UNIT 2 PARTIAL DIFFERENTIATION

Structure

2.1 INTRODUCTION

In this unit, you will learn how to extend concepts you have already learnt concerning functions of one variable to functions of more than one variable. You will also learn some analogues results when functions of one variable are extended to functions of more than one variable in general and two or three variables in particular.

Objectives

After studying this unit, you should be able to

- find partial derivatives of functions of two or three variables,
- find the total differential of functions of two or three variables,
- find directional derivative in any given direction,
- find where a function of two variables is having relative maximum or relative minimum, and
- evaluate Jacobians of type

$$\frac{\partial\,(x,\,y)}{\partial\,(u,\,v)} \text{ and } \frac{\partial\,(x,\,y,\,z)}{\partial\,(u,\,v,\,w)}.$$

2.2 FUNCTIONS OF SEVERAL VARIABLES

In this section, you will know how to extend the concept of function of one variable to a function of two, three or more variables.

2.2.1 Functions of Two Variables

R denotes the set of real numbers.

Suppose D is a collection of pairs of real numbers (x, y) (D is a subset of R^2). Then a real valued functions of two variables of f is a rule that assigns to each point (x, y) in D a unique real number denoted by $f(x, y)$.

The set D is called the domain of f.

The set $\{f(x, y) : (x, y) \in D\}$, which is the set of values the function f takes on, is called the range of f.

We generally use the letter z to denote the values that a function of two variables takes.

Then we have

$$z = f(x, y)$$

We call the symbol z the dependent variable of f and f as the function of two independent variables x and y.

Example 2.1

The volume of a right circular cone of base radius r and height h is given by

$$V = \frac{1}{3} \pi r^2 h$$

Here we call V as the dependent variable and r and h as the independent variables.

We now define a function of three variables as follows :

Suppose D be a collection of triple of real numbers (x, y, z). (D is a subset of R^3). Then a real valued function of three variables f is a rule that assigns to each point (x, y, z) in D a unique real number denoted by $f(x, y, z)$.

The set D is called the domain of f.

The set

$$\{f(x, y, z) : (x, y, z) \in D\},$$

which is the set of values the function f takes on, is called the range of f.

We often use the letter w to denote the values that a function f and x, y, z the independent variables of f take.

Then we have

$$w = f(x, y, z)$$

Here we call the symbol w the dependent variable of function f and x, y, z the independent variables of f.

Example 2.2

The volume of a rectangular box of sides x, y, z is given by

$$V = x y z$$

Here V is the dependent variable and x, y, z are the independent variables.

In general, we can define a function of n variables $x_1, x_2, \ldots, x_n$ in a similar manner.

In this unit, we consider functions of two, three or more variables. These functions are obviously single valued.

2.3 LIMITS AND CONTINUITY

In this section, you will learn about the concepts of limit and continuity of functions of several variables. In this unit, we confine most of our discussions of functions of several variables to those of two variables. We can easily extend these ideas in a similar manner to functions of more than two variables.

You have learnt the concepts of open and closed intervals in connection with functions of one variable. Now, we extend these ideas in connection with functions of two variables.

Let (x_0, y_0) be a point in D, which is a subset of R^2.

Consider the equation

$$|(x, y) - (x_0, y_0)| = r \qquad \ldots \text{(2.1)}$$

Since (x, y) and (x_0, y_0) are in R^2, we get

$$|(x, y) - (x_0, y_0)| = |(x - x_0, y - y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2} \qquad \ldots \text{(2.2)}$$

From Eqs. (2.1) and (2.2), we get

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = r \qquad \ldots \text{(2.3)}$$

Squaring both sides of Eq. (2.3), we get

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \qquad \ldots \text{(2.4)}$$

We note that Eq. (2.4) represents a circle with centre (x_0, y_0) and radius r.

We also see that the set of points whose coordinates (x, y) satisfy the inequality

$$|(x, y) - (x_0, y_0)| < r \qquad \ldots \text{(2.5)}$$

is the set of points in R^2 interior to the circle given by Eq. (2.4) as shown in Figure 2.1.

Figure 2.1

From the above discussion, we formulate the following definition :

(i) The open disc D centred at (x_0, y_0) with radius r is the subset of R^2 given by $\{(x, y) : |(x, y) - (x_0, y_0)| < r\}$.

(ii) The closed disc D centred at (x_0, y_0) with radius r is the subset of R^2 is given by $\{(x, y) : |(x, y) - (x_0, y_0)| \le r\}$.

(iii) The boundary of the open or closed disc defined in (i) or (ii) is the circle $\{(x, y) : |(x, y) - (x_0, y_0)| = r\}$.

(iv) A neighbourhood of a point (x_0, y_0) in R^2 is an open disc centred at (x_0, y_0).

We note that an open disc does not contain any point on its boundary just as an open interval does not contain its end points. A closed disc contains all points on its boundary just as the closed interval contains all its boundary points.

Limit

If the values of the function $z = f(x, y)$ can be made as close as we like to a fixed number L by taking the point (x, y) close to the point (x_0, y_0), but not equal to (x_0, y_0), then we say that L is the limit of f as the point (x, y) approaches the point (x_0, y_0).

We write this in symbols as

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$$

We read this as : the limit of f as (x, y) approaches (x_0, y_0) is L.

If the cartesian distance of (x, y) from (x_0, y_0) namely $\sqrt{(x - x_0)^2 + (y - y_0)^2}$ is small in some sense, then we mean that (x, y) is close to (x_0, y_0).

Since $$|x - x_0| = \sqrt{(x - x_0)^2} \le \sqrt{(x - x_0)^2 + (y - y_0)^2} \qquad \ldots (2.6)$$

and $$|y - y_0| = \sqrt{(y - y_0)^2} \le \sqrt{(x - x_0)^2 + (y - y_0)^2} \qquad \ldots (2.7)$$

We have : If the inequality $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$ holds, then $|x - x_0| < \delta$ and $|y - y_0| < \delta$ conversely, if for some $\delta > 0$, both $|x - x_0| < \delta$ and $|y - y_0| < \delta$ hold, then

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} < \sqrt{\delta^2 + \delta^2} = \sqrt{2}\,\delta,$$

which will be small if δ is sufficiently small. Therefore when we calculate limits, we may think either in terms of distance in the plane or in terms of differences in coordinates as shown in Figure 2.2 below.

Figure 2.2

The open square $|x - x_0| < \delta$, $|y - y_0| < \delta$ lies inside the open disc $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \sqrt{\delta^2 + \delta^2} = \sqrt{2}\,\delta$, which also contains the open disc $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.

Now we have two equivalent definitions of limit as given below.

Definition 1

The limit of $f(x, y)$ as (x, y) approaches (x_0, y_0) is the number L if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all points either

(i) $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$ *implies that* $|f(x, y) - L| < \varepsilon$, *or*

(ii) $0 < |x - x_0| < \delta$ and $0 < y - y_0| < \delta$ *implies that* $|f(x, y) - L| < \varepsilon$.

Example 2.3

If $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$ for $(x, y) \neq (0, 0)$ show that $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Solution

Here the point (x, y) can approach the origin infinitely in many ways.

For example, if we approach the origin along x-axis, then $y = 0$. In this case,

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{-x^2}{x^2} = -1$$

On the other hand, if we approach the origin along y-axis, then $x = 0$ and in this case, we get

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{+y^2}{y^2} = 1$$

Thus we get different values of limits depending on how we approach the origin. Hence we have shown that for any open disc centred at the origin, there are points at which takes on the values + 1 and – 1. Therefore f cannot have the limit as $(x, y) \to (0, 0)$.

This example leads us to a general rule for non-existence of the limit, which we can state as follows.

If we get two or more different values for

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y)$$

as we approach (x_0, y_0) along different paths (path is another name for a curve joining (x, y) to a given point (x_0, y_0)), then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y)$$

does not exist.

We state the following theorem without proof.

