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UNIT 1 FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

Structure

1. INTRODUCTION

You have read about the motion of a projectile and the motion of a rocket in your Elementary Mechanics **(PHE-01)** course. You know **that** the velocity of the projectile is affected due to air resistance and the velocity of the rocket is affected by the quantity of fuel burnt. But can we find out mathematically, how the velocity of a projectile is affected due to air resistance? And *how* does the quantity of fuel burnt by a rocket affect its velocity? A similar question may also be raised regarding an environmental issue. You may be aware that the oil slick formed in the Persian Gulf during the Gulf War of **1991** posed the danger of serious environmental pollution. Now, how long will it take the Persian Gulf to return to its natural state once the oil slick is completely checked? We can obtain an answer to these and many other questions pertaining to different situations by framing and subsequently solving what is **known** as the first order ordinary differential equation for the concerned system. In this unit you will be **studying** first order differential equations as they find many applications in physics. You have already dealt with a few differential equations in the "Oscillations and Waves" course **(PHE-02)**.

Here we will first discuss what is meant by a differential equation (henceforth referred to as a DE) through some simple examples. You will then learn to classify **DEs** in various ways. Next you will learn what is meant by the solution of a DE.

Our ultimate aim is to learn the methods of solving **DEs.** In this unit we will discuss **various methods** for solving first order ordinary differential equations. You will learn to solve them by the method of separation of variables and the method of substitution, You will also learn to solve exact equations. Next you will learn the technique of converting an inexact equation into an exact equation. This will enable you to solve first order linear **ODEs.** In Unit 4, you will study about the **applications** of some of these **equations** in physics.

In the next unit we will **take** up the **study of** second **order** ordinary differential equations.

Objectives

After studying this unit you should be able to

- define the general solution and the particular solution of a differential equation
- solve first order ordinary differential equations reducible to separable forms
- solve an exact equation
- solve first order linear ordinary differential equations by the method of integrating factors
- solve ordinary differential equations reducible to first order.

1.2 WHAT IS A DIFFERENTIAL EQUATION?

You must have read about the phenomenon of radioactivity in your school science courses. The principle of radioactive decay was discovered by the French scientist Henri Becquerel in the year 1896. He was able to establish experimentally that the rate at which the atoms of a radioactive substance disintegrate is proportional to the number of atoms (N) present in it. Now let us try to express this idea mathematically. We can express the rate of disintegration of atoms as $\left(-\frac{dN}{dt}\right)$, where t represents time. The negative sign appears because N decreases with t (so that $\frac{dN}{dt}$ is negative). Now according to Rutherford this rate is proportional to N. So we have

Ernest Rutherford (1871–1937) did much of the early work on characterising radioactivity. Recall from Unit 8 of PHE-01 that Rutherford also proposed the nuclear model of the atom.

$$-\frac{dN}{dt} = \lambda N, \qquad \text{where } \lambda \text{ is a constant}$$

or

$$\frac{dN}{dt} + \lambda N = 0 \tag{1.1}$$

Using Eq. (1.1) we may obtain a relation between the independent variable t and the dependent variable N. Notice that Eq. (1.1) contains terms involving N and its ordinary derivative with respect to time. You have also come across equations involving higher order ordinary derivatives of the dependent variable with respect to the independent variable. For example, you have studied the one-dimensional equation of motion of a linear harmonic oscillator in Block 1 of PHE-02 given as

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{1.2}$$

where m is the mass of the oscillator and k is the force constant. Eq. (1.2) has terms involving x and its second derivative w.r.t. time. Notice that Eqs. (1.1) and (1.2) involve only ordinary derivatives. Let us consider another example. Suppose a current i flows through an electric circuit for an infinitesimal duration of tune, dt. Then the charge that flows during this tune is given by

w.r.t. stands for 'with respect to'

$$dq = i\,dt \tag{1.3}$$

Eq. (1.3) involves the differentials dq and dt. Equations like (1.1) to (1.3) are called ordinary differential equations (ODEs). More precisely,

> An equation which contains differentials or only ordinary derivatives of one or more dependent variables w.r.t. a *single independent variable* is said to be an ordinary **differential** equation.

