UNIT 2 LIMIT, CONTINUITY AND DIFFERENTIABILITY

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2.1 INTRODUCTION

In this unit we lay a foundation for the calculus. This foundation will consist of definitions and theorems centring around limit, continuity and differentiation. We will now begin the study of calculus, starting with the concept of "**limit**". Next, we shall study **continuity**, and, thereafter, **differentiation.**

Without limits calculus simply does not exist. Early in your study of mathematics you were exposed to problems whose solutions involved the use of limits, although you were unaware of it at that time. To cite one example, according to a formula of geometry, the area of a circle of radius r is πr^2 where π is a number approximately equal to 3.1416. How is such a formula obtained ? The usual way is to inscribe regular polygons in the circle, find the areas of these polygons and then determine the "limiting value" of these areas as the number of sides of polygons increases without bound. Thus even such a seemingly simple formula as that for the area of a circle depends on the concept of limit for its derivation.

Every single notion of calculus is a limit in one sense or another.

Objectives

After you have completed this unit, you should be able to

* evaluate the limit of a function at a point,
* identify points of continuity and discontinuity of a function, and
* obtain the derivative (or differential coefficient) of a function at a point.

2.2 LIMITS

In Unit 1 we introduced you to the concept of a function and discussed several useful types of functions. In this section we consider the value "to which $f(x)$ approaches as x gets closer and closer to some number a ". The phrase in inverted commas is to be understood intuitively and through practice. We do not give a formal definition here. We call such a value the limit of $f(x)$ and denote it by $\lim_{x \to a} f(x)$. Sometimes, this value may not exist, as you will see in an example later.

Example 1 :

Consider the function $f: \boldsymbol{R} \to \boldsymbol{R}$ given by $f(x) = x + 4$. We want to find $\lim_{x \to 2} f(x)$

Look at Tables 1 and 2. These give values of $f(x)$ as x gets closer and closer to 2 through values less than 2 and through values greater than 2, respectively.

Table 1

x	1	1.5	1.9	1.99	1.999
$f(x)$	5	5.5	5.9	5.99	5.999

Table 2

x	3	2.5	2.1	2.01	2.001
$f(x)$	7	6.5	6.1	6.01	6.001

From the above tables it is clear that as x approaches 2, $f(x)$ approaches 6. In fact, the nearer x is chosen to 2, the closer $f(x)$ will be to 6. Thus, 6 is the limit of $x + 4$ as x approaches 2, that is, $\lim_{x \to 2}(x + 4) = 6$.

In the above example the value of $\lim_{x \to 2}(x + 4)$ coincides with the value $x + 4$ when $x = 2$, that is, $\lim_{x \to 2} f(x) = f(2)$.

Numbers x near 2 fall into two natural categories; those which are <2, that is, those that lie to the left of 2, and those which are >2, that is, those which lie to the right of 2.

We write

$$\lim_{x \to 2^-} f(x) = 6$$

to indicate that as x approaches 2 from the left, $f(x)$ approaches 6.

We shall describe this limit as the *left-hand limit of* $f(x)$ *as* x *approaches* (or tends to) 2.

Similarly

$$\lim_{x \to 2^+} f(x) = 6$$

indicates that as x tends to 2 from the right, $f(x)$ approaches 6.

We shall call this limit as the *right-hand limit of* $f(x)$ *as* x *approaches 2.*

The left and right-hand limits are called *one-sided* limits.

It is clear now that

$$\lim_{x \to 2} f(x) = 6$$

if, and only if, both

$$\lim_{x \to 2^-} f(x) = 6 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = 6.$$

In the above example the value of $\lim_{x \to 2}(x + 4)$ coincides with the value of $x + 4$ when $x = 2$ that is,

$$\lim_{x \to 2} f(x) = f(2).$$

Likewise

$$\lim_{x \to 1}(x^2 - 3x + 1) = -1.$$

as also

$$\lim_{x \to 1^-}(x^2 - 3x + 1) = -1 = \lim_{x \to 1^+}(x^2 - 3x + 1).$$

Now consider another function $f: \boldsymbol{R} \setminus \{2\} \to \boldsymbol{R}$ given by $f(x) = \dfrac{x^2 - 4}{x - 2}$. This function is not defined at the point $x = 2$, since division by zero is undefined. But

$f(x)$ is defined for values of x which approach 2. So it makes sense to evaluate $\lim_{x \to 2} \frac{x^2-4}{x-2}$. Again, we consider the following Tables 3 and 4 which give the values of $f(x)$ as x approaches 2 through values less than 2 and through values greater than 2, respectively.

Table 3

x	1	1.5	1.9	1.99	1.999
$f(x)$	3	3.5	3.9	3.99	3.999

Table 4

x	3	2.5	2.1	2.01	2.001
$f(x)$	5	4.5	4.1	4.01	4.001

As you can see

$$\lim_{x \to 2^-} f(x) = 4 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = 4.$$

Hence, we shall say

$$\lim_{x \to 2} f(x) = 4.$$

Now we are in a position to define the limit of a function.

Let f be a function and let a be a real number. We do not require that f be defined at a, but we do require that f be defined on a set of the form $(a-p, a) \cup (a, a+p)$.

(This guarantees that we can form $f(x)$ for all $x \neq a$ that are "sufficiently close" to a.)

Definition 1 :

$f(x)$ is said to tend to the limit l as x approach a, written as

$$\lim_{x \to a} f(x) = l,$$

if, and only if,

$$\lim_{x \to a^-} f(x) = l \quad \text{and} \quad \lim_{x \to a^+} f(x) = l.$$

There is another definition of the limit of a function, equivalent to the above definition.

Definition 2 :

Let f be a function defined on some set $(a-p, a) \cup (a, a+p)$. Then

$$\lim_{x \to a} f(x) = l$$

if, and only if, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ when $0 < |x - a| < \delta$. Note that we do not deny that f may be possibly defined at the point a. All we are saying is that the definition does not require it.

We illustrate the above definitions with the help of examples.

Example 2 :

Show that

$$\lim_{x \to a} (mx + n) = ma + n; \; m \neq 0.$$

Solution :

Let $\varepsilon > 0$. We seek a number $\delta > 0$ such that, if $0 < |x - a| < \delta$, then $|(mx + n) - (ma + n)| < \varepsilon$.

What we have to do first is establish a connection between

$$|(mx + n) - (ma + n)| \text{ and } |x - a|.$$

The connection is simple:

$$|(mx+n)-(ma+n)| = |m|\,|x-a|.$$

To make $|(mx+n)-(ma+n)| < \varepsilon$, we need only make $|x-a| < \frac{\varepsilon}{|m|}$.

This suggests that we choose $\delta = \frac{\varepsilon}{|m|}$.

Thus, if $0 < |x-a| < \delta$, then $|(mx+n)-(ma+n)| < \varepsilon$.

Hence

$$\lim_{x \to a}(mx+n) = ma+n.$$

Example 3 :

Given $f(x) = \begin{cases} 3, & x \text{ an integer} \\ 1, & \text{otherwise} \end{cases}$, evaluate $\lim_{x \to 2} f(x)$, if it exists.

Solution :

$$\lim_{x \to 2^-} f(x) = 1 = \lim_{x \to 2^+} f(x).$$

Hence

$$\lim_{x \to 2} f(x) = 1.$$

Example 4 :

Consider the function $f: R \setminus \{0\} \to R$ given by $f(x) = \frac{1}{x}$. As you see, $f(0)$ is not defined. In contrast to Ex 1,2,3 where f is defined at the point a where we are evaluating the limit. We try to evaluate $\lim_{x \to 0} f(x)$. Look at the following tables.

x	1/2	1/10	1/100	1/1000
$f(x)$	2	10	100	1000

x	–1/2	–1/10	–1/100	–1/1000
$f(x)$	–2	–10	–100	–1000

We see that $f(x)$ does not approach any fixed number as x approaches 0. In this case we say that $\lim_{x \to 0} f(x)$ does not exist.