Theorem

If $\displaystyle \lim_{(x,y)\to(x_0,y_0)} f(x, y) = L_1$ **and** $\displaystyle \lim_{(x,y)\to(x_0,y_0)} g(x, y) = L_2$

then

(i) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)+g(x,y)] = L_1 + L_2$,

(ii) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)-g(x,y)] = L_1 - L_2$,

(iii) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)\,g(x,y)] = L_1 L_2$,

(iv) $\lim_{(x,y)\to(x_0,y_0)} [K f(x,y)] = K L_1$, k **being any constant, and**

(v) $\lim_{(x,y)\to(x_0,y_0)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{L_1}{L_2}$ **if** $L_2 \neq 0$**.**

Definition 2

A function $f(x, y)$ *is said to be continuous at a point* (x_0, y_0) *if*

(i) f *is defined at* (x_0, y_0)

(ii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ *exists, and*

(iii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

Example 2.4

Determine the points at which $f(x, y)$ is continuous if

$$f(x,y) = \begin{cases} \frac{x\,y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Solution

We approach (0, 0) along the line $y = m\,x$, m being an arbitrary constant. Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x\,m\,x}{x^2+m^2\,x} = \lim_{(x,y)\to(0,0)} \frac{m\,x^2}{x^2\,(1+m^2)} = \frac{m}{1+m^2}$$

Here we get different limits for different values of m.

Therefore, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

So f is not continuous at (0, 0).

When $(a, b) \neq (0, 0)$, $a^2 + b^2 \neq 0$ and $\lim_{(x,y)\to(a,b)} f(x,y) = \frac{ab}{a^2+b^2} = f(a,b)$.

Therefore, f is continuous at all points $(a, b) \neq (0, 0)$. So we conclude that f is continuous at all points except the origin.

Here we make a note : if one or more of three conditions in the definition of continuity of $f(x, y)$ fails to hold, then f is discontinuous at the point under consideration.

SAQ 1

(a) Test the continuity at the origin for the function

$$f(x,y) = \begin{cases} \frac{x\,y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 1, & \text{if } (x,y) = (0,0) \end{cases}$$

(b) Determine the points of continuity of the function

$$f(x, y) = \begin{cases} \dfrac{2x - y}{x + y}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

2.4 PARTIAL DERIVATIVES

In this section, you will learn how to extend the concept of derivative of a function of one variable to functions of more than one variable. For simplicity, we consider here the functions of two variables only. Based on similar arguments, we can get the corresponding results involving functions of three or more variables.

Suppose $w = f(x, y)$ is a function of two variables x and y defined for values of (x, y) in some region D of the $x\,y$ plane.

Take two points $P_0\ (x_0, y_0)$ and $P_1\ (x_1, y_1)$ of D with value w_0 at P_0 and w_1 at P_1 respectively as shown in Figure 2.3. Then the increment in w in going from w_0 at P_0 to w_1 at P_1 is

$$\Delta w = w_1 - w_0 = f(x_1, y_1) - f(x_0, y_0) \qquad \ldots (2.8)$$

corresponding to $\Delta x = x_1 - x_0$ and $\Delta y = y_1 - y_0$. We keep P_0 fixed and let P_1 approach P_0 along some specific smooth curve in the $x\,y$ plane. We now suppose that P_1 approaches P_0 along a straight line L making an angle ϕ with the x-axis.

Figure 2.3

Then, if

$$\frac{dw}{ds} = \lim_{P_1 \to P_0} \frac{\Delta w}{\Delta s} = \lim_{P_1 \to P_0} \frac{f(x_1, y_1) - f(x_0, y_0)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \qquad \ldots (2.9)$$

exists, its value is called the directional derivative of $w = f(x, y)$ at (x_0, y_0) in the direction of L. Here we use the objective 'directional' because the answer in Eq. (2.9) depends on the function at P_1, the point P_0 and the direction in which P_1 approaches P_0. We will take up two special cases of the directional derivative. In Section 2.7, we discuss the general use of the directional derivative in any direction.

We first make P_1 approach P_0 along the line $y = y_0$ parallel to the x-axis. Secondly we make P_1 approach P_0 along the line $x = x_0$ parallel to the y-axis. In the first case where P_1 approaches P_0 along the line $y = y_0$, we get

$$f_x\ (x_0,\ y_0) = \lim_{\Delta x \to 0} \frac{f\ (x_0 + \Delta x,\ y_0) - f\ (x_0,\ y_0)}{\Delta x} \quad \ldots$$

(2.10)

We call the resulting limit the partial derivative of $w = f\ (x, y)$ with respect to x at $P_0\ (x_0, y_0)$. From Eq. (2.10), we note that this is just the ordinary derivative with respect to x of the function $f\ (x) = f\ (x, y_0)$ obtained from $f\ (x, y)$ by holding the variable y as constant. This also measures the instantaneous rate of change at P_0 of the function $w = f\ (x, y)$ per unit change in x. We use the notation $\frac{\partial w}{\partial x}$ or f_x to denote the partial derivative of w with respect to x. By deleting the subscript 0 everywhere in Eq. (2.10), we obtain the partial derivative of w at (x, y) and the same is given by

$$\frac{\partial w}{\partial x} = f_x\ (x,\ y) = f_x = \lim_{\Delta x \to 0} \frac{f\ (x + \Delta x,\ y) - f\ (x,\ y)}{\Delta x} \quad \ldots$$

(2.11)

We also note that in order to calculate such a partial derivative from the equation for w, we simply apply the rules of ordinary differentiation, treating y as constant.

Considering the second case and proceeding on similar lines, we get the partial derivatives of $w = f\ (x, y)$ with respect to y, denoted by f_y or $\frac{\partial w}{\partial y}$ as

$$\frac{\partial w}{\partial y} = f_y = f_y\ (x,\ y) = \lim_{\Delta y \to 0} \frac{f\ (x,\ y + \Delta y) - f\ (x,\ y)}{\Delta y} \quad \ldots$$

(2.12)

We make the following observations :

(i) The definition allows us to calculate partial derivatives much in the same way as we calculate the ordinary derivatives by allowing only one of the variables to vary at a time. So we can use all those formulae learnt on differentiation of functions of one variable for partial derivatives.

(ii) The partial derivatives $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ give us the rate of change of w with respect to each of the variable x and y with the other one held constant. We will discuss in Section 2.7 about how w changes when both x and y change simultaneously.

(iii) Even though we calculate $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ by holding one of the variables as constant, we note that each is a function of both the variables x and y.

Now we can easily obtain the partial derivatives of a sum, difference, product and quotient of two functions much in the same way as that of ordinary derivatives in view of observation (i) above. We can now easily extend the definition of partial derivatives to functions of three or more variables.

Higher Order Partial Derivatives

The partial derivatives of a given function of two variables are also functions of two variables and we can some times differentiate them further to get higher order partial derivatives.

Let $w = f(x, y)$ and its partial derivatives be defined over a domain D. Then for $(x, y) \in D$, we define the second order partial derivatives as :

$$\frac{\partial^2 w}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right), \frac{\partial^2 w}{\partial x \partial y} = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right),$$

$$\frac{\partial^2 w}{\partial y \partial x} = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right), \text{ and } \frac{\partial^2 w}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right).$$

Let $w = f(x, y)$ and its first and second order partial derivatives are defined over a domain D, then for $(x, y) \in D$, we define third order partial derivatives as :

$$\frac{\partial^3 w}{\partial x^3} = f_{xxx} = \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}\right), \frac{\partial^2 w}{\partial x \partial y \partial x} = f_{xyx} = \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial y \partial x}\right),$$

$$\frac{\partial^3 w}{\partial x^2 \partial y} = f_{yxx} = \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x \partial y}\right), \frac{\partial^2 w}{\partial x \partial y^2} = f_{yyx} = \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial y^2}\right),$$

$$\frac{\partial^3 w}{\partial y^3} = f_{yyy} = \frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial y^2}\right), \ldots$$

We call the partial derivatives $\frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y \partial x}$ as mixed partial derivatives of second order and $\frac{\partial^3 w}{\partial x \partial y \partial x}, \frac{\partial^3 w}{\partial x \partial y^2}, \frac{\partial^3 w}{\partial y^2 \partial x}, \frac{\partial^3 w}{\partial y \partial x \partial y}$ etc. as the mixed partial derivatives of third order.

