UNIT 3 ELECTRIC POTENTIAL

Structure

3.1 INTRODUCTION

In Unit 1 of this Block, you have learnt that the force between any two charges is explained by the Coulomb's Law. The force experienced by a unit positive charge also defines the strength of the electric field E at a point. The computation of E directly or through the use of Gauss's law was the topic of discussion in the first two units of this block.

In most problems in electrostatics, our aim is to calculate E. Since E is a vector quantity, its computation requires calculation of each of its component. Many a time, to make this computation easier, we first calculate a scalar quantity known as the electrostatic potential ϕ, from which E can be calculated by a simple relation. Since ϕ is a scalar, its computation in most cases is not so difficult as in the case of electric field. The concept of potential is also important because potential is closely linked to the work done by the charged particles and their energies. The study of potential, its difference between any two points, and its connection with E are the main topics of discussion of this Unit. However, for developing the concept of ϕ and for obtaining its relation with E, we introduce two new mathematical concepts in this Unit. These are: (i) the gradient of a scalar function, and **(ii)** the line integral of a vector.

In the next Unit, we will further discuss various related topics like potential due to continuous charge distributions, **equipotential** surfaces and electrostatic potential energy.

Objectives

After studying this unit, you should be able to:

- compute the work done in taking a charge q from one point to another,
- show that the line integral of the electric field over a closed path is equal to zero,

- compute the electric potential at a point due to a single charge,
- relate the electric potential and electric field, and thereby compute the electric field at a point knowing the electric potential;
- compute the electric potential at a point due to a dipole and a quadrupole, and
- compute the torque experienced by an electric dipole in a uniform electric field.

3.2 MATHEMATICAL BACKGROUND

The concepts of gradient of a scalar function and of line integral of a vector have been used in this Unit. These concepts are useful in describing the physical topics of this Unit. Let us, therefore, understand these concepts. If you have recently mastered the concepts given in the course of Mathematical Methods in Physics-I (PHE-04), you may skip this section.

3.2.1 Gradient of a Scalar

You know there are three different products involving vectors: the multipiication of a vector A by a scalar **(kA),** the scalar or dot product of two vectors **(A·B)** and the vector or cross product of two vectors (A x **B**). Similarly, there are a number of differential operations involving vectors, all with different uses. The simplest of these is the differentiation of a radius vector with respect to a scalar such as time denoted as $\frac{d\mathbf{r}}{dt}$. The other differential operations connected with vectors involve the vector differential operator, del denoted as ∇. This, in expanded form, in Cartesian coordinates, is:

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \qquad ...(3.1)$$

Here $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are partial derivatives. If, say, $f(x, y, z)$ is a function of x, y and z, then $\frac{\partial f}{\partial x}$ is a partial derivative **of** f with respect to x, keeping both y and z as constant. **Similarly,** $\frac{\partial f}{\partial y}$ is a partial derivative **of** f with respect to y, when both x and z are kept constants.

You may note that ∇ is a vector, but is also an operator which has to operate on something which appears on its right. You may remember that since ∇ is an operator, it cannot appear on the right of something on which it is supposed to operate.

When ∇ operates on a scalar field or function $f(x, y, z)$, it determines its derivative with respect to the space coordinates x, y and z.

The gradient of a scalar field, say, potential ϕ gives a vector field, as follows:

$$\nabla\phi = \hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z} \qquad ...(3.2)$$

The magnitude of $\nabla\phi$ determines the maximum spatial rate of change of ϕ. The direction of $\nabla\phi$ is that for which there is maximum change in ϕ. It is always perpendicular to surfaces of constant ϕ.

To understand the physical meaning of gradient operation, consider a temperature in a **room** that is being heated by **sun from** one side. The temperature in the room varies from point **to point** and can be written as a function of coordinates (x,y,z).

Let this be denoted by $T(x, y, z)$. Now suppose we ask about its rate of vaiiation in space. Clearly, in different directions, it varies with different magnitudes. If we move from a point (x_0, y_0, z_0) through a distance Δr, the change in temperature ΔT will be different in different directions. If Δr is small, we can use Taylor's expansion in three variables to calculate the temperature difference. In Unit 2, Section 2.5, Taylor's expansion of a function of a single variable has been mentioned. If the function depends on three variables, as in this case, we can write.

$$T(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) = T(x_0, y_0, z_0) + \Delta x \frac{\partial T}{\partial x_0}(x_0, y_0, z_0) + \Delta y \frac{\partial T}{\partial y_0}(x_0, y_0, z_0) + \Delta z \frac{\partial T}{\partial z_0}(x_0, y_0, z_0) + \ldots \qquad \ldots(3.3)$$

As Δx, Δy and Δz are very small, the higher order terms are neglected in this expansion.

Note that Δx, Δy, Δz form the vector components of the displacement vector Δr, so we may write

$$T(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) = T(x_0, y_0, z_0) + \Delta \mathbf{r} \cdot \nabla T(x_0, y_0, z_0)$$

$$\because \nabla T = \hat{\mathbf{i}} \frac{\partial T}{\partial x_0} + \hat{\mathbf{j}} \frac{\partial T}{\partial y_0} + \hat{\mathbf{k}} \frac{\partial T}{\partial z_0}$$

Or more compactly

$$T(\mathbf{r}_0 + \Delta \mathbf{r}) = T(\mathbf{r}_0) + \Delta \mathbf{r} \cdot \nabla T(\mathbf{r}_0) \qquad \ldots(3.4)$$

Thus, the knowledge of ∇T enables us to know temperature variation in an arbitrary direction.

From this equation, we can clearly understand the meaning of the direction of the gradient operation. Suppose we ask, in which direction does the T–field varies most. From Eq. **(3.4)**, it is clear that $\Delta T = T(\mathbf{r}_0 + \Delta \mathbf{r}) - T(\mathbf{r}_0)$ is maximum when $\Delta \mathbf{r}$ is parallel to ∇T. Thus, ∇T has the direction in which the variation of T field is most rapid. Another point about gradient is the following. In the above example, we can draw surfaces in the room on which the temperature is constant. For example, through $\mathbf{r}_0$, we can find a surface along which the temperature has a fixed value which is the value at $\mathbf{r}_0$. Such surfaces are called, in general, equipotential **surface** (isothermals for temperature field, isobars for pressure field). Now, by definition, the component of ∇T along the surface in any direction is zero. As $\Delta T = \Delta \mathbf{r}_s \cdot \nabla T = 0$ where $\mathbf{r}_s$ is along the surface ∇T must be perpendicular to the equipotential surface.

To make certain that you can really calculate the gradient of a scalar function, we are giving a solved example.

Example 1

Find grad ϕ, if $\phi = Ax^2 yz^3$, where A is a constant,

Solution

$$\text{grad } \phi = \left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z}\right) A\,(x^2 yz^3)$$

$$\text{grad } \phi = \hat{\mathbf{i}} \frac{\partial}{\partial x}(Ax^2 yz^3) + \hat{\mathbf{j}} \frac{\partial}{\partial y}(Ax^2 yz^3) + \hat{\mathbf{k}} \frac{\partial}{\partial z}(Ax^2 yz^3)$$

Now, for finding $\frac{\partial}{\partial x}(x^2 yz^3)$, we assume yz^3 as constant, and find the derivative of x^2 with respect to x. We follow a similar procedure for the other two terms. This gives:

$$\nabla \phi = 2Ax\, yz^3\, \hat{\mathbf{i}} + Ax^2 z^3\, \hat{\mathbf{j}} + 3Ax^2 yz^2\, \hat{\mathbf{k}}$$

Note that $\nabla \phi$ is a vector.

3.2.2 Line Integral of a Vector

A line integral is an important vector field operation. It means integral along a **curve** (or line), that is, a single integral as contrasted to an area integral over a surface or a **volume** integral over a volume. The essential point to understand about a line **integral** is that only one independent variable needs to be varied for moving along a curve because in any dimensions the equation of a curve can be written in terms of a single independent variable.

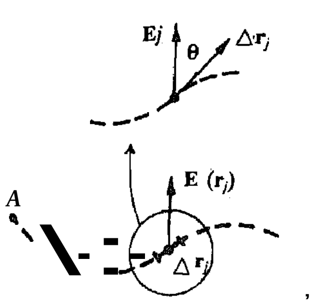

Fig. 3.1: Line integral of a vector field. The **direction** of **vector** Δr_j is along the tangent to the point under consideration on the curve C.

Suppose we want to evaluate the line integral of a vector field **E** from a point A to a point B along a curve C in a particular region of space. See Fig. 3.1. Divide the curve C into a large number of small parts $\Delta\mathbf{r}_1$, $\Delta\mathbf{r}_2$,... $\Delta\mathbf{r}_3$, ..., $\Delta\mathbf{r}_n$. These $\Delta\mathbf{r}$'s are **along** tangents to the curve C at various points. Take the scalar product of the vector field E at the jth part and the **displacement** vector $\Delta\mathbf{r}_j$, of the jth part, and then sum over all parts, **i.e.,**

$$\mathbf{E}(\mathbf{r}_1)\cdot\Delta\mathbf{r}_1+\mathbf{E}(\mathbf{r}_2)\cdot\Delta\mathbf{r}_2+\ldots\ldots+\mathbf{E}(\mathbf{r}_n)\cdot\Delta\mathbf{r}_n$$

$$=\sum_{j=1}^{n}\mathbf{E}(\mathbf{r}_j)\cdot\Delta\mathbf{r}_j \qquad \ldots(3.5)$$

Since the curve C is continuous, the limit of this expression when $n\to\infty$ and $|\Delta\mathbf{r}_j|\to 0$ is called the line integral of the vector field E along the path C, **i.e.,**

$$\int_{A\,C}^{B}\mathbf{E}\cdot d\mathbf{r}=\underset{n\to\infty,\,\Delta r_j\to 0}{\text{Limit}}\sum_{j=1}^{n}\mathbf{E}(\mathbf{r}_j)\cdot\Delta\mathbf{r}_j \qquad \ldots(3.6)$$

A line integral is thus an integral over the product of the component of E along the direction of dr (**i.e.,** along the tangent to the curve) and dr.

$$\int_{A\,C}^{B}\mathbf{E}\cdot d\mathbf{r}=\int_{A\,C}^{B}|\mathbf{E}|\;|d\mathbf{r}|\cos\theta$$

The value of line integral depends on the relative orientation of E and $d\mathbf{r}$ which, in general, varies in space. If they are parallel at each point of space, then $\theta=0$, and $\int_{A\,C}^{B}\mathbf{E}\cdot d\mathbf{r}$ is **simply equal** to $\int_{A\,C}^{B}|\mathbf{E}|\;|d\mathbf{r}|$. If, at each point, they are perpendicular to each other, **i.e.,** $\theta=90°$, then $\int_{A}^{B}\mathbf{E}\cdot d\mathbf{r}=0$.