Example 5 :

Evaluate the following limits:

(i) $\lim_{x \to 3} \frac{x^2-9}{x-3}$

(ii) $\lim_{x \to -3} \frac{x^3-27}{x+3}$

Solution :

(i) $$\lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x+3)(x-3)}{(x-3)}$$

$$= \lim_{x \to 3}(x+3) = 6$$

(ii) $$\lim_{x \to -3} \frac{x^3-27}{x+3}$$

$$= \lim_{x \to -3} \frac{(x+3)(x^2-3x+9)}{(x+3)}$$

$$= \lim_{x \to -3}(x^2-3x+9) = (-3)^2-3(-3)+9 = 27$$

We state below a theorem giving six properties of limits.

Theorem :

Properties of limits

Let f and g be two functions of x. Then

1) For any constant c, $\lim_{x \to a} c = c$.

2) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

3) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

4) $\lim_{x \to a} [kf(x)] = k \lim_{x \to a} f(x)$, k any real number

5) $\lim_{x \to a} [f(x)\, g(x)] = [\lim_{x \to a} f(x)][\lim_{x \to a} g(x)]$

6) $\lim_{x \to a} [f(x)/g(x)] = [\lim_{x \to a} f(x)]/[\lim_{x \to a} g(x)]$

provided $\lim_{x \to a} g(x) \neq 0$.

Using the above properties, we try a few examples.

Example 6 :

Evaluate $\lim_{x \to 2} (x^2 - 4)(x^2 + x + 1)$

Solution :

$$\lim_{x \to 2} (x^2 - 4)(x^2 + x + 1) = [\lim_{x \to 2} (x^2 - 4)][\lim_{x \to 2} (x^2 + x + 1)]$$

$$= (4 - 4)(4 + 2 + 1) = 0.$$

Example 7 :

Evaluate $\lim_{x \to 0} \dfrac{x^3 + 2x^2 + x}{x^2 + 2x}$

Solution :

First we reduce $\dfrac{x^3 + 2x^2 + x}{x^2 + 2x}$ by cancelling the common factor.

$$\frac{x^3 + 2x^2 + x}{x^2 + 2x} = \frac{x(x+1)^2}{x(x+2)} = \frac{(x+1)^2}{x+2}$$

Hence, $\lim_{x \to 0} \dfrac{x^3 + 2x^2 + x}{x^2 + 2x} = \lim_{x \to 0} \dfrac{(x+1)^2}{x+2}$

$$= \frac{\lim_{x \to 0} (x+1)^2}{\lim_{x \to 0} (x+2)} = \frac{1}{2}.$$

We now turn to another limit, the importance of which will become clear in this unit.

Theorem 2 :

1) $\lim_{x \to 0} \dfrac{\sin x}{x} = 1.$

2) $\lim_{x \to 0} \cos x = 1.$

We are now in a position to evaluate a variety of trigonometric limits.

Example 8 :

Find $\lim_{x \to 0} \frac{\sin 4x}{x}$.

Solution :

You have seen that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

From this it follows that

$$\lim_{x \to 0} \frac{\sin 4x}{4x} = 1.$$

and

$$\lim_{x \to 0} \frac{\sin 4x}{x} = \lim_{x \to 0} (4) \left(\frac{\sin 4x}{4x} \right) = 4.$$

Example 9 :

Find

$$\lim_{x \to 0} x \cot 4x.$$

Solution :

We write

$$x \cot 4x = x \frac{\cos 4x}{\sin 4x}.$$

Since

$$\lim_{x \to 0} \frac{\sin x}{x} = 1,$$

it follows that

$$\lim_{x \to 0} \frac{x}{\sin x} = \lim_{x \to 0} \frac{1}{\sin x / x} = \frac{1}{1} = 1.$$

Thus

$$\lim_{x \to 0} x \cot 4x = \lim_{x \to 0} \left(\frac{1}{4} \right) \cos 4x \left(\frac{4x}{\sin 4x} \right) = \left(\frac{1}{4} \right) (1)(1) = \frac{1}{4}.$$

Example 10 :

Find

$$\lim_{x \to 0} \frac{x^2}{\sec x - 1}$$

Solution :

We cannot deal with the expression $\frac{x^2}{\sec x - 1}$ as it stands since both the numerator and denominator tend to zero with x. As such we rewrite the expression in a certain form which will make things easy.

$$\lim_{x \to 0} \frac{x^2}{\sec x - 1} = \lim_{x \to 0} \frac{x^2 \cos x}{1 - \cos x} = \lim_{x \to 0} \frac{x^2 \cos x}{2 \sin^2 \frac{x}{2}}$$

$$= \lim_{x\to 0}(2)(\cos x)\left(\frac{x/2}{\sin x/2}\right)^2$$

$$= (2)(1)(1) = 2.$$

E1

Evaluate the limits that exist:

(i) $\lim_{x\to -2}\dfrac{x^2-x-6}{(x+2)^2}$ (ii) $\lim_{x\to -4}\left(\dfrac{3x}{x+4}+\dfrac{8}{x+4}\right)$

(iii) $\lim_{x\to 3}\dfrac{(x^2+x-12)^2}{x-3}$ (iv) $\lim_{x\to 0}\dfrac{\tan x}{x}$

(v) $\lim_{x\to 0}\dfrac{x^2-2x}{\sin 3x}$ (vi) $\lim_{x\to 0}\dfrac{\sin^2 x}{x(1-\cos x)}$

(vii) $\lim_{x\to a}\dfrac{\sin(x-a)}{(x-a)^2}$ (viii) $\lim_{x\to 0}\dfrac{\sqrt{3+x}-\sqrt{3}}{x}$

(ix) $\lim_{x\to 1}\dfrac{x^{1/3}-1}{x^{1/2}-1}$ (x) $\lim_{x\to 0}\dfrac{\sin \alpha x}{\sin \beta x}$

2.3 CONTINUITY

In ordinary language, to say that a certain process is "continuous" is to say that it goes on without interruption and without abrupt changes. In mathematics, continuity of a function can also be interpreted in a similar way.

Like limits, the idea of continuity is basic to calculus. First we introduce the idea of continuity at a point (or number) a, and then about continuity on an interval.

2.3.1 Continuity at a Point

Definition 3 :

Let f be a function defined at least on an open interval $(a-p, a+p)$.
We say f is continuous at a if, and only if,

$$\lim_{x\to a} f(x) = f(a).$$

or, equivalently,

$$\lim_{h\to 0} f(a+h) = f(a).$$

If the domain of f contains an interval $(a-p, a+p)$, then f can fail to be continuous at a for one of the following two reasons:

1. Either $\lim_{x\to a} f(x)$ does not exist.
2. $\lim_{x\to a} f(x)$ exists but is not equal to $f(a)$.

The function graphed in Figure 2.1 is discontinuous at a because it does not have a limit at a.

The function depicted in Figure 2.2 does have a limit at a. It is discontinuous at a only because $\lim_{x \to a} f(x)$ is different from $f(a)$, the value of f at a. The discontinuity at a is

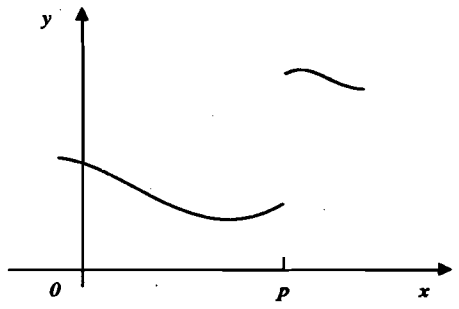

Figure: 2.1

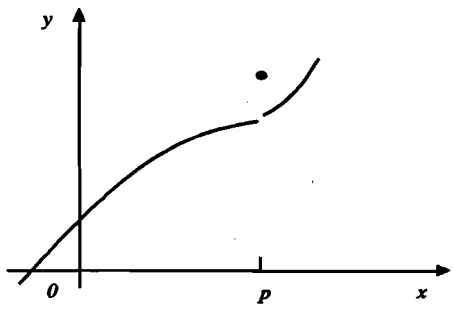

Figure : 2.2

removable; it can be removed by lowering the dot into place (or by redefining f at a).

An $\varepsilon - \delta$ characterization of continuity at a reads as follows:

Definition 4 :

> f is continuous at a if, and only if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. Clearly, δ will depend both on ε and a; if any of these are changed the same δ may not work.
> Continuity of the function $f(x)$ at an end point of an interval $[a, b]$ of its domain is defined below:
>
> (i) $f(x)$ is continuous at the left end point 'a' $\lim_{x \to a^+} f(x) = f(a)$
>
> (ii) $f(x)$ is continuous at the right end point 'b' $\lim_{x \to b^-} f(x) = f(b)$.
>
> If a function is continuous at each point of an interval, it is continuous over the interval.