On similar lines, we can define partial derivatives of orders higher than three.

Example 2.5

Obtain all the second order partial derivatives for $w = x^3 y^2 - x y^5$.

Solution

$$\frac{\partial w}{\partial x} = 3x^2 y^2 - y^5, \frac{\partial w}{\partial y} = 2x^3 y - 5xy^4,$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = 6xy^2,$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = 2x^3 - 20xy^3,$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = 6x^2 y - 5y^4,$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = 6x^2 y - 5y^4.$$

We note that if $w = f(x, y)$ and its partial derivatives are continuous, the order of differentiation is immaterial, that is,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

Example 2.6

Find all the second order partial derivatives of

$$z = x \cos y - y \cos x.$$

Solution

$$\frac{\partial z}{\partial x} = \cos y + y \sin x, \; \frac{\partial z}{\partial y} = - x \sin y - \cos x,$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = y \cos x,$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = - \sin y + \sin x,$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = - x \cos y,$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = - \sin y + \sin x.$$

We note here that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

SAQ 2

(a) Find u_x, u_y, u_{xx}, u_{yy}, u_{xy} and u_{yx} if $u = \log (ax + by)$, where a and b are arbitrary constants.

(b) Obtain all the nine second order partial derivatives and verify that all the three pairs of mixed partial derivatives are equal if $u = xy^3 - zx^5 + x^2 yz$.

2.5 THE TOTAL DIFFERENTIAL

We first prove the increment theorem for the function of two variables.

Theorem

Let the function $w = f(x, y)$ be continuous and possesses partial derivatives f_x, f_y throughout the region $R : |x - x_0| < h, |y - y_0| < k$ of the xy plane. Let f_x and f_y be continuous at (x_0, y_0). Let

$$? w = f(x_0 + ? x, y_0 + ? y) - f(x_0, y_0) \quad \ldots$$

(2.13)

Then $\quad ?\, w = f_x\,(x_0\ y_0)\ ?\, x + f_y\,(x_0\ y_0)\ ?\, y + e_1 + e_2\ ?\, y$ . . . **(2.14)**

where ε_1 and $\varepsilon_2 \to 0$ when Δx and $\Delta y \to 0$.

Proof

We note here that the region R is a rectangle with centre at (x_0, y_0) with sides $2h$, $2k$. We choose Δx and Δy to be small so that the points (x_0, y_0), $(x_0 + \Delta x,\ y_0 + \Delta y)$, $(x_0 + \Delta x,\ y_0)$, $(x_0,\ y_0 + \Delta y)$ all lie in R. We also assume that f is continuous and has partial derivatives f_x and f_y throughout the rectangle R.

We can use mean value theorem of a function of one variable as all conditions required are now satisfied.

We note that the increment Δw is the change in f from $A\,(x_0, y_0)$ to $B\,(x_0 + \Delta x,\ y_0 + \Delta y)$ in R.

We write $\quad \Delta w = \Delta w_1 + \Delta w_2$ . . . (2.15)

where $\quad \Delta w_1 = f\,(x_0 + \Delta x,\ y_0) - f\,(x_0,\ y_0)$ . . . (2.16)

and $\quad \Delta w_2 = f\,(x_0 + \Delta x,\ y_0 + \Delta y) - f\,(x_0 + \Delta x,\ y_0)$ . . . (2.17)

Here Δw_1 is the change w from A to C and Δw_2 is the change in w from C to B.

Figure 2.4

In Δw_1, we hold $y = y_0$ fixed and have an increment of a function of x that is continuous and differentiable. Applying mean value theorem, we get

$$\Delta w_1 = f\,(x_0 + \Delta x,\ y_0) - f\,(x_0,\ y_0) = f_x\,(x_1, y_0) \quad \text{. . . (2.18)}$$

for some x_1 between x_0 and $x_0 + \Delta x$, i.e. $x_0 < x_1 < x_0 + \Delta x$.

Similarly applying mean value theorem for Δw_2 after holding $x + x_0 + \Delta x$ fixed and having an increment of a function y that is both continuous and differentiable, we get

$$\Delta w_2 = f\,(x_0 + \Delta x,\ y_0 + \Delta y) - f\,(x_0 + \Delta x,\ y_0) = f_y\,(x_0 + \Delta x,\ y_1)\ \Delta y \quad \text{. . . (2.19)}$$

for some y_1 between y_0 and $y_0 + \Delta y$, i.e. $y_0 < y_1 < y_0 + \Delta y$.

So we get

$$\Delta w = f_x\,(x_1, y_0)\,\Delta x + f_y\,(x_0 + \Delta x, y_1)\,\Delta y \quad \ldots (2.20)$$

for some x_1 in $x_0 < x_1 < x_0 + \Delta x$ and for some y_1 in $y_0 < y_1 < y_0 + \Delta y$.

Using the fact that f_x and f_y are continuous at (x_0, y_0), we get

$$f_x\,(x_1, y_0) \to f_x\,(x_0, y_0) \quad \ldots (2.21)$$

and $$f_y\,(x_0 + \Delta x, y_1) \to f_y\,(x_0, y_0) \quad \ldots (2.22)$$

as Δx and Δy approach zero.

So we can now write

$$f_x\,(x_1, y_0) = f_x\,(x_0, y_0) + \varepsilon_1 \quad \ldots (2.23)$$

and $$f_y\,(x_0 + \Delta x, y_1) = f_y\,(x_0, y_0) + \varepsilon_2 \quad \ldots (2.23(a))$$

where ε_1 and ε_2 both approach zero as Δx and Δy approach zero.

Making use of Eq. (2.23) and (2.23(a)), in Eq. (2.20), we get

$$\Delta w = f_x\,(x_0, y_0)\,\Delta x + f_y\,(x_0, y_0)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as Δx and $\Delta y \to 0$.

Now we can extend Eq. (2.14) easily to functions of three variables as follows :

Let $w = f\,(x, y, z)$ be continuous and possess partial derivatives f_x, f_y, f_z at and in some neighbourhood of the point (x_0, y_0, z_0) whose partial derivatives are continuous at that point. Then we have

$$\begin{aligned} \Delta w &= f\,(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f\,(x_0, y_0, z_0) \\ &= f_x\,(x_0, y_0, z_0)\,\Delta x + f_y\,(x_0, y_0, z_0)\,\Delta y + \\ &\quad f_z\,(x_0, y_0, z_0)\,\Delta z + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y + \varepsilon_3\,\Delta z \quad \ldots (2.24) \end{aligned}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \to 0$ as $\Delta x, \Delta y, \Delta z \to 0$.

Now we define the **differential** of $w = f\,(x, y)$ by the formula

$$dw = f_x\,\Delta x + f_y\,\Delta y$$

We can write, in view of Eq. (2.15), that

$$\Delta w = dw + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y,$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$.

If the increments Δx and Δy are small, then the differential dw is a good approximation for the change Δw in the value of $w = f\,(x, y)$ as we move from the point (x, y) to $(x + \Delta x, y + \Delta y)$, since

$$\frac{\Delta w - dw}{\sqrt{\Delta^2 x + \Delta^2 y}} = \frac{\varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y}{\sqrt{\Delta^2 x + \Delta^2 y}} \to 0 \text{ as } \Delta x, \Delta y \to 0.$$

As in the case of functions of one variable, writing Δx and Δy as dx and dy respectively, we get

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy \qquad \ldots (2.25)$$

It is of interest to note here that the function $w = f(x, y)$ is differentiable in R whenever Eq. (2.25) holds.