Now, you have also studied the one-dimensional wave equation in Unit 6 of PHE-02 given by

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \tag{1.4}$$

Here ψ is a wave function and v is the wave speed. Eq. (1.4) involves second order partial derivatives of ψ with respect to the variables x and t. Equations like (1.4) are called partial

You can know more about partial derivatives by studying Units 2 and 3 of PHE-04 (Mathematical Methods in Physics-I) or Sec. 5.2 of Block 2 of this course.

differential equations (PDEs) as they involve partial derivatives of one or more dependent variables w.r.t. *two or more independent variables.* In this block we shall deal only with ODEs. We shall take up the study of PDEs in Block 2.

We can now give a general definition of a differential equation:

> An equation containing the derivatives or differentials of one or more dependent variables with respect to *one* or *more independent variables* is said to be a differential equation.

You must now identify a few ODEs and PDEs in the following SAQ.

SAQ 1

Spend 5 min

Ten equations from various areas of physics are listed below. Identify the ordinary and partial differential equations.

i) $\frac{d^2y}{dt^2} = -g$

ii) $y = u_0t - \frac{1}{2}gt^2$

iii) $\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$

iv) $\frac{\partial^2T}{\partial x^2} + \frac{\partial^2T}{\partial y^2} = 0$

v) $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t)$

vi) $\frac{\partial^2u}{\partial x^2} + \frac{\partial^2u}{\partial y^2} + \frac{\partial^2u}{\partial z^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$

vii) $u = A\sin(x - \omega t) + B\cos(x - \omega t)$

viii) $\frac{dT}{dt} = K(T - T_0)$

ix) $m\frac{dv}{dt} = mg - kv$

x) $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2u}{\partial\theta^2}\right) + \frac{\partial^2u}{\partial z^2} = 0$

Now there are three major aspects in the study of ordinary differential equations. These are the formation of an ODE, its solution and its application These aspects are different for different kinds of ODEs. So before you study these aspects you must know how ODEs are classified. This is the subject of Sec. 1.3. We expect that after studying Sec. 1.3 you ought to be able to classify an ODE just by looking at it. So study it carefully and thoroughly.

1.3 CLASSIFICATION OF ORDINARY DIFFERENTIAL EQUATIONS

Ordinary differential equations are classified in a number of ways as shown in Table 1.1. We will be using Table 1.1 quite a lot in our discussion. So, first a word about how to use it. As you can see, the second column of the table lists several examples of ODEs. And the first row lists the various ways in which ODEs are classified. Now as you study this section it would be better if you concentrate on the particular example and the way of classification being discussed in the text. During that discussion, ignore the rest of the information presented in the table. Do you see the blank spaces in the table? You will be asked to fill them up once you have studied this section! Let us now continue our study.

The most fundamental way of classifying ODEs is on the basis of their order and degree.

Order and degree of an ordinary differential equation

> The order of an ODE is the order of the highest derivative appearing in it.
>
> The degree of an ODE is the power of the highest order derivative appearing in the equation, after it has been expressed in a form such that no derivatives have **fractional** or negative powers.

Let us consider **equation(4) in Table** 1.1. What are the order and degree of this equation? The highest derivative is the second order derivative y''. So its order is 2. Now let us remove the fractional power **1/2** in the equation by squaring it. Then, the power of y'' is 2, so that the degree of this equation is 2. Likewise you can verify the order and degree of the equations (1) and (3) in Table 1.1.

So you have learnt to classify an ODE in terms of its order and degree. We can also classify ODEs as linear or non-linear.