Remark :

If the interval is closed, the limit in the continuity test over the interval is two-sided at an interior point and the appropriate one-sided at the end points.

Example 11 :

Prove that $f(x) = \sin x$ is continuous at $x = 0$.

Solution :

(i) $f(0) = \sin 0 = 0$

(ii) $\lim_{x \to 0} f(x) = \lim_{x \to 0} \sin x = 0$ and

(iii) $\lim_{x \to 0} f(x) = f(0) = 0$

Therefore, $f(x) = \sin x$ is continuous at $x = 0$. (In fact $\sin x$ and $\cos x$ are continuous at each real x.)
We give below a theorem which gives some results concerning continuity of functions.

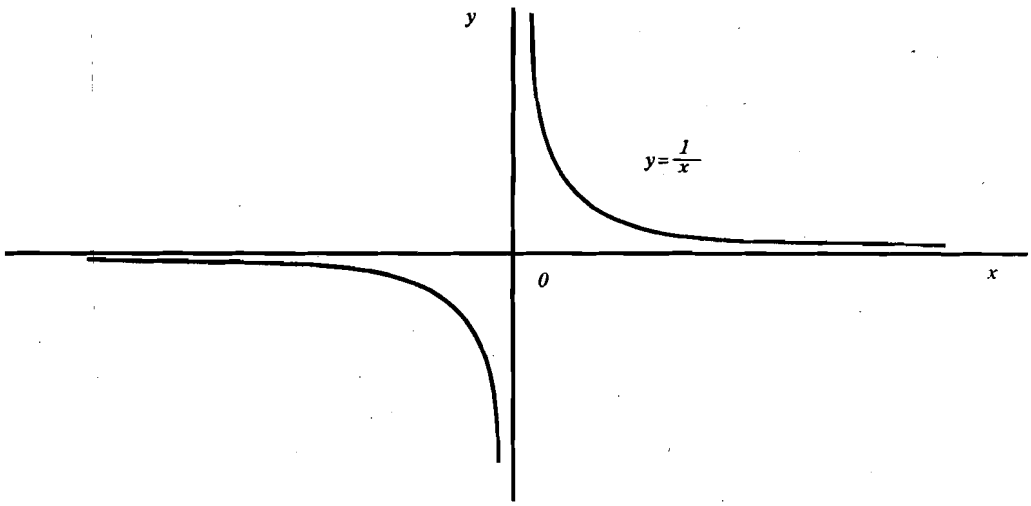

Figure 2.3

Example 12 :

Examine the continuity of the function $f(x) = \frac{1}{x}$.

Solution :

The function is defined for all non-zero real values of x. It is not defined at $x = 0$.

Also $$\lim_{x \to x_0} f(x) = \frac{1}{x_0} = f(x_0), \text{ if } x_0 \neq 0$$

Therefore, the function is continuous for all real $x \neq 0$. It fails to be continuous at $x = 0$ (why?) [see Figure 2.3]. This function is continuous on any interval which does not include $x = 0$ as an element of it.

Theorem 3 :

If the functions $f(x)$ and $g(x)$ are continuous at $x=a$, then

(i) $f(x) \pm g(x)$,

(ii) $f(x)\,g(x)$, and

(iii) $\frac{f(x)}{g(x)}$, $g(a) \neq 0$,

are continuous at $x = a$.

These results are in fact the immediate corollaries of the corresponding limit theorems discussed in Section 1.3. We list below a few more important results on continuous functions without proof. They will be very useful later.

(1) If $f(x)$ is continuous at $x = x_0$ and if $g(y)$ is continuous at $y = y_0 = g(x_0)$ then the composite function $F(x) = g(f(x))$ is continuous at $x = x_0$.

(2) If $f(x)$ is continuous on $[a, b]$, there exists a number B such that $|f(x)| \leq B$ for all $x \in [a, b]$ (if there exists a number B such that $|f(x)| \leq B$ for all $x \in [a, b]$ then we say $f(x)$ is bounded on that interval).

(3) A function continuous on a closed interval $[a, b]$ has a maximum value M (i.e., $M \geq f(x)$ for all $x \in [a, b]$) and a minimum value m (i.e., $m \leq f(x)$ for all $x \in [a, b]$) there. That is, there are numbers $\alpha, \beta \in [a, b]$ such that $f(\alpha) = m, f(\beta) = M$ and $f(\alpha) \leq f(x) \leq f(\beta)$ for all $x \in [a, b]$.

(4) A function continuous on an interval assumes every number between any two values that it can take on that interval. (Intermediate value theorem.)

If a function is not continuous at a point x_0, it is said to be *discontinuous* at x_0.

Example 13 :

Let $f(x) = \frac{\sin x}{x}$ $(x \neq 0)$, Now f is not defined at $x = 0$. If we define $f(0) = 1$

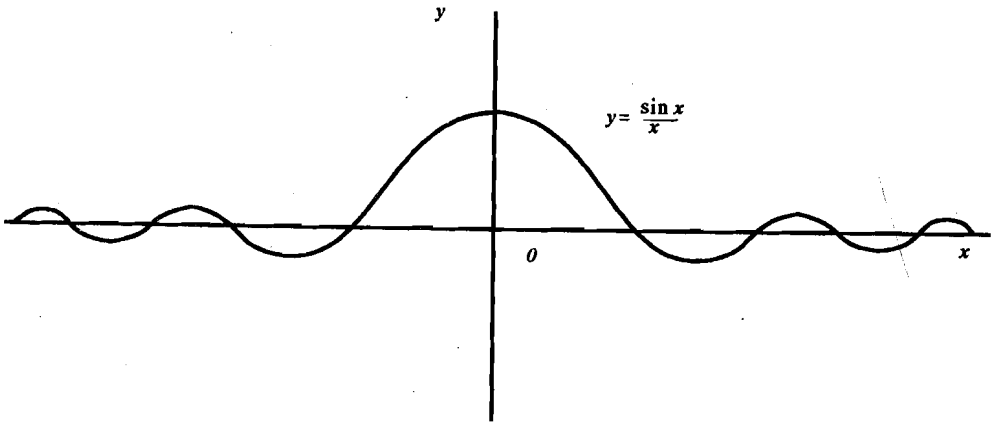

Figure 2.4

which is same as $\lim_{x \to 0} \frac{\sin x}{x}$, then f is continuous at $x = 0$.

Example 14 :

The function f defined by

$$f(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x \leq 0, \end{cases}$$

is discontinuous at $x = 0$.

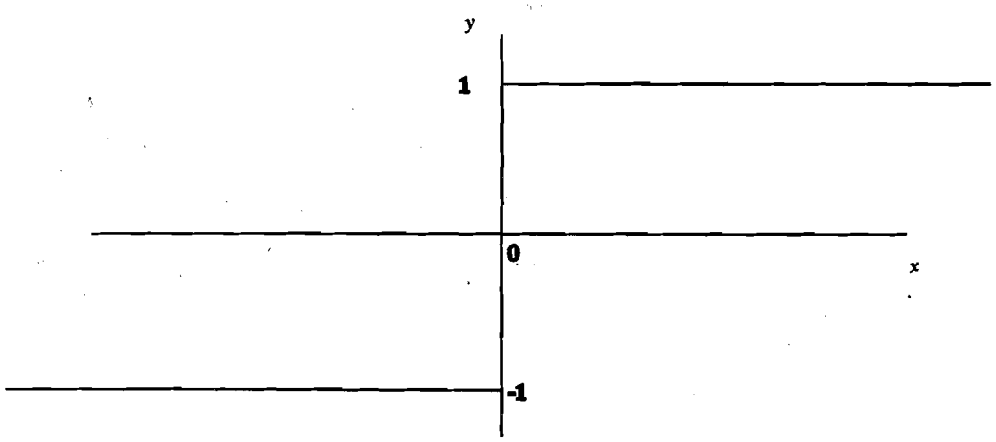

Figure 2.5

Example 15 :

Draw the graph of the function

$$f(x) = \begin{cases} -1 + x^2, & -1 \leq x < 0 \\ x, & 0 \leq x < 1 \\ 2, & x = 1 \\ -x + 2, & 1 < x < 2 \\ 0, & 2 < x \leq 3 \end{cases}$$

and examine its continuity on $[-1, 3]$.