Similarly, for a function of three variables $w = f(x, y, z)$, we define the differential w by

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz \qquad \ldots (2.26)$$

Note :

In Eq. (2.26), the separate terms $\frac{\partial w}{\partial x} dx$, $\frac{\partial w}{\partial y} dy$ and $\frac{\partial w}{\partial z} dz$ are sometimes called partial differentials of w with respect to x, y and z respectively.

We can then call the sum of these partial differentials of w as the **total differential** dw. So dw is called as the **total differential** of w, when w is a function of more than one variable.

Example 2.7

Find the total differential of $w = x \sin y - y \sin x$.

Solution

$$\frac{\partial w}{\partial x} = \sin y - y \cos x, \quad \frac{\partial w}{\partial y} = x \cos y - \sin x$$

So, $$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy = (\sin y - y \cos x)\, dx + (x \cos y - \sin x)\, dy$$

To give you an idea about the use of differentials, we consider a rectangle with sides x and y. The area of this rectangle is

$$A(x, y) = xy.$$

If we increase the dimensions of the rectangle to $x + \Delta x$ and $y + \Delta y$, the change in area is

$$\Delta A = (x + \Delta x)(y + \Delta y) - xy$$

$$= x\,\Delta y + y\,\Delta x + \Delta x\,\Delta y$$

The differential estimate for this change in area is

$$dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy = y\,\Delta x + x\,\Delta y$$

The error of our estimate, the difference between the actual change and the estimated change, is the difference $\Delta A - dA = \Delta x\,\Delta y$.

Example 2.8

Use the differential to estimate

$$\sqrt{27}\,\sqrt[3]{1021}$$

Solution

We know

$$\sqrt{25} = 5, \quad \sqrt[3]{1000} = 10.$$

What we need to find is an estimate for the increase of

$$f(x, y) = x^{1/2} \; y^{1/3}$$

from $x = 25, y = 1000$ to $x = 27, y = 1021$.

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

gives $$df = \frac{1}{2} x^{-1/2} y^{1/3} \, dx + \frac{1}{3} x^{1/2} y^{-2/3} dy$$

Putting $x = 25, y = 1000, dx = 2, dy = 21$, we have

$$df = \frac{1}{2} (25)^{-1/2} (1000)^{1/3} (2) + \frac{1}{3} (25)^{1/2} (1000)^{-2/3} (21) = 2.35$$

Hence $$\sqrt{27} \; \sqrt[3]{1000} \approx \sqrt{25} \; \sqrt[3]{1000} + 2.35 = 52.35.$$

SAQ 3

(a) Find the total differential of $w = x^2 + 2xy - 3y^2$.

(b) Use differentials to find the approximate value of $\sqrt{125} \;\; \sqrt[4]{15}$.

(c) Estimate by a differential the change in the volume of a right circular cylinder if the height is increased from 12 to 12.1 cm and the radius is decreased from 6 to 5.8 cm.

2.6 THE CHAIN RULE

In this section, we obtain the chain rules for finding partial derivatives of functions of two or three variables. In a similar way, we can get the chain rules of partial differentiation for functions of more than three variables.

2.6.1 Chain Rule for Functions of Two Variables

If $w = f(x, y)$ has continuous partial variables f_x and f_y and if $x = x(t), y = y(t)$ are differentiable functions of t, then the composite function $w = f(x(t), y(t))$ is a differentiable function of t.

In this case, we get

$$\frac{dw}{dt} = f_x \; (x(t), \; y(t)) \; x'(t) + f_y \; (x(t), \; y(t)) \; y'(t)$$

or $$\frac{dw}{dt} = \frac{\partial f}{\partial t}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \quad \ldots (2.27)$$

We take t_0 as any value of t. Then $x_0 = x(t_0)$ and $y_0 = y(t_0)$. If we take $\Delta x, \Delta y$ as changes that occur in x and y when t is changed from t_0 to $t_0 + \Delta t$, we then get from Eq. (2.25) that

$$\Delta w = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y \quad \ldots (2.28)$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$.

Dividing both sides of Eq. (2.28) by Δt, we get

$$\frac{\Delta w}{\Delta t} = f_x(x_0, y_0)\frac{\Delta x}{\Delta t} + \Delta y(x_0, y_0)\frac{\Delta y}{\Delta t} + \varepsilon_1\frac{\Delta x}{\Delta t} + \varepsilon_2\frac{\Delta y}{\Delta t} \quad \ldots (2.29)$$

We also note that, as

$$\Delta t \to 0, \frac{\Delta x}{\Delta t} \to \left(\frac{dx}{dt}\right)_{t_0} = x'(t_0) \text{ and } \frac{\Delta y}{\Delta t} \to \left(\frac{\Delta y}{\Delta t}\right)_{t_0} = y'(t_0).$$

So, we get

$$\frac{dw}{dt} = \lim_{\Delta t \to 0}\frac{\Delta w}{\Delta t} = f_x(x_0, y_0)\,x'(t_0) + f_y(x_0, y_0)\,y'(t_0) \quad \ldots (2.30)$$

The equations we want to prove are simply the statements that Eq. (2.30) holds at every admissible value of t, which infact it does.

We extend Eq. (2.30) to functions of three variables as follows :

If $w = f(x, y, z)$ has continuous partial derivatives and $x = x(t), y = y(t), z = z(t)$ are differentiable functions of t, then the composite function $f(x(t), y(t), z(t))$ is a differentiable function of t. In this case, we get

$$\frac{dw}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt} + f_z\frac{dz}{dt}$$

or $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \quad \ldots (2.31)$$

If $w = f(x, y)$ and if $x = x(r, s), y = y(r, s)$, are any functions, then the composite $f(x(r, s), y(r, s))$, when defined, is a function of r and s. If x, y and f have continuous partial derivatives, then the partial derivatives of w with respect to r and s exist and, in this case, we get

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \quad \ldots (2.32)$$

and $$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad \ldots (2.33)$$

We can extend Eqs. (2.32) and (2.33) to functions of three variables as follows :

If $w = f(x, y, z)$ and if $x = x(r, s)$, $y = y(r, s)$, $z = z(r, s)$ are any functions, then the composite function $w = f[x(r, s), y(r, s), z(r, s)]$, when defined, is a function of r and s. If x, y, z and f have continuous partial derivatives, then the partial derivatives of w with respect to r and s exist and in this case, we get

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \quad \ldots (2.34)$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \quad \ldots (2.35)$$

To show what variables are assumed independent in computing a partial derivative, we employ the following notations.

$\left(\frac{\partial w}{\partial x}\right)_y$ means $\frac{\partial w}{\partial x}$ with x and y independent.

$\left(\frac{\partial w}{\partial x}\right)_z$ means $\frac{\partial w}{\partial x}$ with x and z independent.

$\left(\frac{\partial f}{\partial y}\right)_{x,t}$ means $\frac{\partial f}{\partial y}$ with y, x and t independent.

Example 2.9

If $w = x^2 + y - z + \sin t$ and $x + y = t$, find

(i) $\left(\frac{\partial w}{\partial x}\right)_{y,z}$ and

(ii) $\left(\frac{\partial w}{\partial x}\right)_{t,z}$.

Solution

(i) With x, y, z independent, we have

$$t = x + y,\ w = x^2 + y - z + \sin(x + y).$$

Therefore, $\left(\frac{\partial w}{\partial x}\right)_{y,z} = 2x + \cos(x + y)\frac{\partial}{\partial x}(x + y)$

$$= 2x + \cos(x + y)$$

(ii) With x, t, z independent, we have

$$y = t - x,\ w = x^2 + (t - x) + \sin t.$$

Thus $\left(\frac{\partial w}{\partial x}\right)_{t,z} = 2x - 1$

Example 2.10

If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

Solution

$x = r\cos\theta, y = r\sin\theta$

$$\Rightarrow \frac{\partial x}{\partial r} = \cos\theta, \frac{\partial x}{\partial \theta} = -r\sin\theta, \frac{\partial y}{\partial r} = \sin\theta, \frac{\partial y}{\partial \theta} = r\cos\theta.$$

Therefore, we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$= \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial t}$$

Similarly, we have

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$= -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}$$

Hence $$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

2.6.2 Implicit Differentiation

You would recall that the method of implicit differentiation was used in previous unit to find the derivative of a function defined implicitly by an equation. In this section, you will see how the derivative of an implicitly defined function can be obtained through the use of partial derivatives.

