$ implies A is non-singular.

We also show that the inverse of a non-singular matrix can be computed by using determinants. The Laplace expansion of a determinant shows that

* $D(A) = a_{i1}\, C_{i1} + a_{i2}\, C_{i2} + ... + a_{in}\, C_{in}$

* and $0 = a_{k1}\, C_{i1} + a_{k2}\, C_{i2} + ... + a_{kn}\, C_{in}$, $(k \neq i)$,

where C_{ij} is the cofactor of a_{ij}. Let us consider the matrix

$$B = (b_{ij}), \text{ where } b_{ij} = \frac{C_{ji}}{D(A)}, \text{ assuming that } D(A) \neq 0.$$

Then $AB = BA = I$. Thus

$$B = A^{-1} = \text{transpose of } \frac{[C_{ij}]}{D(A)}$$

$$= \frac{1}{D(A)} \begin{bmatrix} C_{11} & C_{12} & ... & C_{1n} \\ C_{21} & C_{22} & ... & C_{2n} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ C_{n1} & C_{n2} & ... & C_{nn} \end{bmatrix}^T$$

$$= \frac{1}{D(A)} \quad (\text{Matrix of cofactors})^{\text{transpose}}$$

Thus $D(A) \neq 0$ implies A^{-1} exists , i.e., A is nonsingular.

16.8.1 Cramer' Rule

Consider the matrix equation

$$AX = b,$$

where A is a n by n non-singular matrix. We know that $D(A) \neq 0$ and

$$X = A^{-1} b$$

is the unique solution of the equation. The Cramer's rule gives this unique solution in terms of $n + 1$ determinant of order n. Let

$$A = [A_1 A_2 \ldots A_n],$$

where A_j is the j th column of A. Then, the equation $AX = b$ is equivalent to

$$x_1 A_1 + x_2 A_2 + \ldots + x_n A_n = b.$$

Let $x_1, x_2, \ldots, x_n$ be the unique solution of this equation. Then

$$D(b, A_2, \ldots, A_n) = D(x_1 A_1 + \ldots + x_n A_n, A_2, \ldots A_n)$$

$$= x_1 D(A_1, A_2, \ldots, A_n) + \sum_{j=2}^{n} x_j D(A_j, A_2, \ldots, A_n)$$

$$= x_1 D(A) + 0 \qquad \text{(using property 6)}.$$

Therefore,

$$x_1 = \frac{D(b, A_2, \ldots, A_n)}{D(A)}$$

Using a similar argument, we get

$$x_j = \frac{D(A_1, A_2, \ldots, A_{j-1}, b, A_{j+1}, \ldots, A_n)}{D(A)}$$

for $j = 1$ to n. Thus, the unique solution of $Ax = b$ can be obtained by evaluating n+1 determinants. The above computation for x_j is known as Cramer's rule.

Note that the numerator of x_j is obtained by replacing A_j by b in $D(A)$. We illustrate the computation in example 13.

Example 13 :

Solve the system of linear equations

$$3x + 2y + 4z = 1,$$

$$2x - y + z = 0,$$

$$x + 2y + 3z = 1.$$

Here

$$D(A) = \begin{vmatrix} 3 & 2 & 4 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -5.$$

Therefore, using the Cramer's rule , we get

$$x = -\frac{1}{5} \begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \frac{-1}{5}$$

$$y = -\frac{1}{5}\begin{vmatrix} 3 & 1 & 4 \\ 2 & 0 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 0.$$

$$z = -\frac{1}{5}\begin{vmatrix} 3 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = \frac{2}{5}.$$

E 43

Determine values of a for which the following system of linear equations has (a) a unique solution , (b) infinite number of solutions :

$$x + y + z = 2$$

$$x + 2y + z = -2$$

$$x + y + (a - 5)z = a - 4.$$

E 44

If $A^{-1} = A^T$ for a matrix A, prove that $D(A) = +1$ or -1.

E 45

let $A = (a_{ij})$ be a triangular matrix. Prove that A^{-1} exists if and only if $a_{ii} \neq 0$ for every i. Also show that the inverse of an invertible upper triangular matrix is a upper triangular matrix.

E 46

Let A and B be square matrices such that $AB + A + I = 0$. Prove that A is non-singular and $A^{-1} = -I - B$.

E 47

If A is a non-singular matrix, prove that

$$(A+B)A^{-1}(A-B) = (A-B)A^{-1}(A+B)$$

for all B.

E 48

Let A be a non-singular matrix and B be any matrix. Show that $A + B$ and $I + A^{-1}B$ are either both non-singulalr or both singular.

E 49

Let A be a non-singular matrix and $B = XY$, where X is a n by 1 matrix and Y is a 1 by n matrix. If $A + B$ is non-singular, prove that

$$(A+B)^{-1} = A^{-1} - \frac{1}{a+1}A^{-1}BA^{-1},$$

where $a = YA^{-1}X$.