SAQ 1

Let A and B refer to **any** two points in a **vector** field E due to a point charge. Show that the line integral of the vector field E from B to A is negative of that from A to B, **i.e.,**

$$\int_{B}^{A}\mathbf{E}\cdot d\mathbf{r}=-\int_{A}^{B}\mathbf{E}\cdot d\mathbf{r} \qquad \ldots(3.7)$$

The most common use of a line integral is the calculation of work by a force **F.** The **work** (W) done by a force **F** in taking an object from a point C to a point D is commonly written as:

$$W=\int_{C}^{D}\mathbf{F}\cdot d\mathbf{r}$$

where $d\mathbf{r}$ is a displacement vector, You will learn about it in next section.

Example 2

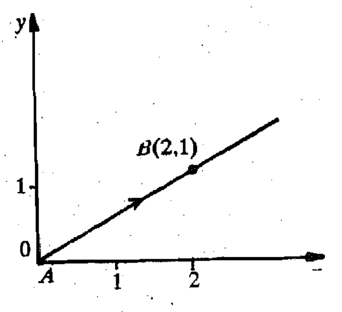

Fig. 3.2: Path AB for motion of object.

Calculate the work done by a force $F=(2y\hat{i}+6xy\hat{j})$ N in moving **an** object along a straight line from $A(0,0)$ to $B(2,1)$ in the xy plane as shown in Fig. 3.2.

Solution

We know that

Work done, $W = \int_A^B \mathbf{F} \cdot d\mathbf{r}$

$$= \int_A^B (2y\hat{\mathbf{i}} + xy\hat{\mathbf{j}}) \cdot (dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}})$$

where $d\mathbf{r} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}}$ in Cartesian coordinates.

$$\therefore W = \int_A^B (2y\,dx + xy\,dy) \qquad \text{as } \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = 1 \text{ and } \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{i}} = 0$$

Since the equation of straight line joining A and B can be written by calculating its slope, we get $y = x/2$ and

$$dy = \frac{dx}{2}.$$

Substituting these in the equation of work done and writing the integral in one variable, i.e., x, we get :

$$W = \int_0^2 \left(x\,dx + \frac{x^2}{4}\,dx \right)$$

Here, the limits on the values of x are those given by the x coordinates of points A and B, respectively. Solving the integrals, we get:

$$W = \left[\frac{x^2}{2} \right]_0^2 + \left[\frac{x^3}{12} \right]_0^2$$

$$\therefore W = 2 + \frac{2}{3} = 2.67\,\text{J}$$

SAQ 2

Calculate the wdrk done by a force $\mathbf{F} = xy\,\hat{\mathbf{i}} - y^2\,\hat{\mathbf{j}}$ in moving an object from (0, 0) to (2,1) along the parabolic path $y = \frac{x^2}{4}$.

3.3 WORK DONE IN MOVING A CHARGE

You have seen in Unit 1 that a single charge (or a combination of charges) produces an electric field in its vicinity. The electric field E at a point is defined as the force experienced by a unit positive charge placed at that point. If, instead of a unit positive charge, we place a charge q' at that point, then force $\mathbf{F}$ experienced by the charge q' in the electric field E is given by $\mathbf{F} = q'\mathbf{E}$. If we move this charge q' against the force $\mathbf{F}$ from a point A to a point B through a small distance $d\mathbf{r}$ as shown in Fig. 3.3, we have to do work against this force. This work done may be written as:

$$dW = -\mathbf{F} \cdot d\mathbf{r} = -|\mathbf{F}|\,|d\mathbf{r}|\cos\theta \qquad \text{...(3.8)}$$

and hence

$$W = -\int_A^B \mathbf{F} \cdot d\mathbf{r} \qquad \text{...(3.9)}$$

Here, $F\cos\theta$ determines the component of **F** along the displacement vector $d\mathbf{r}$. Notice that in order to move the charge q' against the force F produced by another charge, an external force $\mathbf{F}_{ext}$ has to be applied. This $\mathbf{F}_{ext}$ is equal in magnitude but opposite in sign to **F**, i.e., $\mathbf{F}_{ext} = -\mathbf{F}$.

Fig. 3.3: Movement of charge in an electric field, produced by Q.

3.3.1 Line Integral of Electric Field

Now let us extend the foregoing discussion and move the charge q' from A to B along the path shown in Fig. 3.4. The path AB lies within the region of electric field E. Divide this path into a very large number of small segments each with a length dr. The vector $d\mathbf{r}$ represents the direction as well as the length of any segment'of this path. The work done in carrying the charge q' from A to B (see Fig. 3.4) is the negative of the scalar product of the force F and the displacement vector $d\mathbf{r}$, i.e., $-\mathbf{F}\cdot d\mathbf{r}$.

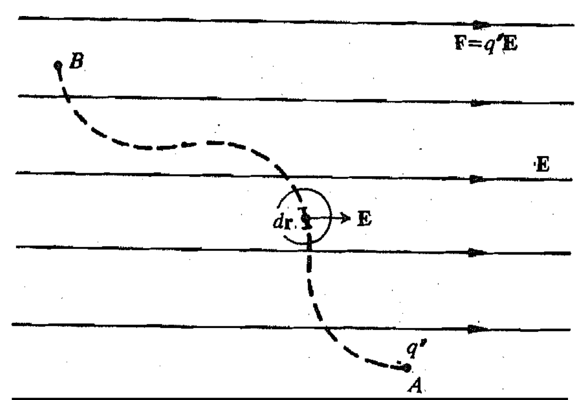

Fig. 3.4: Movement of charge q' from A to B within an electric field E.

If we add the work done in all the infinitesimal segments between A and B, we get the total work done in moving the charge q' from A to B. In the limit, where the number of these segments approaches infinity, work W done in moving the charge q' from A to B is written as

$$W = -\int_A^B \mathbf{F}\cdot d\mathbf{r} = -q'\int_A^B \mathbf{E}\cdot d\mathbf{r} \qquad ...(3.10)$$

Now, what happens if instead of the charge q', we move only a unit positive charge between A and B? You can easily see that the work W' done in that case would be obtained simply by dividing W by q', i.e.,

$$W' = \frac{W}{q'} = -\int_A^B \mathbf{E}\cdot d\mathbf{r} \qquad ...(3.11)$$

The integral on the right-hand side is called the **line integral** of **the electric field.** So the line integral of the electric field along any path is equal in magnitude to the work done in taking a unit positive charge along that path.

Let us now see if the integral given in Eq. (3.11) depends on the path between A and B, or is independent of that. The reason why we are interested in finding it out will be clear to you later in Secs. 3.4 and 3.5.

3.3.2 Path Independence of Line Integral of Electric Field

Let us first consider the field due to a charge q. Let there be two points A and B at distances r_A and r_B from the charge q. Let us try to carry a unit positive charge from A to B along the path shown by the coloured line in Fig. 3.5(a). Eq. (3.11) in that case may be written as:

$$W' = -\int_A^B \mathbf{E} \cdot d\mathbf{r} = -\int_A^{A'} \mathbf{E} \cdot d\mathbf{r} - \int_{A'}^{B} \mathbf{E} \cdot d\mathbf{r} \qquad ...(3.12)$$

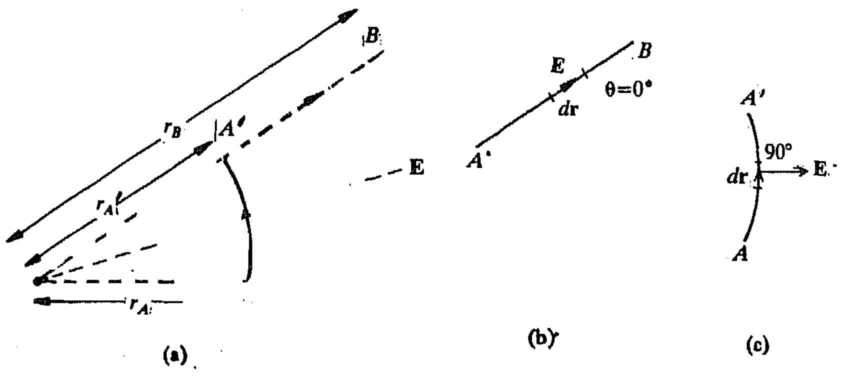

Fig. 3.5: Work done in carrying a unit positive charge from A 'to B along the path shown as coloured.

The first integral on the right-hand-side represents the work done in taking a unit charge from A to A' along the arc of a circle of radius r_A. The second integral represents the work done in taking the same charge from A' to B along the radius of a bigger circle. The first integral $\mathbf{E} \cdot d\mathbf{r}$ is equal to zero as E and $d\mathbf{r}$ are perpendicular to each other as shown in Fig. 3.5(b). The second integral

$$\int_{A'}^{B} \mathbf{E} \cdot d\mathbf{r} = \int_{A'}^{B} |\mathbf{E}|\,|d\mathbf{r}|$$ as both E and $d\mathbf{r}$ are parallel to each other along $A'B$.

Can you tell why it is so? This is because, in the case of first integral, $\cos\theta = 0$ and in the second integral $\cos\theta = 1$.