Let $f(x, y) = 0$ . . . (2.36)

where $y = \phi(x)$

By the chain rule (with $x = x, y = \phi(x)$), we get

$$\frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{d}{dx}(0) = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y} \quad \ldots$$

(2.37)

We have assumed that y is a differentiable function of x.

Example 2.11

If ϕ is a differentiable function such that $y = \phi(x)$ satisfies the equation

$$x^3 + y^2 + \sin xy = 0,$$

find $\frac{dy}{dx}$.

Solution

Let $f(x, y) = x^3 + y^2 + \sin xy$

Then $f_x = 3x^2 + y \cos xy,$

$f_y = 2y + x \cos xy.$

Hence $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{3x^2 + y \cos xy}{2y + x \cos xy}.$

SAQ 4

(a) If $u = f(x, y)$ and $r = \dfrac{x}{y}, s = \dfrac{y}{z}, t = \dfrac{x}{x},$ prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0.$$

(b) If z be a function of x and y and $x = e^u + e^{-v}, y = e^u - e^v,$ prove that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}.$$

(c) If $v = f\left(\dfrac{x}{z}, \dfrac{y}{z}\right)$, prove that

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} + z \frac{\partial v}{\partial z} = 0$$

(**Hint :** Take $\xi = \dfrac{x}{z}, \eta = \dfrac{y}{z}$, then $v = f(\xi, \eta)$.

Find $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial v}{\partial \eta}\dfrac{\partial \eta}{\partial x}, \quad \dfrac{\partial v}{\partial y} = \dfrac{\partial v}{\partial \xi}\dfrac{\partial \xi}{\partial y} + \dfrac{\partial v}{\partial \eta}\dfrac{\partial \eta}{\partial y}$).

(d) If $f(x, y, z) = 0$, prove that

$$\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial x}{\partial y}\right)_z = -1.$$

(e) Find $\dfrac{dy}{dx}$, given that $y = \phi(x)$ satisfies each equation.

(i) $x \sec y + y \sec x - 6 = 0$

(ii) $x^3 \ln y - y^3 + x^2 = 4.$

2.7 MAXIMA AND MINIMA

In this section, you will learn about the maxima and minima of functions of two variables. We give some definitions to start with.

Definition 3

A function of two variables $f(x, y)$ is said to have an ***absolute*** *or* ***global maximum*** *at (x_0, y_0) in a region R if*

$$f(x, y) \le f(x_0, y_0) \text{ for all } (x, y) \text{ in } R$$

An ***absolute*** *or* ***global minimum*** *of f in R occurs at (x_0, y_0) in R.*

$$f(x, y) \ge f(x_0, y_0) \text{ for all } (x, y) \text{ in } R.$$

Definition 4

A ***relative or local maximum*** *occurs at (x_0, y_0) if a circle D about (x_0, y_0) exists with $f(x, y) \le f(x_0, y_0)$ for all (x, y) in the interior of D.*

A ***relative or local minimum*** *of f occurs at (x_0, y_0) if a circle D about (x_0, y_0) exists with $f(x, y) \ge f(x_0, y_0)$ for all (x, y) in the interior of D.*

We refer to both maxima and minima as ***extrema****, and relative maximum and relative minimum as* ***relative extrema****.*

Definition 5

A function f of two variables is said to have a ***critical point*** *at (x_0, y_0) in the domain of f if either*

(i) *both partial derivatives of f are zero at (x_0, y_0), or*

(ii) *at least one of the partial derivatives fails to exist at (x_0, y_0).*

Suppose that a relative maximum value of the function $z = f(x, y)$ occurs at an interior point (a, b) of the domain of the function and that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist at (a, b) as shown in Figure 2.5.

Figure 2.5

Then we observe the following facts :

(i) $x = a$ is an interior point of the domain of the curve $C_1 : z = f(a, b)$ in which the plane $y = b$ cuts the surface $z = f(x, y)$.

(ii) The function $z = f(x, b)$ has a relative maximum at $x = a$.

(iii) The value of the derivative of $z = f(x, b)$ at $x = a$ is therefore zero.

Since this derivative is precisely $\left(\frac{\partial f}{\partial x}\right)_{(a,b)}$, we conclude that $\left(\frac{\partial f}{\partial x}\right)_{(a,b)} = 0$.

On the lines of similar argument with the function $z = f(a, y)$, we get $\left(\frac{\partial f}{\partial y}\right)_{(a,b)} = 0$.

Thus we have $f_x(a, b) = 0 = f_y(a, b)$.

So we have a necessary condition for f to have an extreme value at an interior point (a, b) :

$$f_x(a, b) = 0 = f_y(a, b).$$

Definition 6

*The point where there is no relative maximum or no relative minimum but f_x and f_y are both zero, is called a **saddle point***

We apply the second derivative test to determine whether a function $z = f(x, y)$ has a relative maximum or minimum value at a point $P(a, b)$ where f_x and f_y both vanish.

We assume that f and its first and second order partial derivatives are continuous in some region R about $P(a, b)$ as shown in Figure 2.6. We take $S(a + h, b + k)$ to be a point close enough to P so that PS lies in R. We take the parametric equation of PS to be

$$x = a + th,\ y = b + tk,\ 0 \le t \le 1 \qquad \ldots (2.38)$$

Figure 2.6

We study the value of $F(x, y)$ along PS by considering the function

$$F(t) = f(a + ht, b + tk)$$

We know that F is a differentiable function of t since the first partial derivatives are continuous and $x = a + th$, $y = b + tk$ are differentiable functions of t. Using the chain rule, we get

$$F'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \quad \ldots (2.39)$$

Further, we know that F' is continuous on the closed interval $0 \le t \le 1$ since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous throughout R. Once again, making use of chain rule, we get

$$F''(t) = h\frac{\partial F'(t)}{\partial x} + k\frac{\partial F'(t)}{\partial y}$$

$$= h\left(h\frac{\partial^2 f}{\partial x^2} + k\frac{\partial^2 f}{\partial y\,\partial x}\right) + k\left(h\frac{\partial^2 f}{\partial x\,\partial y} + k\frac{\partial^2 f}{\partial y^2}\right)$$

$$= h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2} \quad \ldots (2.40)$$

Now, $F'(t)$ satisfy the conditions of Taylor's theorem on the closed interval [0, 1], namely, F and F' are continuous in (0, 1) and F' is differentiable in open interval (0, 1). So we have

$$F(1) = F(0) + F'(0)(1-0) + F''(c)\frac{(1-0)^2}{2!},$$

i.e.

$$F(1) = F(0) + F'(0) + \frac{1}{2}F''(c) \quad \ldots (2.41)$$

for some c in $0 < c < 1$.

$$F(1) = f(a+h,\, b+k)$$

$$F(0) = f(a,\, b)$$

$$F'(0) = hf_x(a, b) + kf_y(a, b)$$

$$F''(c) = (h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy})_{(a+ch,\, b+ck)}$$

Substituting these values in Eq. (2.41), we get

$$f(a+h,\, b+k) = f(a, b) + hf_x(a, b) + k f_y(a, b)$$

$$+\frac{1}{2}(h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy})_{(a+ch,\, b+ck)} \quad \ldots (2.42)$$

Supposing $f_x = 0$, $f_y = 0$ at (a, b), we wish to determine whether $f(x, y)$ has a maximum or minimum at (a, b). We write Eq. (2.42) as

$$f(a+h,\, b+k) - f(a, b) = \frac{1}{2}(h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy})_{(a+ch,\, b+ck)}$$

Since a minimum or maximum value of f at (a, b), depends on the sign of $f(a+h,\, b+k) - f(a, b)$, we are led to consider the sign of

$$Q(c) = (h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy})_{(a+ch,\, b+ck)} \quad \ldots (2.43)$$

or, considering c very small,

$$Q(0) = h^2 f_{xx}(a,b) + 2hk\, f_{xy}\ (a,b) + k^2 f_{yy}\ (a,b) \qquad \ldots (2.44)$$

Multiplying both sides of Eq. (2.44) by $f_{xx}(a,b)$ we get

$$f_{xx}\ Q(0) = (h\, f_{xx} + k\, f_{xy})^2 + (f_{xx}\ f_{yy} - f_{xy}{}^2)\, k^2 \qquad \ldots (2.45)$$

The sign of Q (0) can be determined from Eq. (2.44). Consequently, we get the following criteria for the behaviour of $f(x, y)$ at (a, b).