Linear and nonlinear **ordinary** differential equations

Consider equation (3) of Table 1.1. In this equation the function y and its derivatives are all of degree 1. It does not contain **products** like yy', yy', $y'y''$ etc. It also does not involve any transcendental functions like sin y, ln y etc. It is an example of a linear ODE. We call an ordinary differential equation linear when the following conditions are fulfilled:

Any Function which cannot be expressed as a solution of a polynomial equation of the form
$P_0(x)u^n + P_1(x)u^{n-1} + \ldots$
$P_{n-1}(x)u + P_n(x) = 0$
is called a **transcendental function**. The logarithmic, trigonometric, hyperbolic functions and their corresponding inverses are examples of transcendental functions.

> i) The unknown function and its derivatives occur only to the first degree
> ii) In the equation there are no products involving either the unknown function and its derivatives or two or more derivatives.
> iii) There are no transcendental functions involving the unknown function or any of its derivatives.

An nth-order ordinary differential equation, linear in y, may be **expressed** as

$$a_n(x)\,y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \ldots + a_1(x)y' + a_o(x)y = f(x) \qquad (1.5)$$

Here f and the coefficients $a_0, a_1, \ldots, a_n$ are functions of x only, on some interval of x, and $a_n(x) \neq 0$ on that interval. In writing Eq. (1.5), we have adopted the notation

$$y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}, \ldots, y^{(n)} = \frac{d^ny}{dx^n}.$$

A differential equation that is not linear is said to be nonlinear. You can verify that ODE(1) in Table 1.1 is linear and ODEs(4) and (7) are nonlinear. A linear ODE can further be classified as homogeneous or nonhomogeneous.

Homogeneous and nonhomogeneous ordinary differential equations

> If in the RHS of Eq. (1.5), we have $f(x) = 0$, then it is a homogeneous linear ODE and if $f(x) \neq 0$, **then** it is called nonhomogeneous.

For example, ODEs (1) and (6) are nonhomogeneous because f (x) = E and f (x) = e^x, respectively. Since $f(x) = 0$ for ODE (3), it is homogeneous.

Note: The term homogeneous has another meaning when used for a first order ODE. We will explain what a homogeneous first order ODE means in **Sec.** 1.5.2 of this unit.

In many books on ODEs, you will come across the term 'inhomogeneous ODEs'. It has the same meaning as the term 'non-homogeneous ODEs'.

So you have learnt to classify ODEs in four ways: by way of (i) order (ii) degree **(iii) linearity/nonlinearity** (iv) homogeneity/nonhomogeneity. Henceforth, in your study of **ODEs,** you must make it a habit to classify an ODE the moment you see it. This means that you must be able to tell what its order **and degree is,** and whether it is linear or nonlinear. Moreover, if it is linear you must be able to say whether it is homogeneous or **nonhomogeneous. For example,** ODE (1) in Table 1.1 is a linear, nonhomogeneous first order ordinary differential equation of degree 1.

Table 1.1 : Classification of ODEs

No.	ODE	Order	Degree	Linear (L)/ Non-linear (NL)	Non-homogeneous (NH)/ Homogeneous (H)	Remarks (if any)
(1)	$L \frac{di}{dt} + Ri = E$	1	1	L	NH	This can be made homogeneous by taking E to the LHS and making the substitution $i_1 = i - \frac{E}{R}$.
(2)	$m \frac{d^2x}{dt^2} + r \frac{dx}{dt} + kx = F \cos \omega t$					
(3)	$x^2y'' + 2xy' + y = 0$	2	1	L	H	
(4)	$(1 + (y')^2)^{1/2} = y''$	2	2	NL	–	It is nonlinear because its degree is 2. Since it is nonlinear, the question of classifying it as homogeneous or non-homogeneous does not arise.
(5)	$(y'')^3 + xy' - y = 0$					
(6)	$y''' + y = e^x$				NH	It is nonhomogeneous because of e^x on the RHS.
(7)	$y'' + 7y = \sin y$	2	1	NL	–	It is nonlinear because sin y is a transcendental function in y.
(8)	$y'' - 2y' + 3y = 0$					

You may now like to work out an SAQ on what you have learnt so far.