Solution :

The points of discontinuity:

(i) $\lim_{x \to 0^-} f(x) = -1 \neq 0 = f(0) = \lim_{x \to 0^+} f(x)$

Therefore, $x = 0$ is a discontinuity.

(ii) $\lim_{x \to 1^-} f(x) = (\lim_{x \to 1^+} f(x) = 1)$

But $\lim_{x \to 1} f(x) = 1 \neq f(1) = 2$. Therefore, $x = 1$ is a discontinuity.

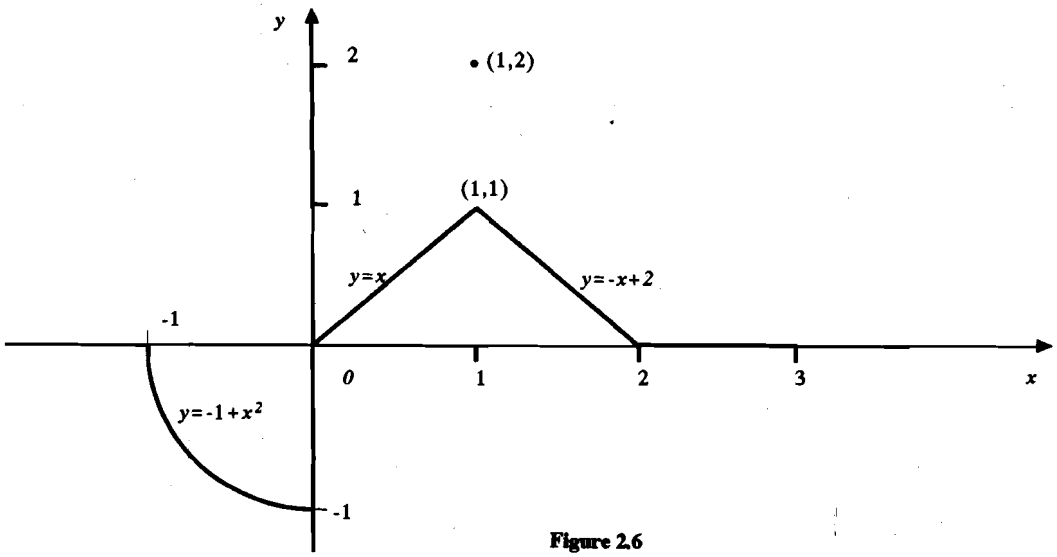

Figure 2.6

(iii) The function is not defined at $x = 2$, but $\lim_{x \to 2} f(x) = 0$ exists.

Therefore, $x = 2$ is also a removable discontinuity.

E2

The bracket function or greatest integer function, $f(x) = [x]$ is defined to be the largest integer $\leq x$. Discuss the discontinuities of (i) $[-x]$ (ii) $[x] + [-x]$ (iii) $[x] + \sqrt{x - [x]}$ (iv) $\dfrac{1}{[x]}$

E3

Is the function

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & \text{when } x \neq 1 \\ 2 & \text{when } x = 1 \end{cases}$$

continuous at $x = 1$? Explain.

E4

Find the value of b for which the function

$$f(x) = \begin{cases} x^3 + 1 & \text{when } x < 2 \\ bx + \dfrac{2}{x} & \text{when } x \geq 2 \end{cases}$$

continuous at $x = 2$.

E5

Does the function $y = x^2$ have a maximum value on $(-1, 1)$?
Does it assume a minimum value on that open interval?

E6

Let $f(x)$ be continuous on $[a, b]$ and let $f(a) < 0$ and $f(b) > 0$. Will there be any $x_0 \in (a, b)$ such that $f(x_0) = 0$? Explain your answer.

2.4 THE DERIVATIVE

You are familiar with notions like velocity, acceleration, slope of a tangent line, etc. You also know that differential calculus is the right mathematical tool to obtain formulae to calculate velocity and acceleration of moving bodies, the slope of the tangent line to a curve, etc. In fact differential calculus deals with the problem of calculating rates of change and also helps to express the physical laws in precise mathematical terms for studying their consequences. The problems associated with a moving body are mainly responsible for development of the concept of derivative in calculus. To illustrate the fact we consider the rectilinear motion of a body. We know that the distance 'S' traversed by the body and measured from a fixed point depends on the time 't'. Let the distance $S = f(t)$ be a function of time 't'. The average velocity during a time interval $(t, t + \Delta t)$ is

$$V_{av} = \frac{\Delta s}{\Delta t}, \text{ where } \Delta s = f(t+\Delta t) - f(t).$$

To obtain the velocity at time t we need to calculate

$$\lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

The above limit, if exists, is called the derivative of $f(t)$ with respect to t and we write

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = f'(t).$$

The derivative $\frac{ds}{dt}$ is the velocity of the moving body at time t.

2.4.1 Derivative of a Function

We now define the derivative of a given function by using the notion of limit.

Definition 5 :

> The function $y = f(x)$ is said to have a *derivative* (or be differentiable) at a point x if, and only if, the limit
>
> $$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
>
> exists and is finite. This limit, written as $\frac{dy}{dx}$ or $f'(x)$ is called the *derivative* of $f(x)$ with respect to x.

If the derivative of a function exists at every point of an interval, then we say that the function is *differentiable in that interval.* However, while considering the derivatives at the end points of an interval, we evaluate suitable one-sided derivatives. For example, if the interval is $[a, b]$, then the derivative at $x = a$ is calculated as

$$\lim_{\Delta x \to 0^+} \frac{f(a+\Delta x)-f(a)}{\Delta x} \text{ and at } b \text{ as } \lim_{\Delta x \to 0^+} \frac{f(b+\Delta x)-f(b)}{\Delta x}$$

If $f'(x)$ is continuous at a point, then we say that $f(x)$ is continuously differentiable at that point

Example 16 :

Let $y = x^n$ (n is a positive integer). Prove that $\frac{dy}{dx} = nx^{n-1}$.

Solution :

$$y = f(x) = x^n.$$

Then

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \frac{(x+\Delta x-x)}{\Delta x}[(x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}x$$

$$+ \ldots + (x+\Delta x)x^{n-2} + x^{n-1}]$$

The above step is derived by using the result

$$a^n - b^n = (a-b)[a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1}].$$

We now have $\frac{\Delta y}{\Delta x} = (x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}x + \ldots + (x+\Delta x)x^{n-2} + x^{n-1}$

Taking the limit $\Delta x \to 0$, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = x^{n-1} + x^{n-1} + \ldots + x^{n-1} \ (n \text{ number of terms})$$

Therefore $\frac{dy}{dx} = nx^{n-1}$.

E7

Show that the derivative of a constant function is zero .

In the following theorem we make an important observation (a necessary condition) about any differentiable function.

Theorem 4 :

If a function $y = f(x)$ is differentiable at some point $x = x_0$, it is continuous at that point.

Proof :

By definition, the derivative of $f(x)$ at $x = x_0$ is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0).$$

We can write the above relation as

$$\lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} - f'(x_0) \right) = 0.$$

or $\varepsilon(\Delta x) = \frac{\Delta y}{\Delta x} - f'(x_0)$ where $\varepsilon = \varepsilon(\Delta x)$, as a function of Δx, tends to zero as $\Delta x \to 0$.

Therefore, $\frac{\Delta y}{\Delta x} = f'(x_0) + \varepsilon(\Delta x)$.

or $\Delta y = f'(x_0)\Delta x + \varepsilon \Delta x$.

Since $f'(x_0)$ is finite, it follows that $\Delta y \to 0$ as $\Delta x \to 0$, which means $\lim_{\Delta x \to 0} [f(x_0 + \Delta x) - f(x_0)] = 0$, implying that $f(x)$ is continuous at x_0.

The converse of the above result is not true. Indeed, there exist functions which are continuous but not differentiable.