(i) If $f_{xx} < 0$ or $f_{yy} < 0$ and $f_{xx}\ f_{yy} - f_{xy}^2 > 0$ at (a, b), then Q (0) < 0 for all small non zero values of h and k and f has a relative maximum value at (a, b).

(ii) If $f_{xx} > 0$ or $f_{yy} > 0$ and $f_{xx}\ f_{yy} - f_{xy}^2 > 0$ at (a, b), then Q (0) > 0 for all small non zero values of h and k and f has a relative maximum value at (a, b).

(iii) If $f_{xx}\ f_{yy} - f_{xy}{}^2 < 0$ at (a, b), then it can be shown that there are combinations of arbitrary small non zero values of h and k for which Q (0) > 0 and Q (0) < 0. Thus, arbitrarily close to point $P_0\ (a, b, f\ (a, b))$ on the surface $z = f\ (x, y)$ there are points above P_0 and also points below P_0. The function f therefore has a saddle point at (a, b).

(iv) Finally, if $f_{xx}\ f_{yy} - f_{xy}{}^2 = 0$, we can draw no conclusion about the sign of Q (0) and we require some other test to settle this case.

Now we give a brief summary of maximum and minimum tests as follows :

If $z = f\ (x, y)$ is continuous, then extreme values of f may occur at

(i) boundary points of the domain of f

(ii) interior points where $f_x = f_y = 0$

(iii) points where f_x or f_y fails to exist.

Further, if f has continuous first and second order partial derivatives on some open disc containing (a, b) and if $f_x\ (a, b) = f_y\ (a, b) = 0$, then

(i) $f_{xx} < 0$ or $f_{yy} < 0$ and $f_{xx}\ f_{yy} - f_{xy}^2 > 0$ at (a, b) implies that f has a local maximum at (a, b).

(ii) $f_{xx} > 0$ or $f_{yy} > 0$ and $f_{xx}\ f_{yy} - f_{xy}^2 > 0$ at (a, b) implies that f has a local minimum at (a, b).

(iii) $f_{xx}\ f_{yy} - f_{xy}^2 < 0$ at (a, b) implies that the point (a, b) is a saddle point of f.

(iv) $f_{xx}\ f_{yy} - f_{xy}^2 = 0$ at (a, b) implies that the test is inconclusive as this case requires some further investigations.

Example 2.12

Find the maximum and minimum values of

$$2(x^2 - y^2) - x^4 + y^4$$

Solution

$$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$$

$$f_x = 4x - 4x^3 \,,\; f_y = -4y + 4y^3$$

$$f_x = 0 \text{ gives } x = 0, \pm 1,\; f_y = 0 \text{ gives } y = 0, \pm 1$$

The likely points where $f(x, y)$ has a maximum or minimum, are

(0, 0), (0, ± 1), (±1, 0)

$$f_{xx} = 4 - 12x^2 \,,\; f_{xy} = 0,\; f_{yy} = -4 + 12y^2$$

The results are summarized in the following table.

Table

Point	$f_{xx}\, f_{yy} - f_{xy}^{\,2}$	f_{xx}	Max/Min/Saddle Point
(0, 0)	– 16	–	Saddle point
(0, 1)	32	4	Minimum value – 1
(0, – 1)	32	4	Minimum value – 1
(1, 0)	32	– 8	Maximum value 1
(– 1, 0)	32	– 8	Maximum value 1

SAQ 5

(a) Find the maximum and minimum values of $xy + 27\left(\frac{1}{x} + \frac{1}{y}\right)$.

(b) Investigate the maximum and minimum values of $x^2 y^2 - 5x^2 - 8xy - 5y^2$.

2.8 JACOBIANS

In this section, you will learn about Jacobians. Jacobians play an important role in evaluation of multiple integrals when the variables of an integral are changed by a suitable transformation.

If $x = g(u, v)$ and $y = h(u, v)$ be differentiable, then the **Jacobian** of x and y with respect to u and v, denoted by $\frac{\partial(x, y)}{\partial(u, v)}$, is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

If $x = g(u, v, w)$, $y = h(u, v, w)$, $z = j(u, v, w)$ are differentiable, then we define the Jacobian of the transformation from a region U in $u\, v$ w-space to a region W in $x\, y\, z$-space, by

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

In a similar manner, we can define Jacobians of order higher than three.

Note

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} \text{ is sometimes denoted by } J\left(\frac{x, y, z}{u, v, w}\right)$$

We now prove some important properties of Jacobians. We prove these properties for Jacobians of second order only. You can get similar such properties for Jacobians of order higher than 2.

Theorem

If $J = \frac{\partial(u, v)}{\partial(x, y)}$ **and** $J' = \frac{\partial(x, y)}{\partial(u, v)}$,

then $JJ' = 1$.

Proof

Consider $u = \phi(x, y)$ and $v = \psi(x, y)$. We can solve x, y in terms of u and v and get $u = \phi_1(u, v)$ and $y = \psi_1(u, v)$.

We also get

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

and

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy.$$

So that we get

$$\frac{\partial u}{\partial u} = 1, \ \frac{\partial u}{\partial v} = 0, \ \frac{\partial v}{\partial v} = 1 \text{ and } \frac{\partial v}{\partial u} = 0$$

Thus

$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u}$$

Now we have

$$JJ' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Theorem

If *u* and *v* are functions of *p* and *q* and *p* of *q* are functions of *x* and *y*, then

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(p, q)}\frac{\partial(p, q)}{\partial(x, y)}$$

Proof

Since u and v are functions of p and q, we get

$$du = \frac{\partial u}{\partial p}dp + \frac{\partial u}{\partial q}dq$$

$$dv = \frac{\partial v}{\partial p}dp + \frac{\partial v}{\partial q}dq$$

We also get

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial q}\frac{\partial q}{\partial x}$$

and $$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial v}{\partial q}\frac{\partial q}{\partial y}$$

Now consider

$$\frac{\partial(u, v)}{\partial(p, q)} \times \frac{\partial(p, q)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial p} & \frac{\partial u}{\partial q} \\ \frac{\partial v}{\partial p} & \frac{\partial v}{\partial q} \end{vmatrix} \times \begin{vmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial p} & \frac{\partial u}{\partial q} \\ \frac{\partial v}{\partial p} & \frac{\partial v}{\partial q} \end{vmatrix} \times \begin{vmatrix} \frac{\partial p}{\partial x} & \frac{\partial q}{\partial y} \\ \frac{\partial p}{\partial x} & \frac{\partial q}{\partial y} \end{vmatrix}$$

by interchanging rows and columns of the second determinant.

$$= \begin{vmatrix} \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} & \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} \\ \frac{\partial v}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial q}\frac{\partial q}{\partial x} & \frac{\partial v}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial v}{\partial q}\frac{\partial q}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial (u, v)}{\partial (x, y)}$$

So we get now

$$\frac{\partial (u, v)}{\partial (x, y)} = \frac{\partial (u, v)}{\partial (p, q)} \times \frac{\partial (p, q)}{\partial (x, y)}$$

Theorem

If the function u and v of two independent variable x and y be such that $f(u, v) = 0$, then $\frac{\partial (u, v)}{\partial (x, y)} = 0$.

