TABLE TOP EXPERIMENT 2
KEPLER'S THIRD LAW OF PLANETARY MOTION

Structure

T2.1 INTRODUCTION

You know that interpreting data is an essential feature of any scientific endeavour. The data could be obtained through measurements made in the laboratory or through observations carried out in natural surroundings. These need to be analysed systematically to arrive at the underlying relationship or law between the variables being measured or observed. You may have studied some methods of data analysis in your physics and mathematics elective courses.

In this table top experiment, we have chosen to illustrate how the method of least squares is used to fit data to various curves and thereby arrive at the correct relationship between experimental variables. As an application of this method, we have chosen the observations on the movements of planets which were interpreted by Kepler to arrive at his famous laws of planetary motion. Specifically, we have provided the data corresponding to Kepler's third law. We expect you to apply the method of least squares to obtain the appropriate relationship and verify this law.

Objectives

After doing this table top expriment you should be able to

- apply the method of least squares, and
- verify Kepler's third law from given data

T2.2 THE BASIC PRINCIPLE

According to Kepler's third law we have $T^2 \propto R^3$, where T is the time-period of revolution of a planet round the sun and R is the mean distance of the planet from the sun. We shall verify this law. For that we assume

$$T = kR^p \qquad \text{(T2.1)}$$

where k and p are constants to be determined. From Eq. (T2.1), we get

$$\log T = \log k + p \log R \qquad \text{(T2.2)}$$

Eq. (T2.2) is of the form

$$y = a + bx \qquad \text{(T2.3)}$$

where $y = \log T$ and $x = \log R$, $\log k = a$, $p = b$.

Our aim is to determine p which is directly given by b. However, we also get k which is given by $k = 10^a$. In order to determine a and b we shall apply the **method of least squares.**

Now, what is the method of least squares? You are familiar with the method of drawing free hand curves representing the variation of a variable, say y, with respect to another variable, say x, when a set of corresponding pairs of values of x and y are given. Two typical cases are shown in Fig. T2.1.

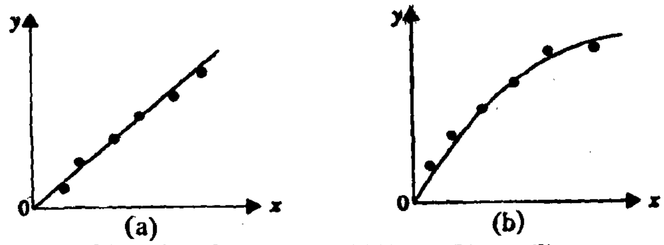

Fig. T2.1 : Free-hand curves (a) linear (b) curvilinear

You must have realised that the drawing of free-hand curves depends entirely on individual judgment. Hence, it has the disadvantage that different individuals will get different kind of variation of y with respect to x. This disadvantage is taken care of in the method of least squares. Let us see how.

Suppose we have n pairs of observations; (x_1, y_1); (x_2, y_2); (x_3, y_3); ......, (x_n, y_n) and we have to fit Eq. (T2.3) to these data. According to Eq. (T2.3) the estimated values of y, when $x = x_1, x_2, \ldots, x_n$ would be $a + bx_1, a + bx_2, \ldots, a + bx_n$. Now, in order that the equation $y = a + bx$ gives a good representation of the relationship between x and y, it is desirable that the estimated values $a + bx_1, a + bx_2, \ldots, a + bx_n$, are, on the whole, sufficiently close to the corresponding observed values $y_1, y_2, \ldots, y_n$.

The deviation in case of the ith value is given by

$$e_i = y_i - a - bx_i \qquad \text{(T2.4)}$$

The deviation may be positive or negative but e^2_i is always positive. As we are aiming at minimising the deviation on the whole, we take the sum-total of the squares of the deviations and squares being all positive, the question of positive and negative deviations is taken care of. Let the sum total be E, i.e.,

$$E = \sum_i e^2_i = \sum_i (y_i - a - bx_i)^2 \qquad \text{(T2.5)}$$

Our problem is thus to find a and b so that E is minimum. This can be obtained from the following conditions

$$\frac{\partial E}{\partial a} = 0 \qquad \text{(T2.6)}$$

and

$$\frac{\partial E}{\partial b} = 0 \qquad \text{(T2.7)}$$

From Eqs. (T2.6) and (T2.7), we get

$$a = \frac{(\sum_i x_i)(\sum_i y_i x_i) - (\sum_i x^2_i)(\sum_i y_i)}{(\sum_i x_i)^2 - n\sum_i x^2_i} \qquad \text{(T2.8)}$$

$$b = \frac{(\sum_i x_i)(\sum_i y_i) - n(\sum_i x_i y_i)}{(\sum_i x_i)^2 - n\sum_i x_i^2} \qquad \text{(T2.9)}$$

SAQ 1

Prove Eqs. (T2.8) and (T2.9).

Now that you have derived Eqs. (T2.8) and (T2.9), you will be able to obtain the values of 'a' and 'b' from a given set of data corresponding to values of x and y.

For your calculations, it would be convenient to prepare a table as given below.

Table T2.1

x_i	y_i	x_i^2	$x_i y_i$	$x_i^2 y_i$
x_1	y_1	x_1^2	$x_1 y_1$	$x_1^2 y_1$
x_2	y_2	x_2^2	$x_2 y_2$	$x_2^2 y_2$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
x_n	y_n	x_n^2	$x_n y_n$	$x_n^2 y_n$
$\sum_i x_i$	$\sum_i y_i$	$\sum_i x_i^2$	$\sum_i x_i y_i$	$\sum_i x_i^2 y_i$

Remember that in your case $x = \log R$ and $y = \log T$.

T2.3 VERIFICATION OF KEPLER'S THIRD LAW

You can now use the data given in Table T2.2 of corresponding values of T and R for the planets and verify Kepler's third law of planetary motion.

Table T2.2 : T vs R data for the Planets

Planet	Period of Revolution in years (T)	Mean distance from the sun in 10^6 km (R)
Mercury	0.241	57.9
Venus	0.615	108.2
Earth	1.000	149.6
Mars	1.881	227.9
Jupiter	11.862	778
Saturn	29.458	1427
Uranus	84.013	2870
Neptune	164.793	4497
Pluto	248.430	5912