Let us now work out the second integral in more detail. We know from Unit 1 that the electric field E for a charge q at a distance r is given by

$\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector along the direction of r. Using this, and

replacing $d\mathbf{r}$ by $\hat{\mathbf{r}}\,dr$, (since dr is a vector element along the direction of $\hat{\mathbf{r}}$), we get:

$$\int_{A'}^{B} \mathbf{E} \cdot d\mathbf{r} = -\int_{r_{A'}}^{r_B} \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}}{r^2}\,(dr) = -\int_{r_A}^{r_B} \frac{q}{4\pi\epsilon_0}\frac{dr}{r^2} \quad \text{since } r_{A'} = r_A$$

$$= -\frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$$

Therefore, from Eq. (3.12)

$$W' = -\int_A^B \mathbf{E} \cdot d\mathbf{r} = -\frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] \qquad ...(3.13)$$

along the path as shown between A and B.

Let us now consider another arbitrary path (shown coloured) to carry a unit positive charge from A to B within the same field as above (See Fig. 3.6.)

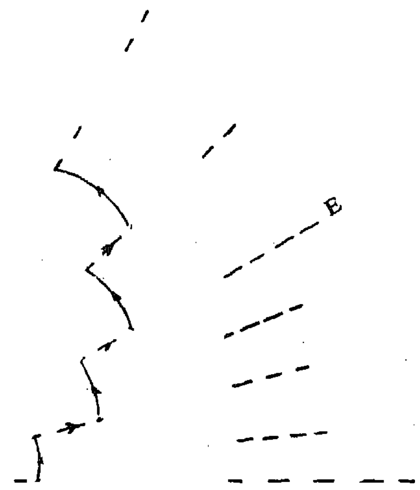

Fig. 3.6: Work done along the path (shown coloured) between A and B in carrying a unit positive charge.

We can calculate the work done in the same manner along this path. The unit charge dong the path goes for a while along the arc of a circle, then radially for a while, then again along an arc, again radially, and so on. We may note that every time when the unit charge goes along the arc, no work is done. However, whenever the unit charge goes **along** the radial path, work is done. So, we may write:

$$-\int_A^B \mathbf{E}\cdot d\mathbf{r} = -\int_{A_1}^{A_2} \mathbf{E}\cdot d\mathbf{r} - \int_{A_3}^{A_4} \mathbf{E}\cdot d\mathbf{r} - \int_{A_5}^{A_6} \mathbf{E}\cdot d\mathbf{r} - \int_{A_7}^{B} \mathbf{E}\cdot d\mathbf{r} \qquad \ldots(3.14)$$

From Eq. **(3.13),** the work done along each of these radial **stretches** is:

$$-\int_{A_1}^{A_2} \mathbf{E}\cdot d\mathbf{r} = -\frac{q}{4\pi\,\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_{A_2}}\right]$$

$$-\int_{A_3}^{A_4} \mathbf{E}\cdot d\mathbf{r} = -\frac{q}{4\pi\,\epsilon_0}\left[\frac{1}{r_{A_2}} - \frac{1}{r_{A_4}}\right]$$

$$-\int_{A_5}^{A_6} \mathbf{E}\cdot d\mathbf{r} = -\frac{q}{4\pi\,\epsilon_0}\left[\frac{1}{r_{A_4}} \quad \frac{1}{r_{A_6}}\right]$$

and

$$-\int_{A_7}^{B} \mathbf{E}\cdot d\mathbf{r} = -\frac{q}{4\pi\,\epsilon_0}\left[\frac{1}{r_{A_6}} - \frac{1}{r_B}\right]$$

Adding all these together, we get

$$W' = -\int_A^B \mathbf{E}\cdot d\mathbf{r} = -\frac{q}{4\pi\,\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] = \frac{q}{4\pi\,\epsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right] \qquad \ldots(3.15)$$

Comparing **Eqs.** (3.13) and (3.15). we note that the work done in carrying a unit positive charge in the field of a point charge q is the same along the two different **paths.**

From the above representative example, we can conclude that we get the same result for any path between A and B. It implies that the line integral of the electric field is independent of the path between A and B.

SAQ 3

Calculate the work done in moving a unit positive charge through a distance d in a uniform electric field parallel to the field direction.

3.3.3 Consequence of Path Independence

We have seen just now that the work done in moving a unit charge against the electrostatic force is independent of the path taken between the initial and final positions. Using Eqs. (3.11) and (3.15), we can write for the work done in moving a charge q' from A to B in the field of another fixed charge q as

$$W' = -\int_A^B q'\,\mathbf{E}\cdot d\mathbf{r} = \frac{qq'}{4\pi\epsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$$\therefore\; W' = (U_B - U_A) \qquad \text{where } U = \frac{qq'}{4\pi\epsilon_0}\left(\frac{1}{r}\right) \qquad ..(3.16)$$

U is the electrostatic potential energy. In such a situation, we can express the work done in terms of difference in potential energies of a charge at two points. Under this condition, we store the energy used in changing the position of the charge and this energy is recovered by allowing the charge to return to its initial position. The potential energy of a charge in an electric field, about which we are talking, is a measure of the energy stored in it by virtue of its position relative to the charges which give rise to the electric field. You will read more about the electrostatic potential energy and the consequence of path independence in the next unit.

3.4 ELECTRIC POTENTIAL

In the last section, we have seen that the work done in moving a unit positive charge from A to B in an electric field E of a point charge q is independent of the path between A and B, and depends only on the end points A and B. We can, therefore, represent it as a difference between two numbers (or scalars). These numbers, from Eq. (3.15), are $\frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_B}\right)$ and $\frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_A}\right)$ respectively. If we denote these numbers by ϕ_B and ϕ_A respectively, then their difference ϕ_{BA} is

$$\phi_{BA} = \phi_B - \phi_A = -\int_A^B \mathbf{E}\cdot d\mathbf{r} \qquad ...(3.17)$$

Let us now see what happens if our initial point A is at infinity. Following the discussion prior to Eq. (3.13), we can write

$$\phi_{BA} = -\int_A^B \mathbf{E}\cdot d\mathbf{r} = -\int_A^B \frac{q\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\,dr}{4\pi\epsilon_0 r^2}$$

$$= -\frac{q}{4\pi\epsilon_0}\int_{A=\infty}^{B}\frac{dr}{r^2} = -\frac{q}{4\pi\epsilon_0}\left|-\frac{1}{r}\right|_{r_A=\infty}^{r_B}$$

$$= -\frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r_B} + \frac{1}{\infty}\right] = \frac{q}{4\pi\epsilon_0 r_B} = \phi_B \qquad ...(3.18)$$

The scalar $\phi_B = \frac{q}{4\pi\epsilon_0 r_B}$ in Eq. (3.18) is usually referred to as the electrostatic potential per unit charge (or simply potential) at a point distant r_B from a charge q. You also know that the negative of the line integral of the electric field gives the work done in taking a unit positive charge from one point to another in an electric field. So, the electric potential at any point B at a distance r_B from a charge q (see Fig. 3.7) is usually defined as the work done in bringing a unit positive charge from infinity up to that point. Therefore, SI unit for potential is the **Joule/Coulomb.** This combination occurs so often that a special unit, the volt (abbreviation V), is used to represent it.

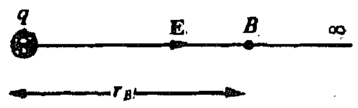

Fig. 3.7: Potential at a point B due to a charge q.

Electric potential ϕ_r due to a point charge q at a distance r from it is:

$$\phi_r = \frac{q}{4\pi\epsilon_0 r} \quad ...(3.19)$$

Before moving further, try to solve the following SAQ using the expression given above.

SAQ 4

What is the electric potential at the surface of a gold nucleus? The radius of a gold nucleus is 6.6×10^{-15}m and the atomic number of gold is Z = 79. Assume, the nucleus acts as a point charge, and electronic charge e = 1.6×10^{-19}C.

Have you figured out why we have involved infinity in our definition of electric potential? If you have not, let us tell you. If, for r_B we insert ∞ in Eq. **(3.18),** we find $\phi_B = 0$. This means that we define the potential at any point relative to a point at ∞ where the potential due to any charge is equal to zero.

From Eq. **(3.18),** we note that, for a positive charge, the potential at a point'is positive; while for a negative charge, it is negative. Do you know why? This is because, for a positive charge, the work has to be done in bringing the unit positive charge from infinity **against** the **repulsive** force of a positive charge, and hence it is positive. For a negative charge, on the other hand, the work is done by the electric field while bringing the test charge from infinity (this work done is negative).

You can notice from this discussion that, when work is done against the force (in this **case** electric field), potential (energy) of the system increases. This can be easily understood **by** taking an example in the case of **gravitational** field. When a body of mass 'm' is raised to a height 'h', without giving any acceleration, against the force of gravity mg acting downwards, then the potential energy of the body increases. Here, work is done against gravity. When work is done by the force of gravity as in the free fall of a body, the potential (energy) decreases. The difference in potential gets converted into kinetic energy of the freely falling **object.**

3.4.1 Potential due to a System of Charges

If, instead of a single charge, we have a system of charges, we have to use the superposition principle. That is, the resultant potential ϕ_P at a point P due to a system of charges $q_1, q_2, ..., q_N$ is equal to the **sum** of the potentials due to the **individual** charges at that point. If $r_1, r_2, ..., r_N$ are the distances of the charges $q_1, q_2, ..., q_N$ respectively from the point P, the potential at that point is:

$$\phi_P = \frac{q_1}{4\pi\epsilon_0 r_1} + \frac{q_2}{4\pi\epsilon_0 r_2} + ... + \frac{q_N}{4\pi\epsilon_0 r_N}$$

You should note that here **each** charge is acting **as** if the other charge is not present. The potential at point P may be written in a summation form as:

$$\phi_P = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i} \quad ...(3.20)$$

As a caution, you may, keep in mind that the sum given in Eq. (3.20) is an algebraic sum and not a vector sum as the potential at a point is a scalar quantity.

Example 3

The following point charges are placed on the x-axis: $2\mu C$ at $x = 20cm$, $-3\mu C$ at $x = 30cm$, $-4\mu C$ at $x = 40cm$. Find the potential on the x-axis at $x = 0$.