Proof

Since $f(u, v) = 0$, we have

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0$$

and $$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0.$$

Eliminating $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ from the above two equations, we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial u}{\partial y} \end{vmatrix} = 0$$

i.e., $$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0.$$

by interchanging rows and columns of the determinant.

Therefore, $$\frac{\partial (u, v)}{\partial (x, y)} = 0.$$

Example 2.13

If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, find $\frac{\partial (x, y, z)}{\partial (r, \theta, z)}$.

Solution

$$\frac{\partial (x, y, z)}{\partial (r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta) = r(\cos^2\theta + \sin^2\theta) = r$$

SAQ 6

(a) If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(x, y)}{\partial(r, \theta)}$ and $\frac{\partial(r, \theta)}{\partial(x, y)}$ and also verify that $\frac{\partial(x, y)}{\partial(r, \theta)} \cdot \frac{\partial(r, \theta)}{\partial(x, y)} = 1.$

(b) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, find $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$.

(c) If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$, find $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}$. Also find $\frac{\partial(x_1, x_2, x_3)}{\partial(y_1, y_2, y_3)}$.

(d) If $u = x + y + z$, $y + z = uv$, $z = uvw$, find $\frac{\partial(x, y, z)}{\partial(u, v, w)}$ and $\frac{\partial(u, v, w)}{\partial(x, y, z)}$.

2.9 SURFACES

In this section, you will learn about some basic ideas about surfaces in general.

We first give its definition.

Definition 7

*A **surface** is the locus of a point whose cartesian coordinates x, y, z are functions of two independent parameters u and v.*

So we get

$$x = f(u, v), \; y = g(u, v), \; z = h(u, v)$$

to be the parametric equations of a surface. When we represent a surface in this way, we call this as the Gaussian form of representing a surface; we also call the two parameters u and v as the curvilinear coordinates of a current point on a surface.

For any point (x, y, z) on the surface, we can uniquely determine the values of u and v and we refer to this as (u, v).

If we eliminate parameters u and v in the parametric equations of a surface, then the relation so obtained is of the form

$$F(x, y, z) = 0$$

and this also represents a surface.

When a surface is represented in this form, we call this as the constraint equation of a surface. Consider

$$x = u, \; y = v, \; z = u^2 - v^2 \qquad \ldots (2.46)$$

Eliminating u, v in Eq. (2.46), we get

$$x^2 - y^2 = z$$

Next take

$$x = u + v, \; y = u - v, \; z = 4uv \qquad \ldots (2.47)$$

Again on elimination of u and v in Eq. (2.47), we get

$$x^2 - y^2 = z$$

Finally consider

$$x = u \cosh v, \; y = u \sinh v, \; z = u^2 \qquad \ldots (2.48)$$

on elimination of u and v in Eq. (2.48), we get

$$x^2 - y^2 = z$$

We see that the different parametric Eqs. (2.46), (2.47) and (2.48) give rise to the same surface $x^2 - y^2 = z$.

Therefore we note that the parametric equations of a given surface are not necessarily unique. Further, we observe that the constraint equation $x^2 - y^2 = z$ of the surface represents the whole of it. The parametric equations of the surface represent a portion of it.

For example, the parametric Eq. (2.48) of a surface represents only the part of the surface $x^2 - y^2 = z$ for which $z \geq 0$ as u is real.

Therefore we note here that the parametric equation and the constraint equation to a surface are not always equivalent.

If we write the equation to a surface in the form $z = f(x, y)$, we call this as Monge's form of the equation to a surface.

In Monge's form of equation to a surface, we can regard x and y themselves as parameters.

$$x = x, \; y = y, \; z = f(x, y).$$

Tangent Plane and Normal Line at a Point on a Surface

Let

$$F(x, y, z) = 0 \qquad \ldots (2.49)$$

represents a surface. We take $P\,(x, y, z)$ and $Q\,(x+\Delta x,\; y+\Delta y,\; z+\Delta z)$ as two points close to each other on the surface represented by Eq. (2.49). Let arc PQ be Δs and chord PQ be Δc. We know that

$$\lim_{Q \to P} \frac{\Delta s}{\Delta c} = 1$$

Figure 2.7

From our knowledge of coordinate geometry of three dimension, we know that the direction ratios of PQ are

$$\frac{x+\Delta x - x}{\Delta c}, \frac{y+\Delta y - y}{\Delta c}, \frac{z+\Delta z - z}{\Delta c},$$

i.e.
$$\frac{\Delta x}{\Delta c}, \frac{\Delta y}{\Delta c}, \frac{\Delta z}{\Delta c}.$$

We write the above as

$$\frac{\Delta x}{\Delta s}.\frac{\Delta s}{\Delta c}, \frac{\Delta y}{\Delta s}.\frac{\Delta s}{\Delta c}, \frac{\Delta z}{\Delta s}.\frac{\Delta s}{\Delta c}.$$

We also note that as $\Delta s \to 0$, $Q \to P$, PQ tends to a tangent line PT at P.

Now noting that the coordinates of any point on arc PQ are functions of s only, we get the direction cosines of PT to be

$$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}.$$

Now differentiating both sides of Eq. (2.49) with respect to s, we get

$$\frac{\partial F}{\partial x}\frac{dx}{ds}+\frac{\partial F}{\partial y}\frac{dy}{ds}+\frac{\partial F}{\partial z}\frac{dz}{ds}=0 \quad \ldots$$

(2.50)

Again using our knowledge of coordinate geometry of three dimensions, we get from Eq. (2.50) that line with direction cosines $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ is perpendicular to the line having direction ratios as $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$.

Since we can take different curves joining Q to P, we get a number of tangent lines at P and the line with direction ratios $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ will be perpendicular to all such tangent lines at P. Thus we see that all tangent lines at P lie in a plane through P perpendicular to the line with direction ratios

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, \text{ i.e. } F_x, F_y, F_z.$$

So we get the equation of the tangent plane to surface given by the Eq. (2.49) at point P to be

$$(X-x)\frac{\partial F}{\partial x}+(Y-y)\frac{\partial F}{\partial y}+(Z-z)\frac{\partial F}{\partial z}=0,$$

where (X, Y, Z) are the current coordinates of any point on the tangent plane.

Note

From what we learnt in coordinate geometry of three dimensions, we get the equation of a plane through a point $(x_0\, y_0\, z_0)$ having a line with direction ratios a, b, c as normal to be

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0.$$

We also get the equations of a normal to the surface at P which is a line through P perpendicular to the tangent plane at P as

$$\frac{X-x}{F_x}=\frac{Y-y}{F_y}=\frac{Z-z}{F_z}$$

Example 2.14

Find the equations of the tangent plane and the normal to the surface $xyz = a^3$ at $(x_1\, y_1\, z_1)$ where a is a constant.

Solution

Here $\quad \frac{\partial F}{\partial x}=yz, \frac{\partial F}{\partial y}=zx, \frac{\partial F}{\partial z}=xy$

At $(x_1\ y_1\ z_1)$, $\quad \frac{\partial F}{\partial x}=y_1 z_1, \frac{\partial F}{\partial y}=z_1 x_1, \frac{\partial F}{\partial z}=x_1 y_1.$

So we get the equation of the tangent plane at any point (x_1, y_1, z_1) as

$$y_1 z_1(x-x_1)+z_1 x_1(y-y_1)+x_1 y_1(z-z_1)=0$$

i.e. $xy_1 z_1 + x_1 y z_1 + x_1 y_1 z = 3x_1 y_1 z_1$

The equation of normal at $(x_1 y_1 z_1)$ becomes

$$x_1 (x - x_1) = y_1 (y - y_1) = z_1 (z - z_1).$$

SAQ 7

Find the equations of the tangent plane and the normal to the surface

$$36x^2 + 9y^2 + 5z = 72 \text{ at } (0, 2, 3).$$

2.10 SUMMARY

- $f(x, y)$ is said to tend to the limit L as $(x, y) \to (x_0 y_0)$, written as $\lim_{(x,y)\to f(x,y)} = L$,

  if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that either

  $$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \Rightarrow |f(x, y) - L| < \varepsilon$$

  or $\quad 0 < |x - x_0| < \delta$ and $0 < |y - y_0| < \delta \Rightarrow |f(x, y) - L| < \varepsilon$.