16.9 SUMMARY

In this unit, we have covered the following points:

1. A matrix of order $m \times n$ is an array of mn numbers arranged in m ordered rows and n ordered columns. When $m = n$, it is called a square matrix, otherwise a rectangular matrix.
   a) A square matrix $A = [a_{ij}]$ is called
   i) a **null** matrix if $a_{ij} = 0$ for all i,j,.
   ii) a **unit** matrix if $a_{ij} = 0$ if $i \neq j$ and $a_{ij} = 1$ for all i, j .
   iii) A diagonal matrix if $a_{ij} = 0$ for $i \neq j$ iv) triangular m atrix if $a_{ij} = 0$ for $i > j$ or $i < j$ for all i, j.
   b) Two matrices $A = [a_{ij}], B = [b_{ij}]$, $i = 1, 2, ..., m; j = 1, 2, ..., n$, are said to be equal if $a_{ij} = b_{ij}$ for all i, j

   c) To find the **sum of two matrices,** add the corresponding entries of the two matrices provided they are of the same order.
   d) **Transpose** of a matrix A is obtained by interchanging its rows and columns and is denoted by A^T.
   e) If $A = A^T$, then A is called **symmetric** and if $A = A^{-T}$, then it is called **Hermitian.**
   f) If $A = -A^T$, then A is called **skew - symmetric** and if $A = -A^{-T}$, then it is called **skew - Hermitian**
   g) The matrix obtained on deleting some rows or columns of matrix A is called **sub - matrix** of A .

2. Matrices $A = [a_{ij}]$ and $B = [b_{jk}]$ are compatible for multiplication if the number of columns of pre - factors equals number of rows of post - factor and $C = [c_{ik}]$ is their multiplication, where

$$AB = C = [c_{ik}] \text{ with } c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}, \quad \begin{matrix} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n \\ k = 1, 2, \ldots, p \end{matrix}$$

3. A mapping $X \to AX$ from R^n to R^m, associated with any matrix A and column matrix X, is called a linear transformation.

4. a) Rank of a matrix A is the number of linearly independent rows (or columns) of A.
   b) Rank of a matrix A is unchanged by i) interchange of two rows (or columns) of A and ii) adding any multiple of some row (or column) to another row (or column) of A.

5. a) A transformation T is invertible if and only if, it is one - one onto.
   b) A square matrix A is invertible if there exists a matrix B such that, $AB = BA = I$ and B is called inverse of A and is denoted by A^{-1}.
   c) To compute inverse of A , write post - factor as unit matrix of order of A and perform operations $R_i \leftarrow \alpha R_i + \beta R_j$ till A is reduced to unit matrix and the resulting matrix of I is then A^{-1}.

6. a) Determinant of matrix A of order nxn is given by

$$D(A) = a_{11} M_{11} - a_{12} M_{12} + \ldots + (-1)^{n+1} a_{1n} M_{1n}$$

$$= a_{11} M_{11} - a_{21} M_{21} + \ldots + (-1)^{n+1} a_{n-1} M_{n1}$$

   b) $A^{-1} = \dfrac{1}{D(A)}$ (Matrix of cofactors) ^{T}

   c) For system of equations $AX = b$, Cramer's rule gives solution as

$$x_j = \frac{D(A_1, A_2, \ldots, A_{j-1}, b, A_{j+1}, \ldots, A_n)}{D(A)} \text{ for } j = 1, 2 \ldots, n.$$

16.10 SOLUTIONS/ANSWERS

E 4. (a) αA is Hermitian

(b) αA is not Hermitian

E 5. $B = \begin{vmatrix} 1 & 0 & 5 \\ 0 & 0 & 4 \\ 5 & 4 & 5 \end{vmatrix}$, $C = \begin{vmatrix} 0 & 2 & -1 \\ -2 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}$

E 7. (a) No, A is not necessarily symmetric.

(b) Yes, P, Q, R, and S are symmetric whenever A is symmetric.

E 8. (a) $A \begin{vmatrix} 1 & , & 3 \\ 2 & , & 3 \end{vmatrix} = \begin{vmatrix} 2-i & 3+i \\ 5+2i & 6 \end{vmatrix}$

(b) $A[1,3] = \begin{vmatrix} i & 3+i \\ 4 & 6 \end{vmatrix}$

E 9. $[2\ 3]A = [1\ 2\ 3]$.

E 10. $2A^4 - A^2 + A - 5I = \begin{bmatrix} 15 & 26 & -28 \\ 35 & 3 & 56 \\ 11 & -34 & 9 \end{bmatrix}$.

E 15 Yes.

E 16 AB is symmetric.

E 18 Yes.

E 24. Rank of A is 2.

E 27. Rank of A is 2.

E 29. (a) $\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$

(b) $\begin{vmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{vmatrix}$.

E 31. A is non-singular.

E 40. No change.

E 43. (a) $a \neq 6$, the unique solution is $(x, y, z) = (5, -4, 1)$.

(b) $a = 6$, $(x, -4, 6-x)$ is a solution for any x.