Solution

We know that potential is a scalar and using the superposition principal, it is written as:

$$\phi = \frac{1}{4\pi \epsilon_0} \sum_{i=1}^{3} \frac{q_i}{r_i}$$

On substituting the numerical values of q_i and r_i, we get

$$\phi = 9 \times 10^9 \, Nm^2 C^{-2} \left[\frac{2 \times 10^{-6}C}{0.20m} - \frac{3 \times 10^{-6}C}{0.30m} - \frac{4 \times 10^{-6}C}{0.40m} \right]$$

$$= 9 \times 10^9 \, Nm^2 C^{-2} \, [10^{-5} \, Cm^{-1} - 10^{-5} Cm^{-1} - 10^{-5} Cm^{-1}]$$

$$\phi = -9 \times 10^4 \, NmC^{-1} = -9 \times 10^4 V.$$

If the charge distribution is continuous (like that on a charged sphere) rather than being a collection of various charges, the sum in Eq. (3.20) gets replaced by an integral. In that case, we may write:

$$\phi = \frac{1}{4\pi \epsilon_0} \int_{volume} \frac{\rho \, dV}{r} \qquad \ldots(3.21)$$

where ρ is the volume density of charge, dV is the element of volume, and r is a variable giving the distance of each point in the volume element to the point where the potential is being calculated. Eq. (3.21) can, however, be evaluated only when explicit expressions for the charge density and position for the entire charge distribution are available.

3.4.2 Potential Difference

The way we have defined the potential at a point allows us to define another very **useful** quantity **called** the **potential difference** (or the difference of potential) between two points.

Let us write down the amount of work done in bringing a unit positive charge from infinity first to point A and then to point B (see Fig. 3.8) within the field, of charge q. We have from Eq. (3.18)

$$\phi_A = W_1 = \frac{4}{4\pi \epsilon_0 r_A}$$

and

$$\phi_B = W_2 = \frac{q}{4\pi \epsilon_0 r_B}$$

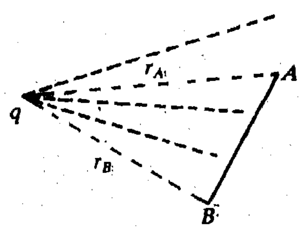

Fig. 3.8: Work done in taking a unit charge from A to B.

The difference of these two (**i.e., $\phi_B - \phi_A$**) is **the work done in taking a unit charge from A to B, and it is called the potential difference between the two points B and A,** It is written as:

$$\phi_{BA} = \phi_B - \phi_A = W_2 - W_1$$

$$= \frac{q}{4\pi \epsilon_0} \left[\frac{1}{r_B} - \frac{1}{r_A} \right] \qquad \ldots(3.22)$$

This is so because the work done in carrying the charge in an electric field is independent of path. It is just this path independence that enables us to define the concept of potential. If, instead of the unit positive charge, we transport a charge q between A and B, then the work W done is given by:

$$W = q'\phi_{BA} = q'(\phi_B - \phi_A) \quad ...(3.23)$$

The potential difference is a very important concept in the field of electrostatics and current electricity. Its knowledge helps us in determining the exact value of the current which flows between any two points in an electric circuit, provided the resistance between the two points is known.

SAQ 5

How much work is required to transport an electron from the positive terminal of a 12V battery to its negative terminal?

3.5 RELATION BETWEEN ELECTRIC FIELD AND ELECTRIC POTENTIAL

We have seen in Eq. 3.17 that the electric potential ϕ is related to the electric field E through a line integral. Thus, by knowing the electric field and evaluating the line integral, the potential at a point can be found out. We are now looking for a reverse relation: where knowing 4, E can be found out. This we do in the discussion which follows:

We have seen in Eq. (3.22) that the difference of potential ϕ_{BA} between two points B and A in a field of charge q is given by:

$$\phi_{BA} = \phi_B - \phi_A = \frac{q}{4\pi\,\epsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

The right-hand side of this equation is also equal to the negative of the line integral of the electric field E between the same two points (see Eq. 3.15). So, we get:

$$\phi_{BA} = -\int_{A}^{B} \mathbf{E}.\, d\mathbf{r} \quad ...(3.24)$$

Now, using Eq. (3.8), we can write the potential difference $d\phi$ between any two points separated by $d\mathbf{r}$ as

$$d\phi = -\mathbf{E}\cdot d\mathbf{r} \quad ...(3.25)$$

or $d\phi = -E\cos\theta\,|d\mathbf{r}|$

$$\text{or} \quad -E\cos\theta = \frac{d\phi}{|d\mathbf{r}|} \quad ...(3.26)$$

Because of the presence of $\cos\theta$ factor, we find that the electric field is a special kind of derivative of the potential. We call it the directional derivative.

In sub-section 3.2.1, you have studied how the variation of temperature in a room in different places in different directions can be written by using the gradient operator. In a similar manner, we can write the difference in electric potential, $d\phi$, between two neighbouring points in an electric field in terms of a gradient operator. This can then be related to the electric field E at a point. Thus

$$d\phi = \phi(x + dx, y + dy, z + dz) - \phi(x, y, z)$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(\hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z}\right)\cdot(\hat{\mathbf{i}}\,dx + \hat{\mathbf{j}}\,dy + \hat{\mathbf{k}}\,dz)$$

$$= \nabla\phi\,.\,d\mathbf{r} \quad ...(3.27)$$

Comparing this equation with Eq. (3.25), we get

$$\mathbf{E} = -\nabla\phi = -\left(\hat{\mathbf{i}}\,\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\,\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial\phi}{\partial z}\right) \quad ...(3.28)$$

The components of E along x, y, z directions are

$$E_x = -\frac{\partial\phi}{\partial x},\ E_y = -\frac{\partial\phi}{\partial y} \text{ and } E_z = -\frac{\partial\phi}{\partial z} \quad ..(3.29)$$

Thus, the electric field E is the negative of the gradient of the potential ϕ at any point. Since by gradient, we mean the slope, the value of the electric field (a vector) is found by evaluating the rate of change of potential (a scalar) along the direction of the field.

In the plane polar coordinate system, we use (r, θ) coordinates. This coordinate system will be used in finding the electric field due to a dipole in the next section. Therefore, the gradient (del) operator in this coordinate system has been derived in Appendix at the end of this unit. It is better that you go through this derivation for a better insight; however, you will not be examined on it.

Example 4

The electric potential at a point is given by the relation $\phi = Ax + By - Cz$ where A, B and C are constants. Find the value of electric field at that point.

Solution

Since $\phi = Ax + By - Cz$

and $\mathbf{E} = \left(-\hat{\mathbf{i}}\,\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\,\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial\phi}{\partial z}\right)$

$\therefore\ \mathbf{E} = -[A\hat{\mathbf{i}} + B\hat{\mathbf{j}} - C\hat{\mathbf{k}}]$

Now apply this method yourself to SAQ 6.

SAQ 6

The potential at any point is given by $\phi = x(y^2 - 4x^2)$. Calculate the electric field E at that point.

To further illustrate the relation between the electric field and potential, let us discuss in detail the case of a dipole and quadrupole.

3.6 ELECTRIC FIELD AND POTENTIAL OF AN ELECTRIC DIPOLE AND QUADRUPOLE

A pair of equal and opposite charges, $\pm q$, separated by a vector distance a is called a dipole (Fig. 3.9). The vector a, which is also along the axis of the dipole, is drawn from the negative to the positive charge. A molecule consisting of a positive and negative ion is an example of electric dipole in nature. An atom consists of equal amounts of positive and negative charges whose centres coincide; hence, it is neutral for all points outside the atom, In the presence of an external electric field, the centres of positive and negative charges get separated. It then becomes a dipole. The electric field and potential in the vicinity of a dipole forms the first step in understanding the behaviour of dielectrics under the influence of an external electric field. By arranging two oppositely directed dipoles in a line, we get a linear quadrupole. Its field and potential are more complicated as compared to those of a dipole. Let us first study the electric field and potential due to a dipole.

a) Electric field at a point P along the **axis** of the dipole

Let the distance between the mid-point of the dipole and the point P which is along the axis be equal to r (see Fig. 3.9). We shall evaluate the electric field at P. The electric field at P due to $+q$ is

$$\mathbf{E}_+ = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{(r - a/2)^2}$$

and that due to $-q$ is

$$\mathbf{E}_- = \frac{-4}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{(r + a/2)^2}$$

The resultant field at P is

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$$

$$= \frac{q\,\hat{\mathbf{r}}}{4\pi\,\epsilon_0}\left[\frac{1}{(r - a/2)^2} - \frac{1}{(r + a/2)^2}\right]$$

$$= \frac{q\,\hat{\mathbf{r}}\,2ar}{4\pi\,\epsilon_0\,(r^2 - a^2/4)^2}$$

$$\mathbf{E} \approx \frac{2\mathbf{p}}{4\pi\,\epsilon_0\,r^3} \text{ for } r >> a \qquad ...(3.30a)$$

where $\mathbf{p} = qa\hat{\mathbf{r}}$ ...(3.30b)

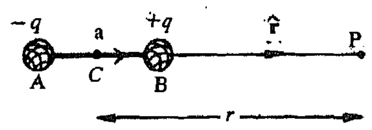

Fig. 3.9: Electric dipole *AS* with centre C and axis a (separation between positive and negative charges). The point *P* is along the axis.

Here, in the denominator, we have neglected $a^2/4$ as compared to r^2, since $a << r$ in actual physical problems. In Eq. (3.30a), $qa\hat{\mathbf{r}}$ has been replaced by p known as the dipole moment. In atomic and molecular dipoles, $a = 10^{-10}$m and $r >> a$.

(b) Electric field *at a point P on* the perpendicular bisector of the dipole axis

Let the distance between the centre of the dipole and the point P in this case be equal to *r*. See Fig. 3.10.