SAQ 2

Spend 10 min

Fill the blank boxes in Table 1.1. You need not fill in the 'Remarks' column.

So far you have learnt the basic terminology associated with ordinary differential equations. In the process you have also learnt to classify ODEs. Now your major goal in this course is to be able to solve differential equations appearing in physics. But before trying to solve DEs you must understand what is meant by the solution of a differential equation.

1.4 WHAT IS A SOLUTION OF A DIFFERENTIAL EQUATION?

Let us consider the following ODE:

$$y'' + y = 0 \qquad (1.6)$$

Now if we put

$$y = \sin x \qquad (1.7)$$

we have

$$y' = \cos x \quad \text{and} \quad y'' = -\sin x$$

and Eq. (1.6) becomes an identity. In that case we call the function $y = \sin x$ a solution of Eq. (1.6). This solution exists for every x in the interval $(-\infty, \infty)$. Now consider another example. The equation

$$y^2 + x = 4 \qquad (1.8)$$

is a solution of the ODE $2yy' = -1$. We can verify this by differentiating Eq. (1.8). We get

$$2yy' + 1 = 0$$

which is identical to the given ODE. Now Eq. (1.8) may also be expressed as

$$y = \pm\sqrt{4-x}$$

which essentially defines two functions. Each one of them is defined for every x in the interval $(-\infty, 4)$. You can see that this interval is different from the previous one. So depending on the context, the interval on which the solution of a DE exists can be any of the intervals $(-\infty, \infty)$, (a, ∞), $(-\infty, a)$, $[a, b]$, (a, b), and so on. We can now define the solution of an ODE as follows:

> A function
>
> $$y = \phi(x)$$
>
> is called a solution of a differential equation in y on some interval, say, $a \le x \le b$, if $\phi(x)$ is defined and differentiable throughout that interval and is such that the equation becomes an identity when y is replaced by ϕ (x) in the DE.

We also say that the differential equation is satisfied by $y = \phi(x)$. The two types of solutions, (1.7) and (1.8), are typical of those we encounter in ODEs. In Eq. (1.7), we have y given as an explicit function of x. Such a solution is called an explicit solution. And Eq. (1.8) is an implicit relation between x and y. We say that Eq. (1.8) is an implicit solution. In other words, a solution of a differential equation in the form

$$G(x,y) = 0 \qquad (1.9)$$

is called an implicit solution.

Now if you look back at Eq. (1.6), you will be able to verify easily that $y = \cos x$ is also a solution of that differential equation. In fact, a differential equation may have many solutions. The principal task of the theory of differential equations is to find all the solutions of a given differential equation. Then we investigate the physical significance of these solutions. Let us study about that in some detail now.

1.4.1 General Solution and Particular Solution

We have already illustrated through Eq. (1.6) that a differential equation may have many solutions. Let us take another example. We consider the differential equation

$$y' = \cos x \qquad (1.10)$$

You may easily verify that each of the functions

$$y = \sin x,\ y = \sin x + 5,\ y = \sin x - 9,\ y = \sin x + \frac{5}{8}$$

is a solution of Eq. (1.10). You can express them generally as

$$y = \sin x + C \qquad (1.11)$$

where C is an arbitrary constant.