Example 17 :

Examine the differentiability of the function

$$y = \begin{cases} x & \text{whenever } x > 0 \\ -x & \text{whenever } x \leq 0 \end{cases}$$

Solution :

The function is continuous for $-\infty < x < \infty$. The graph of the function is given below.

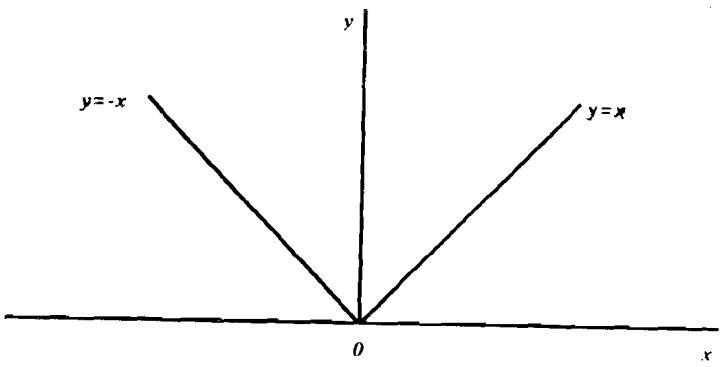

Figure 2.7

Further,

$$\lim_{x \to 0^+} \frac{\Delta y}{\Delta x} = \lim_{x \to 0^+} \frac{\Delta x}{\Delta x} = 1.$$

and $\lim_{\Delta x \to 0^-} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1.$

One-sided limits are not equal.

Therefore, limit of $\frac{dy}{dx}$ at $x = 0$, as $\Delta x \to 0$ i.e., $f'(0)$, does not exist.

Remark :

The above example will help you to conclude that a function can never have a derivative at a point of discontinuity. In the light of this remark, look at the following example.

Example 18 :

Examine the differentiability of the function

$$y = \begin{cases} x & , \ -\infty < x < 0 \\ 1 & , \ 0 \le x < 2 \\ 3-x & , \ 2 \le x . \end{cases}$$

Solution :

From Figure 2.8 we observe as follows:

i) The function is not continuous at $x = 0$. Hence it is not differentiable there.

ii) Though the function is continuous at $x = 2$, the one-sided limits

$$\lim_{\Delta x \to 2^+} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 2^+} \frac{1 - \Delta x - 1}{\Delta x} = -1$$

and $\lim_{x \to 2^-} \frac{\Delta y}{\Delta x} = \lim_{\Delta \to 0^-} \frac{1-1}{\Delta x} = 0$ are not equal.

Therefore, the derivative does not exist at $x = 2$.

At all other points ($x \neq 0$ and $x \neq 2$) the derivatives exist.

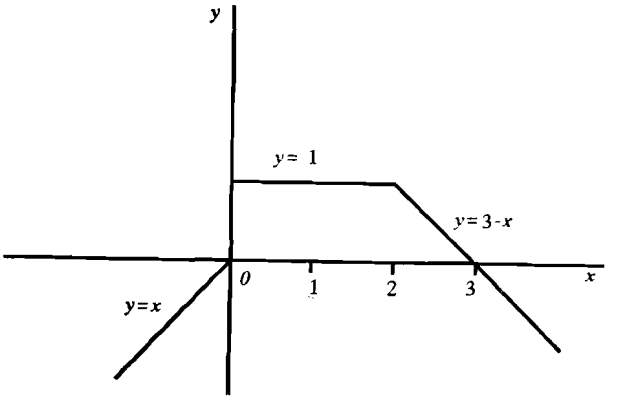

Figure 2.8

E8

Show that $y = |x|$ is differentiable at all points except at $x = 0$.

E9

Find $\frac{dy}{dx}$ wherever it exists, for each of the following functions :

a) $y = x^2$ b) $y = |x-1|$ c) $y = \sqrt{3x+1}$

2.4.2 Algebra of Derivatives

We now state some important rules regarding the derivatives of the sum, difference, product and quotient of functions in terms of their derivatives.

Theorem 5 :

If $f(x)$ and $g(x)$ are differentiable functions at a point x, then

(i) $\frac{d}{dx}[f(x) \pm g(x)] = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$

(ii) $\frac{d}{dx}(f(x).g(x)) = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx}$

(iii) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{df(x)}{dx} - f(x)\frac{dg(x)}{dx}}{[g(x)]^2}$

provided $g(x) \neq 0$.

We shall give the proof of the result (ii) below. Other results we leave as an exercise for you.

Proof of (ii) :

Let $y = f(x).g(x)$.

Then $\Delta y = f(x+\Delta x)g(x+\Delta x) - f(x)g(x)$

$= f(x+\Delta x)[g(x+\Delta x) - g(x)] + g(x)[f(x+\Delta x) - f(x)]$

$\Rightarrow \frac{\Delta y}{\Delta x} = f(x+\Delta x)\frac{[g(x+\Delta x)-g(x)]}{\Delta x} + g(x)\frac{[f(x+\Delta x)-f(x)]}{\Delta x}$

On taking the limit

$$\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \lim_{\Delta x \to 0} f(x+\Delta x) \lim_{\Delta x - 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}$$

$$+ g(x).\lim_{\Delta x \to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

That is,

$$\frac{dy}{dx} = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx},$$

Use of the product rule (ii) and the result which states that 'derivative of a constant is zero' gives

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x).$$

where c is a constant.

Example 19 :

Find the derivative of $y = 5x^2 \sin x$

Solution :

$$\frac{dy}{dx} = 5[x^2\frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)]$$

We derive $\frac{d}{dx}(\sin x)$ as under:

$$\frac{d}{dx}(\sin x) = \lim_{\Delta x \to 0}\frac{\sin(x+\Delta x) - \sin x}{\Delta x}$$

$$= \lim_{\frac{\Delta x}{2} \to 0} \cos\left(x + \frac{\Delta x}{2}\right) \frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$$

$$= \lim_{\frac{\Delta x}{2} \to 0} \cos\left(x + \frac{\Delta x}{2}\right) \cdot \lim_{\frac{\Delta x}{2} \to 0} \frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$$

$$= \cos x \,.\, 1 = \cos x.$$

and $\frac{d}{dx}(x^2) = 2x.$

Hence $\frac{dy}{dx} = 5\,[\,x^2 \cos x + 2x \sin x\,].$

Example 20 :

Find $\frac{dy}{dx}$, if

$$y = \frac{\cos x}{x^2}.$$

Solution :

By formula (iii)

$$\frac{dy}{dx} = \frac{x^2 \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(x^2)}{x^4}$$

To obtain $\frac{d}{dx}(\cos x)$ we have

$$\frac{d}{dx}(\cos x) = \lim_{\Delta x \to 0} \frac{\cos(x + \Delta x) - \cos x}{\Delta x}$$

$$= \lim_{\frac{\Delta x}{2} \to 0} \left[-\sin\left(x + \frac{\Delta x}{2}\right) \lim_{\frac{\Delta x}{2} \to 0} \frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}} \right]$$

$$= -\lim_{\frac{\Delta x}{2} \to 0} \left[\sin\left(x + \frac{\Delta x}{2}\right) \right] \lim_{\frac{\Delta x}{2} \to 0} \frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$$

$$= -\sin x \,.\, 1 = -\sin x$$

and $\frac{d}{dx}(x^2) = 2x$

Therefore, $\frac{dy}{dx} = -\frac{x \sin x + 2 \cos x}{x^3}$

E10

Differentiate :

(a) $\frac{3x+1}{2x-5}$

(b) $\frac{1}{x^3 + 3x - 8}$

(c) $(x^2+x+6)\sin x$

(d) $(x+2)(x^2+1)+\cos x$.

2.4.3 Derivative of a Composite Function (Chain-Rule)

Theorem 6 :

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ as a function of x is differentiable and

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

Proof :

Let $u = g(x)$ be differentiable at x_0 and $y = f(u)$ be differentiable at u_0 where $u_0 = g(x_0)$.