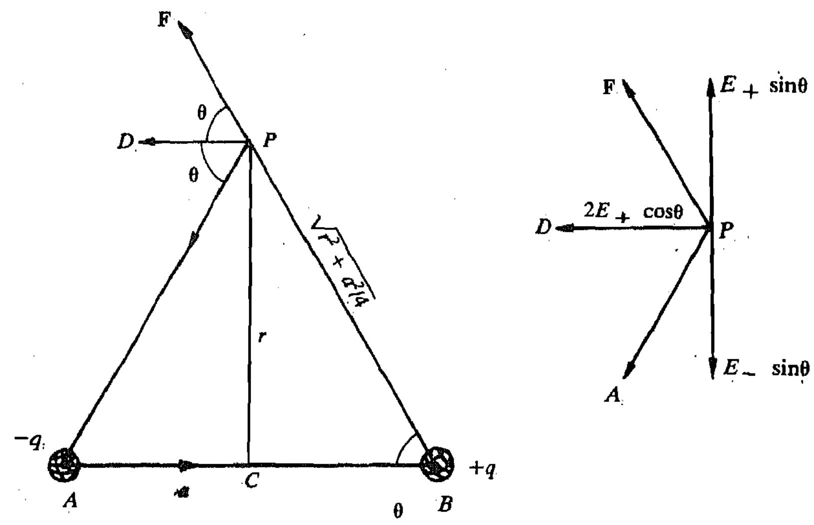

Fig. 3.10 : Electric field due to a dipole at a point P.

The electric field due to $+q$ at P is along PF and of magnitude

$$E_+ = \frac{q}{4\pi\,\epsilon_0}\left[\frac{1}{(r^2 + a^2/4)}\right]$$

The electric field due to $-q$ at P is along PA and of magnitude

$$E_- = \frac{q}{4\pi\,\epsilon_0}\left[\frac{1}{(r^2 + a^2/4)}\right]$$

In order to find the resultant electric field, draw PD perpendicular to CP. Now, if you resolve these two electric fields along *PD* and in the direction perpendicular to *PD,* you will notice that the components $E_+ \sin\theta$ and $E_- \sin\theta$ cancel each other, whereas the components $E_+ \cos\theta$ and $E_- \cos\theta$ add up along PD (shown in Fig. 3.10). As both are equal in magnitude, the resultant field is along PD and has magnitude

$$2E_+ \cos\theta = \frac{2q}{4\pi\epsilon_0}\left[\frac{\cos\theta}{(r^2 + a^2/4)}\right]$$

where $\cos\theta = \dfrac{a/2}{(r^2 + a^2/4)^{1/2}}$

The resultant field at P is along *PD* which is **antiparallel** to p. Thus, the vector field is given by:

$$\mathbf{E} = \frac{-\mathbf{p}}{4\pi\,\epsilon_0}\,\frac{1}{(r^2 + a^2/4)^{3/2}} \qquad ...(3.31)$$

$$\approx -\frac{\mathbf{p}}{4\pi\,\epsilon_0}\,\frac{1}{r^3} \quad \text{for } r >> a$$

(c) Potential due to a dipole

The electric field can also be evaluated from the electrostatic potential ϕ. To do this, let us evaluate the potential ϕ at P at a distant r from the mid-point *C* of the dipole. (See Fig. 3.11). The line joining *P* to *C* makes an angle θ with **a**. The potential at *P* is evaluated by using Eq. (3.19) for the two charges $-q$ and $+q$ of the dipole.

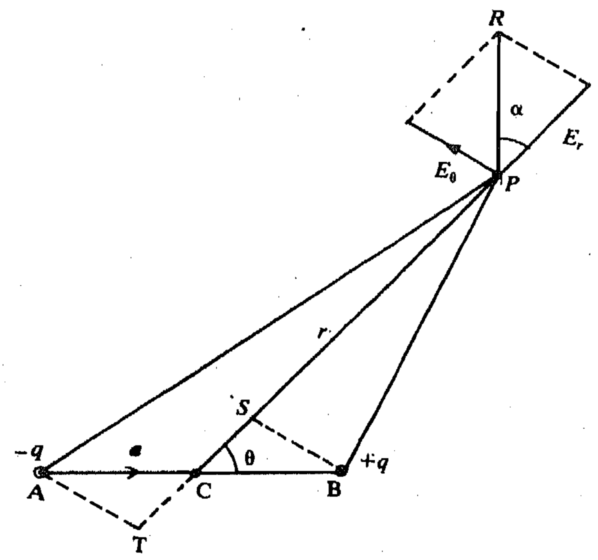

Fig. 3.11: Calculation of potential and field around dipole. BS, AT are perpendiculars to PC.

The distances of P from $-q$ and $+q$ are AP and BP respectively.

From the geometry, you will notice that BP $= SP = PC - CS = r - \frac{a}{2}\cos\theta$

and $AP = TP = TC + CP = r + \frac{a}{2}\cos\theta$

Thus, the potential at P is equal to:

$$\phi = \frac{q}{4\pi\,\epsilon_0}\left[\frac{1}{(r - \frac{a}{2}\cos\theta)} - \frac{1}{(r + \frac{a}{2}\cos\theta)}\right]$$

$$\phi = \frac{q\,a\cos\theta}{4\pi\,\epsilon_0\,(r^2 - a^2/4\,\cos^2\theta)}$$

When *P* is far away, r^2 is large compared to $\frac{a^2}{4}\cos^2\theta$ and neglecting $\frac{a^2}{4}\cos^2\theta$ in the denominator, we can write

$$\phi = \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\,\epsilon_0\,r^2}$$

$$\text{or } \phi = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} \quad \left[\because \mathbf{p}\cdot\hat{\mathbf{r}} = qa\,\hat{\mathbf{r}}\,.\,\hat{\mathbf{r}} = qa\cos\theta\right] \qquad \text{...(3.32)}$$

To find the electric field at P we-use the gradient operator in polar coordinates. (See Appendix at the end of this Unit). Thus,

$$\mathbf{E} = -\nabla\phi = -\left[\hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial\theta}\right]$$

$$\text{or } \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left[\hat{\mathbf{r}}\,(2\cos\theta) + \hat{\boldsymbol{\theta}}\sin\theta\right] \qquad \text{...(3.33)}$$

The resultant E is in the direction PR shown in Fig. 3.11 and has magnitude

$$|\mathbf{E}| = \frac{p}{4\pi\epsilon_0 r^3}\sqrt{3\cos^2\theta + 1}$$

$\mathbf{E}$ makes an angle α with $\hat{\mathbf{r}}$ given by:

$$\tan\alpha = \frac{1}{2}\tan\theta \qquad \text{...(3.34)}$$

For $\theta = 0$, point P lies along the axis of the dipole; in this case, only the radial component is present and we get the same result as in Eq. (3.30). For $\theta = \pi/2$, point P is on the perpendicular bisector of the dipole axis, the radial component is zero, $\cos\frac{\pi}{2} = 0$ and we get same result as in Eq. (3.31) as $\sin\frac{\pi}{2} = 1$ and θ is antiparallel to $\mathbf{p}$ for this point. From these you can conclude that:

i) the dipole potential varies as $1/r^2$ and the field as $1/r^3$ as compared to a point charge for which the potential varies as $1/r$ and the field as $1/r^2$. Thus, the potential and field decreases more rapidly with r for a dipole than for a point charge.

ii) the dipole potential vanishes on points which lie on the perpendicular bisector of the dipole axis; hence, no work is done in moving a test charge along the perpendicular bisector.

Let us now find the electric potential and electric field due to a quadrupole.

Consider the arrangement of 4 charges as shown in Fig. 3.12.

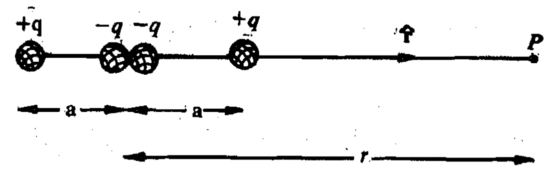

Fig. 3.12: Potential at P due to a quadrupole.

It is one form of a quadrupole. The potential at P due to this quadrupole is:

$$\phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r+a} - \frac{2}{r} + \frac{1}{r-a}\right]$$

$$\phi = \frac{q}{4\pi\epsilon_0}\left[\frac{2a^2}{r\,(r^2 - a^2)}\right]$$

In the denominator, one can neglect a^2 as compared to r^2 provided $r >> a$. This gives;

$$4 = \frac{2q\,a^2}{4\pi\epsilon_0 r^3} \qquad \text{...(3.35)}$$

The electric field in this case has only radial component, hence it is given by

$$\mathbf{E} = -\hat{\mathbf{r}}\frac{\partial\phi}{\partial r} = \frac{6q\,a^2}{4\pi\epsilon_0 r^4}\hat{\mathbf{r}} \qquad \text{...(3.36)}$$

This can also be proved by direct computation. In the case of a quadrupole, the potential decreases more rapidly, **i.e.,** as $1/r^3$ and the field also decreases rapidly as $1/r^4$ compared to that of the dipole and a point charge.

3.7 DIPOLE IN AN ELECTRIC FIELD

After having discussed the field and potential due to a dipole, we now turn our attention to the effect of an external electric field on a dipole.

Let us imagine a dipole in a uniform external electric field E. A uniform electric field means that its **magnitude** and direction are the same everywhere. Let p makes an angle θ with respect to the field direction as shown in Fig. 3.13.

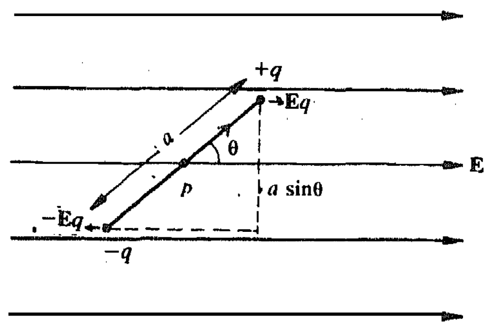

Fig. 3.13: Torque experienced by a dipole placed in a uniform electric field E.