Eq. (1.11) is called a general solution of Eq. (1.10). Eq. (1.11) can yield any number of

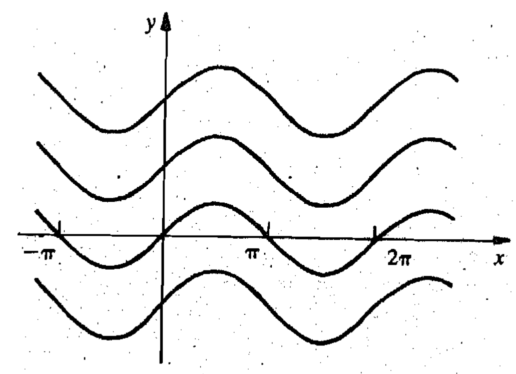

Fig.1.1 : Solutions of the differential equation: $y' = \cos x$

solutions of Eq. (1.10). We have represented a family of such solutions graphically in Fig. 1.1.

Likewise, you may also verify that

$$y = A\cos x + B\sin x \qquad (1.12)$$

where A and B are arbitrary constants, satisfies Eq. (1.6).

> A solution involving arbitrary **constant(s)** is known as the general solutlon.

You must have observed that **Eq.** (1.10) is a first order differential equation and its general solution (1.11) has one arbitrary constant. Eq. (1.6) is a second order differential equation and its general solution (1.12) has two arbitrary constants. So the number of arbitrary constants appearing in the solution of a differential equation is equal to its order.

Let us now impose the following condition on Eq. (1.11): $y = 0$ when $x = 0$. Then we get **from** Eq. (1.11) that

$$0 = 0 + C \quad \text{or} \quad C = 0 \quad \text{and} \quad y = \sin x.$$

So by imposing a condition on Eq. **(1.11),** we can assign a specific value to C. The solution thus obtained is called a particular solution.

You will encounter the term 'particular integral' in Unit 2. Do not confuse it with the term 'particular solution' being discussed here.

> If a definite value can be assigned to each arbitrary constant appearing in a general solution, then we get a particular solution.

For example, $y = \sin x + 2$ is a particular solution of Eq. (1.10) and $y = 2\cos x + 3\sin x$ is a particular solution of Eq. (1.6).

As you have just seen, particular solutions are **determined** from a general solution by imposing **condition(s)** on the solution function. Now, two questions arise in this connection.

(i) Does a particular solution always exist?

(ii) If it exists, is the solution unique?

Let us now discuss these questions briefly. This discussion is just to make you aware of such questions. We will not be going into the details here.

In the text books on ODEs, you will also come across another kind of solution of an ODE, the singular solution. It is that solution of an ODE which contains no arbitrary constant itself. Moreover, it cannot be obtained by assigning any value to the arbitrary constant in the general solution. For example, $y = \frac{x^2}{4}$ is a singular solution of the ODE $y'^2 - xy' + y = 0$. It contains no arbitrary constant and cannot be obtained by imposing a condition on the general solution $y = cx - c^2$ of this ODE.

1.4.2 Existence and Uniqueness of a Particular Solution

In this discussion we will be using certain new **terms** which we would like to explain first. You have learnt that a general solution of an nth-order **ODE** contains n arbitrary constants. So to obtain a particular solution of an nth order **ODE,** we have to impose n conditions on the solution function and its derivatives. We can then solve the n simultaneous linear equations so obtained for the n arbitrary constants. Now there are two common methods of specifying the conditions. We will mention them briefly.

> 1) If the conditions on the solution of a DE, or its derivatives, are specified for a single value of the independent variable, they are **called** initial conditions. The DE with its initial conditions is called an initial-value problem (IVP).
>
> 2) If the conditions on the solution of a DE, or its derivatives are specified for two or more values of the independent variable, they are called boundary conditions. The DE **with** its boundary **conditions** is called a boundary-value problem (BVP).