Let $h(x) = f(g(x))$. The function $g(x)$, being differentiable, is continuous at x_0; so also the function $f(u)$ at u_0 and, therefore, at x_0

$$\Delta u = g(x_0+\Delta x) - g(x_0),$$

where $\Delta u \to 0$ as $\Delta x \to 0$

and at u_0

$$\begin{aligned}\Delta y &= f(u_0+\Delta u) - f(u_0)\\ &= f(g(x_0+\Delta x)) - f(g(x_0))\\ &= h(x_0+\Delta x) - h(x_0).\end{aligned}$$

Assuming $\Delta u \neq 0$ (i.e., $g(x)$ is not a constant function in the neighbourhood of x_0) we write

$$\frac{\Delta h}{\Delta x} = \frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}$$

In the limit we get

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}$$

or

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

You can also show that in the case of $\Delta u = 0$ the above formula holds true. In fact it becomes an identity $0 = 0$. Note that it cannot be divided by Δu in the case.

Example 21 :

If $y = \sin(x^2)$, find $\frac{dy}{dx}$.

Solution :

Let $y = \sin(u)$ and $u = x^2$. By the chain-rule

$$\frac{dy}{dx} = \frac{d}{du}(\sin u)\cdot\frac{du}{dx}$$
$$= (\cos u)(2x)$$
$$= 2x\cos x^2.$$

Example 22 :

Prove that the function defined by $y = x^2 \sin\frac{1}{x}$ if $x \neq 0$, and $y = 0$, if $x = 0$, has derivative for every value of x. Is the derivative of y continuous at $x = 0$?

Solution :

If $x \neq 0$ we get $\frac{dy}{dx} = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ by using the product rule, quotient rule and composite function rule.

If $x = 0$, we write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(0+\Delta x) - y(0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^2}{\Delta x} \sin\left(\frac{1}{\Delta x}\right)$$

$$= \lim_{\Delta x \to 0} \Delta x \sin\left(\frac{1}{\Delta x}\right) = 0$$

(since $\sin\left(\frac{1}{\Delta x}\right)$ is bounded).

Therefore, every $x \in (-\infty, \infty)$ $\frac{dy}{dx}$ exists.

But $\frac{dy}{dx}$ is not continuous at $x = 0$, since both the one-sided limits,

$\lim_{x \to 0\pm}(2x\sin\frac{1}{x} - \cos\frac{1}{x})$ do not exist. The first expression $\lim_{x \to 0} 2x\sin\frac{1}{x} = 0$,

but $\lim_{x \to 0\pm} \cos\frac{1}{x}$ fail to exist.

2.4.4 Second and Higher Order Derivatives

You know now that the derivative of a function $y = f(x)$, (i.e., $f'(x)$) is also a function. Hence we can calculate the derivative of $f'(x)$ at a point of its domain, if it exists, by using the same limiting procedure. That is, if $\lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x}$ exists, it is called the *second order derivative* of $f(x)$ with respect to x. We write it as $\frac{d^2y}{dx^2}$ or $f''(x)$ or $f^{(2)}_{(x)}$. You can also calculate higher order derivatives like 3rd order, 4th order etc., of a function $f(x)$.

Example 23 :

Find the (i) 2nd order and (ii) 5th order derivatives of $y = x^4 - 3x^2 + 5x - 1$.

Solution :

(i) $\frac{dy}{dx} = y' = 4x^3 - 6x + 5$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(4x^3 - 6x + 5) = 12x^2 - 6$$

(ii) $y''' = 24x, y^{(4)} = 24, y^{(5)} = 0.$

2.4.5 Implicit Differentiation

If we are given an equation like $x^6 + 4xy^4 - 3y^3 + 1 = 0$, it is not easy to solve it for y in terms of x. However, it is often possible to calculate $\frac{dy}{dx}$ straight from the equation by the method of implicit differentiation. For example, if we differentiate both the sides of the given equation, we get

$$6x^5 + 4y^4 + 4x\left(4y^3\frac{dy}{dx}\right) - 3\left(3y^2\frac{dy}{dx}\right) + 0 = \frac{d}{dx}(0)$$

$$\Rightarrow 6x^5 + 4y^4 + (16xy^3 - 9y^2)\frac{dy}{dx} = 0$$

Therefore,

$$\frac{dy}{dx} = \frac{-(6x^5 + 4y^4)}{16xy^3 - 9y^2}$$

This holds for all points where $16xy^3 - 9y^2 \neq 0$.

E11

Find the derivatives of the functions

(i) $(x^5 - 5x)^3$ (ii) $(x^2 + 10)^5 + \frac{1}{x^2 + 10}$ (iii) $\frac{(1 - \sin x)^2}{1 + \cos^2 x}$

E12

Find the 2nd order derivatives of the functions

(i) $\sin^2 x$ (ii) $\frac{1}{x^2 + 1}$ (iii) $\left(x^2 + 3x + 1\right)^3$.

E13

A particle moves along a straight line. At any time t the distance s travelled by the particle is given by the formula: $s = 32t^2 + 9$. Find the velocity and acceleration of the particle at time, $t = 2$ units.

E14

Use the method of implict differentiation to calculate $\frac{dy}{dx}$, if

$$x^5 + 3xy^2 - y^4 = 2.$$

2.4.6 Derivatives of Some Elementary Functions

(1) **Power function** $y = x^{\alpha}$ (α *is real*)

We have already proved that the power function $y = x^n$, where n is a positive integer, has the derivative

$$\frac{dy}{dx} = n x^{n-1}.$$

We shall show now that the above rule also holds when $y = x^{p/q}$ (q, p are integers with $q > 0$).

Let us assume $p, q > 0$.

Then $y^q = x^p$. By the method of implicit differentiation we get

or $$q y^{q-1} \frac{dy}{dx} = p x^{p-1}$$

$$\frac{dy}{dx} = \frac{p}{q} \frac{x^{p-1}}{y^{q-1}} = \frac{p}{q} \frac{x^{p-1}}{x^{p-p/q}} = \frac{p}{q} x^{\frac{p}{q}-1} = n x^{n-1}, \text{ if } \frac{p}{q} = n.$$

In fact the formula $\frac{d}{dx}(x^{\alpha}) = \alpha x^{\alpha-1}$ holds true for any real α. We shall accept this result without proof.

(2) **Trigonometric functions**

We have already seen that (in examples 18 and 19)

$$\frac{d}{dx}(\sin x) = \cos x, \frac{d}{dx}(\cos x) = -\sin x.$$

By using the quotient rule you can show that

$$\frac{d}{dx}(\tan x) = \sec^2 x, \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x, \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x.$$

(3) **Logarithmic function (with respect to the base a).**

We shall prove that $\frac{d}{dx}(\log_a x) = \frac{1}{x} \log_a e$. Here $a > 0$ but $a \neq 1, x > 0$ and the number $e = \lim_{h \to 0} (1+h)^{1/h}$.

Let $y = \log_a x$. Then

$$\frac{\Delta y}{\Delta x} = \frac{y(x+\Delta x) - y(x)}{\Delta x} = \frac{1}{\Delta x}[\log_a (x+\Delta x) - \log_a (x)]$$

$$= \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right)$$

$$= \frac{1}{x}\left(\frac{x}{\Delta x}\right) \log_a \left(1 + \frac{\Delta x}{x}\right)$$

$$= \frac{1}{x} \log_a \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{x} \log_a \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}$$

$$= \frac{1}{x} \lim_{\Delta x \to 0} \log_a \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}$$

$$= \frac{1}{x}\log_a \left(\lim_{\frac{\Delta x}{x} \to 0} (1 + \frac{\Delta x}{x})^{x/\Delta x} \right)$$

(since $\log_a x$ is continuous).

$$= \frac{1}{x}\log_a \lim_{h \to 0} (1+h)^{1/h}$$

or

$\frac{dy}{dx} = \frac{1}{x}\log_a e$. Therefore $\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e$.

If $a = e$, we get $\frac{d}{dx}\log_e x = \frac{1}{x}\log_e e = \frac{1}{x}$.