Due to the external electric field, the charge +q experiences a force $\mathbf{F} = \mathbf{E}q$, while the charge $-q$ experiences an equal and opposite force $-\mathbf{E}q$. Since the field is uniform, the resultant force on the dipole is zero, **i.e.,**

$$\mathbf{F}_{total} = \mathbf{E}q - \mathbf{E}q = 0 \qquad ...(3.37)$$

As the resultant force is zero, the dipole is not accelerated, that is, there is no effect on its translatory motion. Does it mean that the external electric field has no effect on the dipole? No, it is not so.

The dipole still experiences a turning effect due to the torque which acts on it. This torque is there because the two equal and opposite forces, which cancel each other as free vectors, are acting at different points. They provide a turning effect. From Fig. 3.13, you can see that this torque has a magnitude $|\mathbf{F}|$ a sin8 = q $|\mathbf{E}|$ a $\sin\theta$. It has a clockwise turning effect; hence, it can be written as $\mathbf{p} \times \mathbf{E}$. Thus, the torque τ, which acts on a dipole, is given by:

$$\tau = \mathbf{p} \times \mathbf{E} \qquad ...(3.38)$$

The unit of torque is clearly Newton metre (N m). Under the action of the torque, the dipole aligns itself along the field direction with dipole moment vector p parallel to E vector.

In this position, the torque on the dipole is zero. The system being in a stable position, the potential energy of the dipole for this position is minimum. Therefore, in rotating the dipole from this position, the work done by an external agency is stored in the form of potential energy in the dipole.

Let us choose the potential energy of a dipole to be zero in an external electric field when angle θ = 90". This is an arbitrary choice to make the final result simpler. The potential energy' U for any other orientation θ is thus:

$$U(\theta) = W(\theta) = \int_{90^\circ}^{3} |\tau| \, d\theta = \int_{90^\circ}^{\theta} pE \sin\theta \, d\theta \qquad ...(3.39)$$

where $W(\theta)$ is the work done in turning the dipole from its reference orientation to angle 8. Evaluating the integral, we get:

$$U(\theta) = -pE \, |\cos\theta|_{90^\circ}^{\theta} = -pE\cos\theta$$

Writing this in vector form, the potential energy of a dipole is

$U(\theta) = -\mathbf{p} \cdot \mathbf{E}$ ...(3.40)

Eq. (3.40) shows that U is minimum (most negative) when the dipole is aligned along the field direction (i.e., $\theta = 0°$), and is maximum (most positive) when it is opposite to the field direction (i.e., $\theta = 180°$).

Let us now sum up what we have learnt in this unit.

3.8 SUMMARY

- The line integral $-\int_A^B \mathbf{E} \cdot d\mathbf{r}$ of the electric field E is equal to the work W done in taking a unit positive charge from the point A to B, i.e.,

  $$W = -\int_A^B \mathbf{E} \cdot d\mathbf{r}$$

- The work done in taking a unit positive charge from one point to another in an electric field is independent of the path chosen between the two points.
- The work done in carrying a unit positive charge from infinity to some point against the electric field is known as the potential at that point.
- The potential ϕ_r at a point at a distance r from a point charge q is given as

  $$\phi_r = \frac{q}{4\pi\epsilon_0 r}$$

- The difference of potential ϕ_{BA} between two points B and A is equal to the work done in taking a unit positive charge from A to B. If a charge q is taker. from A to B, then the work done is:

  $$W = q\,\phi_{BA} = q\,(\phi_B - \phi_A)$$

- The unit of potential difference is volt. The potential difference between points A and B is 1 volt when the work done in carrying unit positive charge between these two points is equal to one Joule.
- The electric field E at a point is the negative gradient of the potential ϕ at that point:

  $$\mathbf{E} = -\nabla\phi$$

- The electric field of a dipole at a point along the axis of the dipole is given by:

  $$\mathbf{E} = \frac{2\mathbf{p}}{4\pi\epsilon_0 r^3}, \quad \text{for } r >> a$$

  And at a point on the perpendicular bisector of the dipole axis is given by:

  $$E = \frac{-\mathbf{p}}{4\pi\epsilon_0 r^3} \quad r >> a$$

  where r is the distance of the point from the centre of dipole and p is dipole moment ,vector.
- The potential at any point P on a line which makes an angle 8 with the axis ot a dipole is given by:

  $$\phi = \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}$$

  where $\hat{\mathbf{r}}$ is a unit vector from the centre of dipole to the point P where field is to be determined.
- A dipole experiences a turning effect in a uniform electric field. The torque τ experienced by the dipole is given by:

  $$\tau = \mathbf{p} \times \mathbf{E}$$

The potential energy of the dipole is given by $U = -\mathbf{p}.\mathbf{E}$. It is minimum when $\mathbf{p}$ is parallel to E.

3.9 TERMINAL QUESTIONS

1) Show that the line integral of the electric field E over a closed path is equal to zero.

2) Show that, in a pair of oppositely charged plane parallel plates, the electric field E is **equal** to the potential difference between the plates divided by their separation. You may assume that the electric field is **confined** to the between the plates as shown in Fig. 3.14.

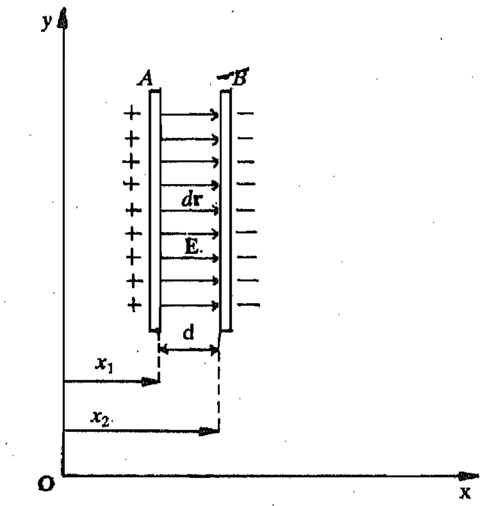

Fig. 3.14

3) Find the potential at two points A and B at distances of **10 cm** and 50cm from a charge of **2μC** as shown in Fig. 3.15. Also find the work needed to be done in bringing a charge of **0.05μC** from B to A.

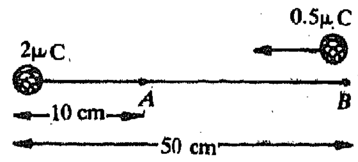

Fig. 3.15

4) Compute the potential difference between points A and *B* assuming that a test charge q_0 is moved without acceleration from A to B along the path shown in Fig. 3.16.

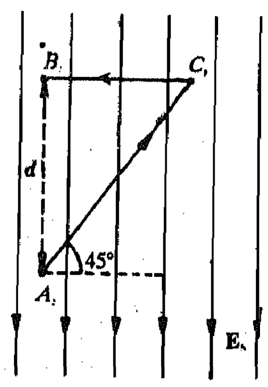

Fig. 3.16

3.10 SOLUTIONS AND ANSWERS

SAQ 1 See Fig. 3.17. The line integral of the electric field E due to a point charge q from a point A to another point B is given by

$$\int_A^B \mathbf{E}\cdot d\mathbf{r} = \lim_{\substack{n\to\infty \\ r_j \to 0}} \sum_{j=1}^{n} \mathbf{E}(r_j) \cdot \Delta\mathbf{r}_j$$

where A $\mathbf{r}_j$'s are along the tangents to the paths from *A* to B. The dot product for a typical line segment of $\mathbf{E}(r_j)$ and $\Delta\mathbf{r}_j$ at any point P is given **by**

$$\mathbf{E}(r_j) \cdot \Delta\mathbf{r}_j = |\mathbf{E}(r_j)|\ |\Delta\mathbf{r}_j| \cos\theta$$

where θ is the angle between $\mathbf{E}(r_j)$ and $\Delta\mathbf{r}_j$. Now, if the line integral from B to *A* is to be evaluated, then the dot products will be negative of the corresponding dot products of the first line integral. This is because the infinitesimal segments will be making angles equal to 180" minus the corresponding angles of the first line integral dot products as shown in Fig.

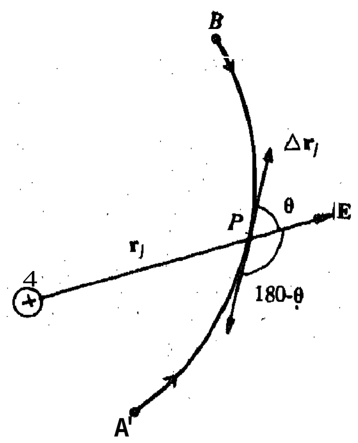

Fig. 3.17

3.17 for a particular line segment. Thus

$$\int_B^A \mathbf{E} \cdot d\mathbf{r} = -\sum_{j=1}^{n} \mathbf{E}(r_j) \cdot \Delta\mathbf{r}_j$$

$$= \int_A^B \mathbf{E} \cdot d\mathbf{r}$$

SAQ 2 The work done by force $\mathbf{F}$ is equal to:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (xy\,\hat{\mathbf{i}} - y^2\hat{\mathbf{j}}) \cdot (\hat{\mathbf{i}}\,dx + \hat{\mathbf{j}}\,dy)$$

$$= \int_C (xy\,dx - y^2\,dy)$$

where C is the parabolic path $y = x^2/4$ from **(0,0)** to **(2,1)**. Substituting for y and dy in terms of x and dx, the integral is given by:

$$\int_0^2 \left(\frac{x^2}{4} - \frac{x^5}{32}\right) dx = \left[\frac{x^4}{16} - \frac{x^6}{192}\right]_0^2 = \frac{2}{3}$$

Here the integral is evaluated between the **limits 0** to 2 for the variable x.