For example,

a) $y' + 2y = 3$, with **the initial** condition $y(0) = 1$, is a first-order initial-value problem.

b) $y'' + 3y = 0$, **with** the initial conditions $y(1) = 2$, and $y'(1) = -8$, is a second-order initial-value problem.

c) $y'' - 2y' + 6y = x^3$ with the boundary conditions $y(0) = 2$, $y(1) = -1$ is a second-order **boundary-value problem.**

Let us now deal with the questions of existence and uniqueness. Let us consider an example. You have seen that the general solution of $y'' + y = 0$ is given by

$$y = A \cos x + B \sin x$$

Now what can we say about the solution of the boundary-value problem $y'' + y = 0$, $y(0) = 0$, $y(\pi) = 2$? Using the given boundary condition in the general solution, we get

$$0 = A \cos 0 + B \sin 0 \quad \text{and} \quad 2 = A \cos \pi + B \sin \pi$$

The first equation yields $A = 0$, while the second equation yields $A = -2$. Since A cannot be equal to both 0 and –2, no solution is possible for this boundary-value problem (see Fig. 1.2a). You may easily verify that the boundary value problem

$$y'' + y = 0, \qquad y(0) = y(\pi) = 0,$$

will yield $A = 0$ unambiguously. However, no value gets assigned to B. Thus, there are infinitely many solutions represented by $y = B \sin x$ (see Fig. 1.2b).

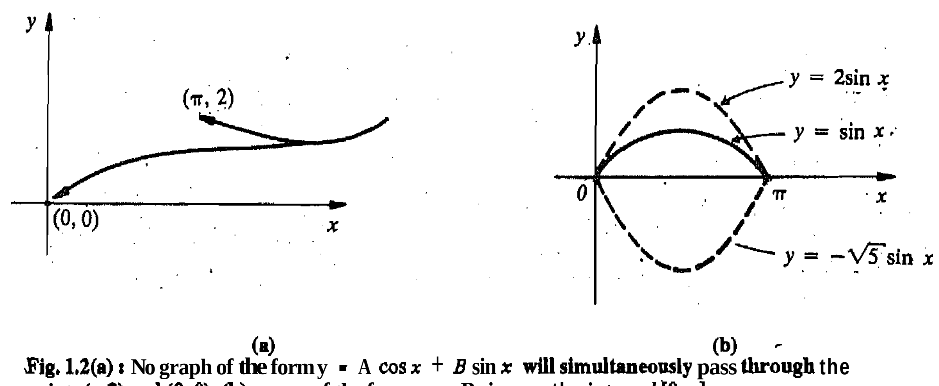

Fig. 1.2(a) : No graph of the form $y = A \cos x + B \sin x$ will simultaneously pass through the points $(\pi, 2)$ and $(0, 0)$; (b) curves of the form $y = B \sin x$ on the *interval* $[0, \pi]$.

From the above examples we understand that a solution may not exist for a boundary-value problem. And if it does, the solution may not be unique. In fact, there is no simple theory which can ensure a unique solution to a boundary-value problem. However, there exists a theorem that specifies necessary conditions for which a unique solution will exist for a first order initial value problem. Since our aim is just to sensitise you to these concepts, we will not go into these details here. If you are interested in such details you may like to study Sec. 1.3 (Unit 1) of the Mathematics course MTE-08 entitled Differential Equations. As a matter of fact, a more advanced course in differential equations at the post graduate level would focus on considerations of existence, uniqueness and general behaviour of solutions of DEs. So, henceforth in this block we shall consider only those DEs for which a solution exists.

So far you have learnt what is meant by the general and particular solution of an ODE. You also have some idea of what is meant by the existence and uniqueness of solutions. We will now discuss some properties associated with the solutions of linear ODEs. You will find these properties very useful when you actually solve linear ODEs.