(4) **Exponential function (with respect to the base a)**

We shall now prove that

$$\frac{d}{dx}(a^x) = a^x \log_e a \,.\, (a > 0 \text{ but } a \neq 1).$$

Let $y = a^x$. Then

$$\log_e y = x \log_e a.$$

By the method of implicit differentiation we get

$$\frac{1}{y}\frac{dy}{dx} = \log_e a.$$

Therefore,

$$\frac{dy}{dx} = y \log_e a = a^x \log_e a.$$

(5) **Inverse function**

Let the function $x = g(y)$ be the inverse of $y = f(x)$ such that $g'(y) \neq 0$ at the point under consideration. Then

$x = g(f(x))$ when differentiated with respect to x gives

$$1 = \frac{dg}{dy} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 1/g'(y) = 1\frac{dg}{dy} = \frac{1}{f'(x)}.$$

Thus the derivatives of one of two reciprocal functions is equal to unity divided by the derivative of the other. For example, if $y = \sin^{-1} x$. Then $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-x^2)}}$. In this formula $\sqrt{(1-x^2)} > 0$ because $y = \sin^{-1} x$ has its range $\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and for these values of y, $\cos y \geq 0$ but we need $y \neq 0$.

Similarly,

$$y = \cos^{-1} x \Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{(1-x^2)}}$$

$$y = \tan^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$y = \cot^{-1} x \Rightarrow \frac{dy}{dx} = -\frac{1}{1+x^2}$$

Example 24 :

Let $y = \cos^{-1}(\tan x), -\frac{\pi}{4} < x < \frac{\pi}{4}$

Find $\frac{dy}{dx}$.

Solution :

$$\frac{dy}{dx} = -\frac{1}{\sqrt{(1-\tan^2 x)}}\frac{d}{dx}(\tan x) = -\frac{1}{\sqrt{(1-\tan^2 x)}}\sec^2 x$$

$$= -\frac{1}{\cos x\sqrt{\cos 2x}}$$

2.4.7 The Derivative of a Function Represented Parametrically

Sometimes we represent a function say $y = f(x)$, $a \le x \le b$ by its parametric equations,

$$\left.\begin{aligned} x &= \phi(t) \\ y &= \psi(t) \end{aligned}\right], \quad \alpha \le t \le \beta,$$

Let us assume $\phi(t)$, $\psi(t)$ are differentiable and $x = \phi(t)$ has an inverse $t = h(x)$. Then we can consider the equation of the function $y = f(x)$ as a composite function.

$$y = \psi(t), t = h(x).$$

Using the differentiation rule for a composite function (chain- rule) we get

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{d\psi}{dt}\frac{dh}{dx} = \frac{d\psi}{dt}\Big/\frac{d\phi}{dt}$$

Since $t = h(x)$ is inverse of $x = \phi(t)$, we arrive at $\frac{dy}{dx} = \frac{d\psi}{dt}\Big/\frac{d\phi}{dt}$

Example 25 :

The parametric equation of a semi-circular arc of radius 'a' with its centre at (0 , 0) is

$$\left.\begin{aligned} x &= a\cos t \\ y &= a\sin t \end{aligned}\right], \quad 0 \le t \le \pi.$$

Find $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$.

Solution :

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{a\cos t}{a(-\sin t)} = -\cot t$$

$$\text{At } t = \frac{\pi}{4}, \frac{dy}{dx} = -\cot \pi/4 = -1.$$

E15

(1) Find (i) $\frac{d}{dx}(\sinh x)$ (ii) $\frac{d}{dx}(\tanh x)$

(2) Find $\frac{d}{dx}(\cosh^{-1} x)$

(3) Find (i) $\frac{d}{dx}(\sin\sqrt{x})$ (ii) $\frac{d}{dx}\log_a(x^2+1)^{1/2}$

2.4.8 Differential

Let $y = f(x)$ be a differentiable function over an interval. We represent its derivative at a point by the symbol

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Then we have (see the proof of theorem in section 2.4.1.)

$$\frac{\Delta y}{\Delta x} - f'(x) = \varepsilon(\Delta x)$$

where the function $\varepsilon(\Delta x) \to 0$ as $\Delta x \to 0$.

We can write

$$\Delta y = f'(x)\Delta x + \varepsilon(\Delta x).\Delta x.$$

Since $\varepsilon(\Delta x).\Delta x$ is a negligible quantity, the expression $f'(x)\Delta x$ is of special interest to us and is called the differential of $f(x)$. We write $f'(x)\Delta x = dy$ symbolically. Now dy is a function of Δx and x and can assume numerical values, if x and Δx take numerical values. Then the corresponding ratio is $\frac{dy}{\Delta x} = f'(x)$

It is convenient to regard the derivative symbol $\frac{dy}{dx}$ as the fraction $\frac{dy}{\Delta x}$ in which we write $\Delta x = dx$ (differential of the independent variable).

The differentials dx and dy are interpreted geometrically in the figure.

In practice the differential dy is easier to compute than the increment Δy. Therefore in many practical applications $dy = f'(x)(\Delta x)$ is a good approximation to Δy ($= dy + \varepsilon \Delta x$).

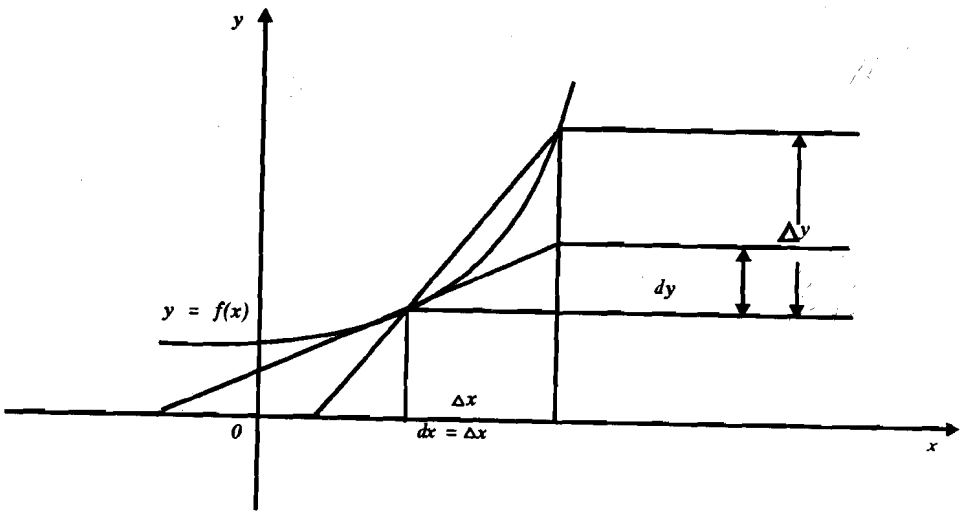

Figure 2.9

Example 26 :

For $y = 3x^2$ find dy and Δy (i) for any $\Delta x = dx$, (ii) for $x = 100$ and $\Delta x = 0.01$.

Solution :

$$y = 3x^2$$

(i) $$\Delta y = 3(x+\Delta x)^2 - 3x^2$$

$$= 6x\Delta x + 3(\Delta x)^2$$

$$dy = f'(x)\Delta x = 6x\Delta x$$

Therefore, $\Delta y = dy + 3(\Delta x)^2$

(Here $\varepsilon = 3\Delta x$.)

(ii) If $x = 100$ and $\Delta x = 0.01$

we get

$$\Delta y = (600)(0.01) + 3(0.0001)$$
$$= 6 + 0.0003 = 6.0003$$

where

$$dy = 6.$$

Note: If Δy is approximated by dy, the error involved is 0.0003 only.

E16

For $y = 2x^2 - 5x$ find dy and Δy in terms of x and Δx. Find dy and Δy if x changes from 5 to 5.1.

E17

For $y = x^3 - 3x^2 + 3x - 1$ use differentials to find approximate change dy in y when x changes from 2 to 2.02. Use this result to compute an approximate value of y corresponding to $x = 2.02$.

2.5 SUMMARY

The important points contained in this unit are as under:

1. $\lim_{x \to a} f(x) = l$, if and only if,

$$\lim_{x \to a^+} f(x) = l = \lim_{x \to a^-} f(x).$$

2. f is continuous at $x = a$ if, and only if,

$$\lim_{x \to a} f(x) = f(a).$$

3. $f(x)$ has the derivative at x if

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

exists.

4. $\frac{d}{dx}[\alpha f(x) + \beta g(x)] = \alpha \frac{d}{dx} f(x) + \beta \frac{d}{dx} g(x).$

5. $\frac{d}{dx}[f(x)\, g(x)] = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x)$

6. $\frac{d}{dx}[f(x)/g(x)] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$.

7. If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}.$$

8. $\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$.