SAQ 3 Let the electric field be E and element of path length be $d\mathbf{r}$. Since both E and $d\mathbf{r}$ are parallel, the angle θ between the two vectors is zero. Then work done

$$W = -\int \mathbf{E} \cdot d\mathbf{r}$$

$$= -\int E\,(\cos 0°)\,dr$$

$$= -\int E\,dr$$

$$\therefore W = -Ed \quad \left(\because \int dr = d\right)$$

SAQ 4 Charge **on** the nucleus $q = Ze = 79 \times 1.6 \times 10^{-19}$C and $r = 6.6 \times 10^{-15}$m

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{q}{r} = \frac{(9\times10^9\ \mathrm{N\,m^2\,C^{-2}})\,(79\times1.6\times10^{-19}\mathrm{C})}{6.6\times10^{-15}\mathrm{m}}$$

$$= 1.7\times10^{7}\ \mathrm{NmC^{-1}} = 1.7\times10^{7}\mathrm{V}\ (\because \mathrm{NmC^{-1}} = \mathrm{JC^{-1}})$$

SAQ 5 In going from the positive **terminal** of a battery to the negative terminal, the **electron** (a negatively charged **particle**) goes from a point at a higher potential to a point at a lower potential, Since

$q = -1.6 \times 10^{-19}$C

and

$\phi_A - \phi_B = -12$v

$\therefore$ Work done, $W = q\,(\phi_A - \phi_B)$

$= (-1.6 \times 10^{-19}\mathrm{C})\,(-12\mathrm{V}) = 1.92 \times 10^{-18}$J

SAQ 6 $\mathbf{E} = -\left[\hat{\mathbf{i}}\,\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\,\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial\phi}{\partial z}\right]$

where $9 = x\,(y^2 - 4x^2)$

$$\therefore\ \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial x}\,[xy^2 - 4x^3] = y^2 - 12x^2$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial y}\,[xy^2 - 4x^3] = 2xy$$

and

$$\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial z}\,[xy^2 - 4x^3] = 0$$

$$\therefore\ \mathbf{E} = -[\hat{\mathbf{i}}\ (y^2 - 12x^2) + \hat{\mathbf{j}}\ (2xy) + \hat{\mathbf{k}}\,(0)]$$
$$= (12x^2 - y^2)\,\hat{\mathbf{i}} - 2xy\,\hat{\mathbf{j}}$$

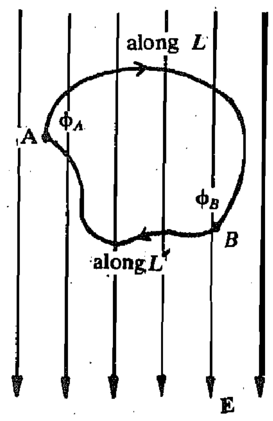

Fig. 3.18:

Terminal Questions

1) Let us consider a closed path starting from and ending at A as shown in Fig. *3.18.* Let B be some point on this closed path. If ϕ_A and ϕ_B are potentials at A and B respectively, we can write

$$-\int_{\substack{A\\ \text{along } L'}}^{B} \mathbf{E}\cdot d\mathbf{r} = \phi_B - \phi_A \qquad \text{...(3.41)}$$

$$\text{also } -\int_{\substack{A\\ \text{along } L'}}^{B} \mathbf{E}\cdot d\mathbf{r} = \phi_B - \phi_A \qquad \text{...(3.42)}$$

Since

$$\int_{\substack{A\\ \text{along } L'}}^{B} \mathbf{E}\cdot d\mathbf{r} = -\int_{\substack{B\\ \text{along } L'}}^{A} \mathbf{E}\cdot d\mathbf{r}$$

$$\therefore\ -\int_{\substack{B\\ \text{along } L'}}^{A} \mathbf{E}\cdot d\mathbf{r} = \phi_A - \phi_B \qquad \text{...(3.43)}$$

Adding Eqs. *(3.41)* and (3.43), we get

$$-\int_{\substack{A\\ \text{along } L}}^{B} \mathbf{E}\cdot d\mathbf{r} - \int_{\substack{B\\ \text{along } L'}}^{A} \mathbf{E}\cdot d\mathbf{r} = \phi_B - \phi_A + \phi_A - \phi_B = 0$$

That is, along a closed path, the line integral of the electric field is equal to zero.

Alternative method: One may also **use** the concept of **path independence**

$$\int_{\substack{A\\ \text{along } L}}^{B} \mathbf{E}\cdot d\mathbf{r} = \int_{\substack{A\\ \text{along } L'}}^{B} \mathbf{E}\cdot d\mathbf{r}$$

$$\text{or } \int_{\substack{A'\\ \text{along } L}}^{B} \mathbf{E}\cdot d\mathbf{r} - \int_{\substack{A\\ \text{along } L'}}^{B} \mathbf{E}\cdot d\mathbf{r} = 0$$

$$\text{or} \int_{A \text{ along } L}^{B} \mathbf{E} \cdot d\mathbf{r} + \int_{B \text{ along } L'}^{A} \mathbf{E} \cdot d\mathbf{r} = 0$$

($L + L'$ implies closed path).

2) Let A and B be two oppositely charged plates separated by a distance d. Let $\mathbf{E}$ be the uniform electric field between the two plates. We then have

$$-\int_A^B \mathbf{E} \cdot d\mathbf{r} = (\phi_B - \phi_A)$$

where ϕ_A and ϕ_B are the potentials at the plates A and B respectively.

In the present case, writing $\int_A^B \mathbf{E} \cdot d\mathbf{r}$ as $\int_{x_1}^{x_2} \mathbf{E} \cdot \hat{\mathbf{i}}\, dx$, and noting that both $\mathbf{E}$ and $\hat{\mathbf{i}}\, dx$ are parallel, we can write

$$\phi_B - \phi_A = -\int_{x_1}^{x_2} \mathbf{E}\, dx = -E\, |x|_{x_1}^{x_2}$$

$$= -E\,(x_2 - x_1) = -E\,d$$

or

$$\phi_A - \phi_B = Ed$$

$$\therefore E = \frac{\phi_A - \phi_B}{d}$$

That is, the magnitude of electric field between two oppositely charged parallel plates is equal to the difference of potential between them divided by their separation.

3) The potential ϕ_r at a point distant $\mathbf{r}$ from a charge q is given by

$$\phi_r = \frac{4}{4\pi\,\epsilon_0\, r} \qquad \text{...(3.20)}$$

where $q = 2\mu C = 2 \times 10^{-6} C$, $\frac{1}{4\pi\,\epsilon_0} = 9 \times 10^9\, Nm^2\, C^{-2}$ and $r = 0.10m$ and $0.50m$

Using Eq. (3.20), we get

$$\phi_{0.10} = (9 \times 10^9\, Nm^2\, C^{-2})\, \frac{2\times10^{-6}C}{0.10m} = 1.8 \times 10^5 V$$

$$\phi_{0.50} = (9 \times 10^9\, Nm^2\, C^{-2})\, \frac{2 \times 10^{-6}C}{0.50m} = 0.36 \times 10^5 V$$

Work done, $W = q'\,(\phi_{0.10} - \phi_{0.50})$, where $q' = 0.05 \times 10^{-6} C$,

$\therefore W = (0.05 \times 10^{-6} C)\,(1.8 \times 10^5 - 0.36 \times 10^5)V$
$= 7.2 \times 10^{-3} J.$

4) We can write

$$\phi_B - \phi_A = (\phi_B - \phi_C) + (\phi_C - \phi_A)$$

$$= -\int_C^B \mathbf{E} \cdot d\mathbf{r} - \int_A^C \mathbf{E} \cdot d\mathbf{r}$$

For path C to B, E and $d\mathbf{r}$ are perpendicular to each other.

Therefore, $\int_C^B \mathbf{E} \cdot dr = |\mathbf{E}|\, dr \cos 90° = 0.$ Thus, $-\int_C^B \mathbf{E} \cdot d\mathbf{r} = 0.$

This gives

$$\phi_B - \phi_A = -\int_A^C \mathbf{E} \cdot d\mathbf{r}$$

For path A to C, the angle between E and dr = 135°.

$$\therefore \int_A^C \mathbf{E} \cdot d\mathbf{r} = \int_A^C E\, dr \cos 135^\circ$$

$$= \int_A^C E\, dr \left(-\frac{1}{\sqrt{2}}\right)$$

$$= -\frac{E}{\sqrt{2}} \int_A^C dr = -\frac{E}{\sqrt{2}} (AC) = -\frac{E}{\sqrt{2}} \sqrt{2}\, d$$

$$= -Ed$$

since AC $= d/\cos 45^\circ = \sqrt{2}d$

$$\therefore \phi_B - \phi_A = -\int_A^C E \cdot d\mathbf{r}$$

$$= Ed.$$

You may note that this is also the value obtained via the direct path from A to B.

Appendix

Polar Coordinates

In addition to the Cartesian coordinates, you will be required to use polar coordinates in two dimensions. The polar coordinates are useful in describing the problems having circular symmetry. In Mechanics, you would have studied circular motion. The mathematical treatment of this problem becomes much simpler if one uses polar coordinates instead of Cartesian coordinates.

A point P (x, y) in two dimensions is represented in polar coordinates in terms of r and θ. Here r is the distance of P from the origin O, and θ is the angle, the line joining O to P makes with the positive x–axis. See Fig. A.1.

A.1

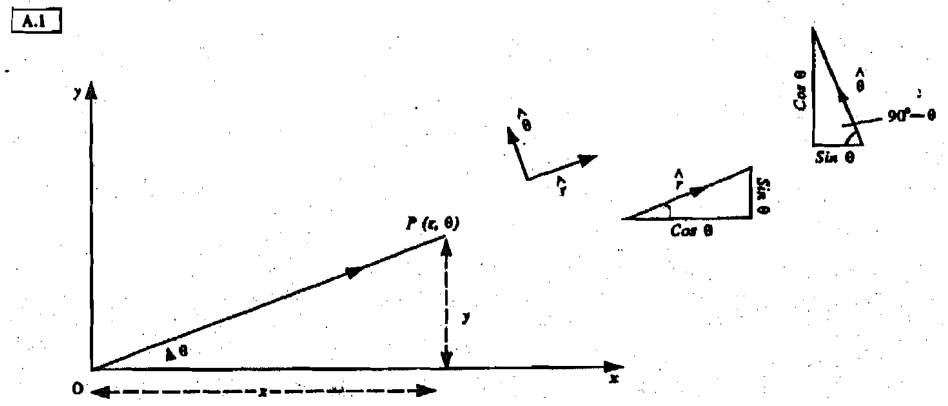

Fig. A.1: Polar coordinates and unit vectors $\hat{r}$ and $\hat{\theta}$ as a sum of two vectors.