- $f(x, y)$ is said to be continuous at a point $(x_0 y_0)$ if

  (i) f is defined at $(x_0 y_0)$

  (ii) $\lim_{(x,y)\to f(x_0, y_0)} f(x, y)$ exists

  (iii) $\lim_{(x,y)\to f(x_0, y_0)} f(x, y) = f(x_0, y_0)$.

- The partial derivative of $w = f(x, y)$ with respect to x, denoted by f_x or $\frac{\partial w}{\partial x}$, is

  $$\frac{\partial w}{\partial x} = f_x = \lim_{\Delta x - 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

  provided the limit exists.

- The second order partial derivatives of $w = f(x, y)$ are $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ where

  $f_{xx} = \frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)$ etc.

- The differential of $w = f(x, y)$ is defined by the formula $dw = f_x\, dx + f_y\, dy$.

- If $w = f(x, y, z)$ and if $x = x(r, s)$, $y = (r, s)$, $z = (r, s)$, then

  $$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r},$$

  $$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

- If $f(x, y) = 0$ and if y is a function of x, then

$$\frac{dy}{dx} = -\frac{f_x}{f_y}.$$

- If $f\,(x, y)$ has continuous first and second order partial derivatives on some open disc containing (a, b) and if $f_x\,(a, b) = 0 = f_y\,(a, b)$, then

  (i) $f_{xx} < 0$ and $f_{xx}\, f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Rightarrow f$ has a local maximum at (a, b).

  (ii) $f_{xx} > 0$ and $f_{xx}\, f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Rightarrow f$ has a local minimum at (a, b).

  (iii) $f_{xx}\, f_{yy} - f_{xy}^2 = 0$ at $(a, b) \Rightarrow$ the point (a, b) is a saddle point of f, i.e. (a, b) is neither a maximum nor a minimum.

  (iv) $f_{xx}\, f_{yy} - f_{xy}^2 = 0$ at $(a, b) \Rightarrow$ the test is inconclusive; this requires further investigation.

- If $x = g\,(u, v, w),\ y = h\,(u, v, w),\ z = j\,(u, v, w)$ are differentiable, then the Jacobian from a region U in uvw-space to a region W in xyz-space, denoted by $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$, is defined as

$$\frac{\partial\,(x, y, z)}{\partial\,(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

- The equation of the tangent plane to the surface $f\,(x, y, z) = 0$ at the point (x, y, z) is

$$(X - x)\ f_x + (Y - y)\ f_y + (Z - z)\ f_z = 0.$$

- The equation of the normal line to the surface $f\,(x, y, z) = 0$ at the point (x, y, z) are

$$\frac{X - x}{f_x} = \frac{Y - y}{f_y} = \frac{Z - z}{f_z}.$$

2.11 ANSWERS TO SAQs

SAQ 1

(a) $f\,(x, y)$ is discontinuous at the origin (0, 0).

(b) $f\,(x, y)$ is continuous at all points except at the origin (0, 0).

SAQ 2

(a) $u_x = \dfrac{a}{ax + by},\ u_y = \dfrac{b}{ax + by},\ u_{xx} = -\left(\dfrac{a}{ax + by}\right)^2$

$$u_{yy} = -\left(\frac{a}{ax + by}\right)^2,\ u_{yx} = -\frac{ab}{(ax + by)^2} \text{ and}$$

$$u_{yx} = -\frac{ab}{(ax + by)^2}.$$

(b) $u_{xx} = -20zx^3 + 2yz,\ u_{yy} = 6xy,\ u_{zz} = 0$

$u_{xy} = 3y^2 + 2xz,\ u_{zx} = -5x^4 + 2xy,\ u_{yz} = x^2,\ u_{yx} = 3y^2 + 2xz,\ u_{zx}$
$= -5x^4 + 2xy$ and $u_{zy} = x^2$.

SAQ 3

(a) $dw = 2(x+y)\,dx + 2\,(x-3y)\,dy$

(b) We know that $\sqrt{121} = 11$ and $\sqrt[4]{15} = 2$

Let $f(x, y) = x^{\frac{1}{2}}\, y^{\frac{1}{4}}$

$$\Rightarrow df = f_x\,dx + f_y\,dy = \frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{4}}\,dx + \frac{1}{4}x^{\frac{1}{2}}y^{-\frac{3}{4}}\,dy$$

For $x = 121,\ y = 16,\ dx = 4,\ dy = -1$

$$df = \frac{1}{2}.\frac{1}{11}.\,24 - \frac{1}{4}.11.\frac{1}{8} = \frac{4}{11} - \frac{11}{32} = \frac{128-121}{352} = \frac{7}{352} \approx 0.02$$

$\therefore\ \sqrt{125}\,\sqrt[4]{15} = \sqrt{121}\,\sqrt[4]{16} + 0.02 = 11\times 2 + 0.02 = 22.02.$

(c) $V = \pi r^2 h \qquad dV = V_r\,dr + V_h\,dh = 2\pi r h\,dr + \pi r^2\,dh$

For $r = 12,\ h = 6,\ dr = 0.1$ and $dh = -0.2,$

$dV = 2\pi\times 12\times 6\times 0.1 - \pi\times 12\times 12\times 0.2$

$= 4.8\pi - 28.8\pi = -20\pi$

SAQ 4

(e) (i) Let $f(x, y) = x\sec y + y\sec x - 6$

$\therefore\quad f_x = \sec y + y\sec x\tan x$

$f_y = x\sec y\tan y + \sec x$

Hence $$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\sec y + y\sec x\tan x}{\sec x + x\sec y\tan y}$$

(ii) Let $f(x, y) = x^3\ln y - y^3 + x^2 - 4$

$\therefore\quad f_x = 3x^2\ln y + 2x$

$f_y = \frac{x^3}{y} - 3y^2$

Hence $$\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{3x^2 y\ln y + 2xy}{x^3 - 3y^3}$$

SAQ 5

(a) The value of the function attains a minimum at $x = y = 3$ and the minimum value is 27.

(b) Function has maximum value 0 at (0, 0) and minimum value – 81 at $(\pm 3, \pm 3)$.

SAQ 6

(a) $\dfrac{\partial(x, y)}{\partial(r, \theta)} = r$ and $\dfrac{\partial(r, \theta)}{\partial(x, y)} = \dfrac{1}{r}$

(b) $\dfrac{\partial(x, y, z)}{(r, \theta, \phi)} = r^2 \sin\theta$

(c) $\dfrac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4$ and $\dfrac{\partial(x_1, x_2, x_3)}{\partial(y_1, y_2, y_3)} = \dfrac{1}{4}$

(d) $\dfrac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 v$ and $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = \dfrac{1}{(x + y + z)^2\,(y + z)}$

SAQ 7

Here $F = 36x^2 + 9y^2 + 4z - 72 = 0$

$\therefore$ $\dfrac{\partial F}{\partial x} = 72x,\ \dfrac{\partial F}{\partial y} = 18y,\ \dfrac{\partial F}{\partial z} = 4$

At (0, 2, 3), $\dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 36,\ \dfrac{\partial F}{\partial z} = 4$

$\therefore$ Equation of tangent plane at (0, 2, 3) is

$0\,(x - x_1) + 36\,(y - y_1) + 4\,(z - z_1) = 0$

i.e. $9\,(y - y_1) + (z - z_1) = 0$

Equation of normal at (0, 2, 3) is

$$\frac{x - 0}{0} = \frac{y - 2}{36} = \frac{z - 3}{4}.$$

$\Rightarrow$ $x = 0$ and $9z - y = 25.$