9. If $x = f(t), y = g(t), \frac{dy}{dx} = \frac{g'(t)}{f'(t)}$.

10. $\frac{d^n y}{dx^n} = \frac{d}{dx}\left[\frac{d^{n-1}y}{dx^{n-1}}\right]$.

11. If $f(x)$, is differentiable at $x = c$, then it is continuous at $x = c$.

12. If $y = f(x)$, then $dy = f'(x)\,dx$, where dx and dy are differentials of x and y, respectively.

2.6 SOLUTIONS / ANSWERS

E1

(i) $\frac{x^2-x-6}{(x+2)^2} = \frac{(x-3)(x+2)}{(x+2)^2} = \frac{x-3}{x+2}$ so the limit dies not exist.

(ii) The limit does not exist.

(iii) $\frac{(x^{2+}-x-12)^2}{x-3} = \frac{(x-3)^2(x+4)^2}{x-3} = (x-3)(x+4)^2$ so limit is 0.

(iv) $\frac{\tan x}{x} = \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right)$: limit is 1.

(v) $\frac{x^2-2x}{\sin 3x} = \left(\frac{x}{\sin 3x}\right)(x-2) = \frac{1}{3}\left(\frac{3x}{\sin 3x}\right)(x-2)$ so limit is $-\frac{2}{3}$

(vi) $\frac{\sin^2 x}{x(1-\cos x)} = \frac{\sin x}{x}\frac{\sin x}{1-\cos x} = \frac{\sin x}{x}\frac{2\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)}{2\sin^2\left(\frac{1}{2}x\right)}$ so the limit does not exist.

(vii) $\frac{\sin(x-a)}{(x-a)^2} = \frac{\sin(x-a)}{x-a}\frac{1}{x-a}$, so the limit dose not exist.

(viii) $\frac{\sqrt{3+x}-\sqrt{3}}{x} = \frac{3+x-3}{x}\frac{1}{\sqrt{3+x}+\sqrt{3}} = \frac{1}{\sqrt{3+x}+\sqrt{3}}$ so the limit is $\frac{1}{2\sqrt{3}}$.

(ix) $\frac{x^{1/3}-1}{x^{1/2}-1}$

(x) $\frac{\sin \alpha x}{\sin \beta x} = \frac{\sin \alpha x}{\alpha x}\,\alpha\,\frac{\beta x}{\sin \beta x}\frac{1}{\beta}$ so the limit is $\frac{\alpha}{\beta}$.

E2

(i) $[-x]$ is not continuous at integer points.

E3

Yes because $\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1}(x+1) = 2 = f(1)$.

E4

At $x = 2, \lim_{x \to 2^-} f(x) = 2^3 + 1 = 9$ and $\lim_{x \to 2^+} = 2b + \frac{2}{2} = 2b + 1$ so $b = 4$.

E5

No maximum value because x^2 is as close to 1 on $(-1, 1)$ as we want but never equal to it. The minimum value is at $x = 0$.

E6

Yes because by intermediate value theorem every value between $f(a) < 0$ and $f(b) > 0$ must be assumed.

E7

Let $y = c = f(x)$. Then $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{c-c}{\Delta x} = \frac{0}{\Delta x}$.

So $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0$.

E8

$\lim_{\Delta x \to 0^+} \frac{|\Delta x|}{\Delta x} = +1$, $\lim_{\Delta x \to 0^-} \frac{|\Delta x|}{\Delta x} = -1$ so these one sided limits are not equal and the derivative does not exist at $x = 0$. At other points
$f'(x) = 1$ if $x > 0$ or -1 if $x < 0$.

E9

(a) $2x$

(b) 1 if $x - 1 < 0, -1$ if $x - 1 < 0$, at $x = 1$ no derivatives exists

(c) $f'(x) = \frac{3}{2} \frac{1}{\sqrt{3x+1}}$ if $x \neq -\frac{1}{3}$.

E10

(a) $\frac{3(2x-5) - 2(3x+1)}{(2x-5)^2} = \frac{6x - 15 - 6x - 2}{(2x-5)^2} = \frac{-17}{(2x-5)^2}$

(b) $-\frac{3x^2 + 3}{(x^3 + 3x - 8)^2}$

(c) $(x^2 + x + 6)\cos x - (2x + 1)\sin x$

(d) $x^2 + 1 + 2x(x+1) - \sin x$

E11

(i) $3(x^5 - 5x)^2 (5x^4 - 5)$

(ii) $5(x^2 + 10)^4 (2x) - \frac{2x}{(x^2 + 10)^2}$

(iii) $\frac{2(1 - \sin x)(-\cos x)(1 + \cos^2 x) - (1 - \sin x)^2 (-2 \cos x \sin x)}{(1 + \cos^2 x)^2}$.

E12

(i) $f'(x) = -2 \sin x \cos x, f''(x) = -2(\cos^2 x - \sin^2 x) = 2(\sin^2 x - \cos^2 x)$

(ii) $f'(x) = \frac{-2x}{(x^2+1)^2} f''(x) = -\frac{2(x^2+1)^2 - (-2x) \cdot 2(x^2+1) \cdot 2x}{(x^2+1)^4}$

$$= \frac{(x^2+1)\left[-2x^2-2+8x^2\right]}{(x^2+1)^4}$$

$$= \frac{(6x^2-2)}{(x^2+1)^3}.$$

(iii) $f'(x) = 3(x^2+3x+1)^2(2x+3)$.

$f''(x) = 6(x^2+3x+1)(2x+3)^2 + 6(x^2+3x+1)^2$

E13

$$v = \frac{ds}{dt}\Big|_{t=2} = 64 \times 2, a = \frac{dv}{dt} = 64.$$

E14

$$5x^4 + 3y^2 + 6xy\frac{dy}{dx} - 4y^3\frac{dy}{dx} = 0 \text{ so } \frac{dy}{dx} = -\frac{5x^4+3y^2}{6xy-4y^3}$$

E15

1. (i) $\sinh x = \frac{e^x - e^{-x}}{2}$ so $\frac{d}{dx}(\sinh x) = \frac{e^x + e^{-x}}{2} = \cosh x$

(ii) $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\cosh^2 x - \sinh^2 x}{(\cosh)^2 x} = \operatorname{sech}^2 x$

2. $\cosh^{-1} x = y$ mean $\cosh y = x$, so $\sinh y \frac{dy}{dx} = 1$ and $\frac{dy}{dx} = \frac{1}{\sinh y}$. Now $\cosh^2 y = \sinh^2 y + 1$ so $\sinh^2 y = x^2 - 1$ so $\sinh y = \sqrt{x^2-1}$. Hence $\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$

3. (i) $\frac{d}{dx}(\sin\sqrt{x}) = \cos(\sqrt{x})\frac{1}{2\sqrt{x}}$

(ii) $\frac{d}{dx}\left[\log_a (x^2+1)^{1/2}\right] = \frac{1}{2}\frac{2x}{\sqrt{x^2+1}} \times \frac{1}{\sqrt{x^2+1}}\log_a e = \frac{x}{x^2+1}\log_a e$

E16

$\Delta y = 2(x+\Delta x)^2 - 5(x+\Delta x) - 2x^2 - 5x = 4x\,\Delta x + 2(\Delta x)^2 - 5\,\Delta x$
$= (4x-5)\Delta x + 2(\Delta x)^2$
$dy = f'(x)\Delta x = (4x-5)\Delta x$
So $\Delta y = dy + 2(\Delta x)^2$
For $x = 5, \Delta x = 0.1$,
$\Delta y = (20-5)(0.1) + 2(0.1)^2 = 1.5 + 0.02 = 1.52.$

E17

$\Delta y = (x+\Delta x)^3 - 3(x+\Delta x)^2 + 3(x+\Delta x) - 1 - x^3 + 3x^2 - 3x + 1$
$= (3x^2 - 6x + 3)\Delta x + (3x - 3 + \Delta x)(\Delta x)^2 = f'(x)\Delta x + \in (\Delta x)^2$
where $\in = 3x - 3 + \Delta x$. Here $dy = f'(x)\Delta x = (3x^2 - 6x + 3)\Delta x$.
If $x = 2, \Delta x = 0.02$ then $\Delta y = (12 - 12 + 3)(0.02) = 0.06$.