OP is known as the position vector of P. Projection of OP on x–axis gives the x coordinate of P and its projection on y–axis gives the y coordinate of the point P. The relations between (x, y) and (r, θ) are

$$x = r\cos\theta \qquad y = r\sin\theta \qquad \text{...(A.1)}$$

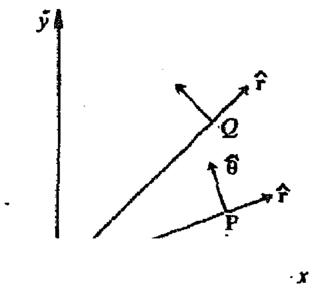

Fig. A.2: Unit vectors at two different points.

The vector **OP** (r) can be written as the sum of two vectors, i.e.,

$$\mathbf{r} = \hat{\mathbf{i}}\, x + \hat{\mathbf{j}}\, y \qquad \text{...(A.2)}$$

Let us define unit vectors in the direction of increasing r at P as $\hat{\mathbf{r}}$ and in the direction of increasing θ as $\hat{\theta}$ as shown In Fig. **A.1**. The direction of $\hat{\theta}$ is the same as the tangent to the circle of radius r at P with O as the centre. $\hat{\mathbf{r}}$ and $\hat{\theta}$ are unit vectors; these can be written as the sum of two vectors. Projecting $\hat{\mathbf{r}}$ in the x direction, one gets $\cos\theta$ as the x component of $\hat{\mathbf{r}}$ and in the y direction, $\sin\theta$ as the y component of $\hat{\mathbf{r}}$. See Fig. **A.1.** Thus

$$\hat{\mathbf{r}} = \hat{\mathbf{i}} \cos\theta + \hat{\mathbf{j}} \sin\theta \qquad \text{...(A.3)}$$

Similarly, projecting $\hat{\theta}$ in the x and y direction, one gets $-\sin\theta$ as the x component and $\cos\theta$ as the y component. See Fig. **A.1** Thus

$$\hat{\theta} = -\hat{\mathbf{i}} \sin\theta + \hat{\mathbf{j}} \cos\theta \qquad \text{...(A.4)}$$

Note the $\hat{\mathbf{r}} \cdot \hat{\theta} = 0$, $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1 = \hat{\theta} \cdot \hat{\theta}$. Referring to Fig. A.2, you will see that the unit vectors $\hat{\mathbf{r}}$ and $\hat{\theta}$ at P are not equal to the unit vectors $\hat{\mathbf{r}}$ and $\hat{\theta}$ at Q. These unit vectors change directions as you go from one point to another. If you differentiate the relations (A.3) and (A.4) of $\hat{\mathbf{r}}$ and $\hat{\theta}$ partially with respect to θ, you get:

$$\frac{\partial \hat{\mathbf{r}}}{\partial \theta} = -\hat{\mathbf{i}} \sin\theta + \hat{\mathbf{j}} \cos\theta = \hat{\theta}$$

and

$$\frac{\partial \hat{\theta}}{\partial \theta} = -(\hat{\mathbf{i}} \cos\theta + \hat{\mathbf{j}} \sin\theta) = -\hat{\mathbf{r}}$$

Any vector A in (x, y) coordinates is written as a sum of vectors in the x, y directions in terms of its components along these directions as follows :

$$\mathbf{A} = \hat{\mathbf{i}} A_x + \hat{\mathbf{j}} A_y$$

Similarly, if A_r and A_θ are the components in the (r, θ) coordinates. then we can write **A** as:

$$\mathbf{A} = \hat{\mathbf{r}} A_r + \hat{\theta} A_\theta$$

In order to write the del (∇) operator in the (r, θ) **coordinates,** we need the infinitesimal changes in **length** in r and θ directions. The infinitesimal change **in** length in $\hat{\mathbf{r}}$ direction is dr and in $\hat{\theta}$ direction, it is $(rd\theta)$. Here $d\theta$, being **only** a change in angle, has to be multiplied by r to *get* the **infinitesimal** change in length in the $\hat{\theta}$ direction at $P\ (r, \theta)$. *See* **Fig. A.3a.**

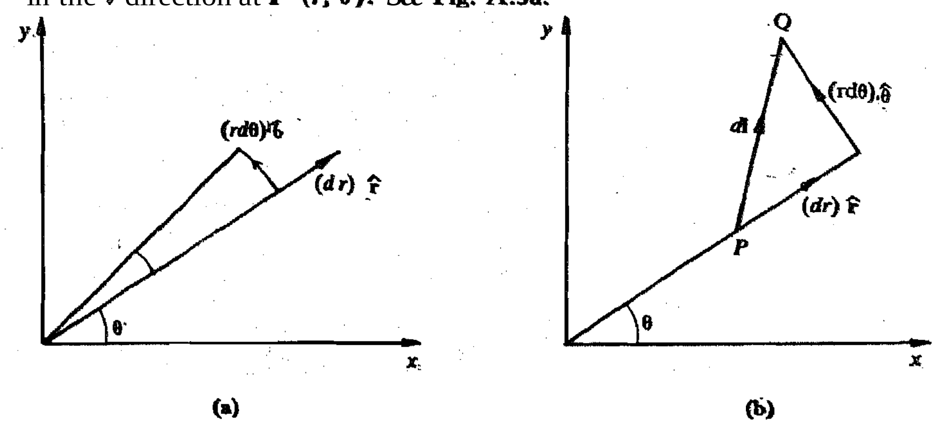

Fig. A.3: a) Elements of lengths in $\hat{\mathbf{r}}$ and $\hat{\theta}$ directions. b) Displacement $d\mathbf{l}$ as a sum of $\hat{\mathbf{r}}\,(dr)$ and $\hat{\theta}\,(rd\theta)$

An infinitesimal **displacement** $d\mathbf{l}$ in (x, y) coordinates is written **as** :

$$d\mathbf{l} = \hat{\mathbf{i}}\, dx + \hat{\mathbf{j}}\, dy$$

Similarly, $d\mathbf{l}$ in (r, θ) coordinates(see **Fig. A.3b) can be** written **as:**

$$d\mathbf{l} = \hat{\mathbf{r}}\, dr + \hat{\theta}\, (rd\theta)$$

It is now possible **to** obtain the **del** operator in (r, θ) coordinates. Consider a *scalar* function ϕ. Suppose $\phi_1\ (r_1, \theta_1)$ is its value at P $(r_1\ \theta_1)$ *and* $\phi_2\ (r_2, \theta_2)$ is *its* **value at** $Q\ (r_2, \theta_2)$. **If** $PQ = d\mathbf{l}$ is an **infinitesimal** displacement, **then**

$\phi_2 (r_2, \theta_2) - \phi_1 (r_1, \theta_1) = d\phi$

The points (r_1, θ_1) and (r_2, θ_2) being close to each other, we can write $d\phi$ as

$$d\phi = \frac{\partial \phi}{\partial r} dr + \frac{\partial \phi}{\partial \theta} d\theta \quad ...(A.5)$$

$$= (\nabla \phi) \cdot d\mathbf{l}$$

Writing $\nabla \phi$ in (r, θ) coordinates, we get

$$(\nabla \phi) = \hat{\mathbf{r}} (\nabla \phi)_r + \hat{\theta} (\nabla \phi)_\theta \quad ...(A.6)$$

where $(\nabla \phi)_r$ and $(\nabla \phi)_\theta$ are r, θ components, respectively.

Hence

$$(\nabla \phi) \cdot d\mathbf{l} = \{\hat{\mathbf{r}} (\nabla \phi)_r + \hat{\theta} (\nabla \phi)_\theta\} \cdot (\hat{\mathbf{r}}\, dr + \hat{\theta}\, r d\theta)$$

$$= (\nabla \phi)_r\, dr + (\nabla \phi)_\theta\, r d\theta$$

Comparing with $d\phi = \frac{\partial \phi}{\partial r} dr + \frac{\partial \phi}{\partial \theta} d\theta$, we get

$$(\nabla \phi)_r = \frac{\partial \phi}{\partial r} \text{ and } (\nabla \phi)_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} \quad ...(A.7)$$

Hence, the del operator in polar coordinates is given by:

$$\nabla = \left\{ \hat{\mathbf{r}} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} \right\} \quad ...(A.8)$$

Example

Just to **illustrate** the convenience in handling a problem using **polar** coordinates, let us consider the motion **of** a particle **in** a **circle** at constant speed. Let the origin be chosen at the centre of the circle of radius r **and** the speed of the particle be v. Then we can write

$\mathbf{r} \cdot \mathbf{r} = r^2 = \text{const.}$
$\mathbf{v} \cdot \mathbf{v} = v^2 = \text{const.}$

Differentiating first of these equations with respect to time, we get

$$2\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0 \quad \text{or} \quad \mathbf{r} \cdot \mathbf{v} = 0$$

an4 differentiating the second equation, **we get**

$$2\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = 0 \quad \text{or} \quad \mathbf{v} \cdot \mathbf{a} = 0 \text{ where } \mathbf{a} = \frac{d\mathbf{v}}{dt}$$

"These show that velocity $\mathbf{v}$ is at right angles to $\mathbf{r}$ at every point and the acceleration a is at **right** angles to $\mathbf{v}$. This means that acceleration a is either parallel to $\mathbf{r}$ or **antiparallel** to $\mathbf{r}$. Differentiating with respect to time $\mathbf{r} \cdot \mathbf{v} = 0$ equation, we get

$\mathbf{r} \cdot \mathbf{a} + \mathbf{v} \cdot \mathbf{v} = 0$
$\therefore v^2 = -\mathbf{r} \cdot \mathbf{a} = -r\, a \cos\theta$

As v^2 is positive, $\cos\theta < 0$ **which means** $\theta = \pi$ **as** r and $\mathbf{a}$ are either parallel or **antiparallel. From this**, it is clear that **acceleration** a is opposite to **r and** its magnitude is

$$a = \frac{v^2}{r}$$

Hence, **the** 'acceleration is **directed** towards centre.