1.4.3 General Properties of the Solutions of Linear ODEs

Let us consider the following ODEs

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1.13a}$$

and

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \tag{1.13b}$$

Eqs. (1.13a) and (1.13b) are both linear second order ODEs. The former is homogeneous and the latter is nonhomogeneous. We shall discuss the properties with reference to these second order ODEs primarily for the sake of simplicity. The other reason behind this is that you will come across linear second order ODEs quite often in physics. You will realise that ***the properties being discussed are true for linear ODEs of any order.***

Properties of the solutions of linear ODEs

i) $y = 0$ is a solution of Eq. **(1.13a)**. This is called **the trivial solution.**

ii) If y_1 and y_2 are linearly **independent** solutions of **Eq.(1.13a)**, then $u = c_1y_1 + c_2y_2$ is also a solution of Eq. **(1.13a)**, where c_1 and c_2 are constants.

iii) If y_1 is a solution of **(1.13a)** and y_2 is a solution of **(1.13b)**, then $a = y_1 + y_2$ is a solution of **(1.13b)**.

iv) The difference $(y_1 - y_2)$ of two solutions y_1 and y_2 of **(1.13b)** is a solution of **(1.13a)**.

Two functions $y_1(x)$ and $y_2(x)$ are said to be **linearly** dependent on an interval I where both functions are defined, if and only if we can find nonzero constants k_1 and k_2 such that $k_1y_1(x) + k_2y_2(x) = 0$ For all x on I. Thus, linearly dependent Functions are proportional on I. If the functions are not proportional on I, they are said to be **linearly independent.** Thus, for linearly independent functions the relation: $c_1y_1 + c_2y_2 = 0$ is satisfied only For $c_1 = c_2 = 0$.

The proofs of these properties are fairly straightforward. You can work them out yourself if time permits. You may now like to work out an SAQ on **what** you have learnt in **this** section.

Spend 10 min

SAQ 3

(a) Verify that $x^2 + y^2 - 1 = 0$ is a solution of the differential equation $yy' = -x$ on the interval **[-1,1]**. State whether this solution is implicit or explicit

(b) Verify that

$$y = Ax + \cos A, \text{ for constant } A$$

is the solution of the ODE

$$y = xy' + \cos y'$$

Identify the type of the solution **(i.e.,** whether **general** or particular).

Now tbat you have learnt the meaning and the basic properties of the solutions of linear **ODEs,** you can study the different methods of solving first order ODEs. We have given several SAQs in the subsequent discussion for the sake of practice. You **must** do them if you want to grasp these methods. We shall start with the **ODEs** that can be reduced to separable **forms.**

1.5 EQUATIONS REDUCIBLE TO SEPARABLE FORM

For several first order **ODEs,** you will find that tbe equation may be rewritten so that tbe concerned variables stand separated. It can then be solved by working **out the** integrals of the separated parts. Let us see how.

1.5.1 Method of Separation of Variables

Let us consider a general first order ordinary differential equation of the **form**

$$y' = f(x, y) \quad (1.14)$$

If we can write $f(x, y)$ as

$$f(x, y) = \frac{M(x)}{N(y)} \quad (1.15)$$

then Eq. (1.14) takes **the** form

$$M(x)dx - N(y)\,dy = 0 \quad (1.16)$$

The **forms** (1.14) and (1.16) are interchangeable. For example, the equations

$$y' = \frac{y}{1+x} \quad \text{and} \quad (1+x)\,dy - y\,dx = 0$$

mean the same thing. An ODE **of the form** $y' = \frac{M(x)}{N(y)}$ **is said to be a separable equation.**

In the **form (1.16)**, the variables x and y are separated. On integrating Eq. **(1.16)**, we get

$$\int M(x)\,dx - \int N(y)\,dy = C \quad (1.17)$$

where C is an arbitrary constant. Eq. (1.17) is the required solution of the ODE and it can be obtained if we can work out the integrals. Let us consider an example.

Let $I = \int \frac{y\,dy}{y^2+2}$

We put $u = y^2 + 2$

$\therefore\ du = 2y\,dy$

$I = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|$

$= \frac{1}{2}\ln|y^2+2|$

Remember that

$\int \frac{dx}{x} = \ln|x|.$

Example 1

Solve the equation $\frac{dy}{dx} = -\frac{5(y^2+2)}{xy}$

Solution

Comparing $f(x, y)$ of this equation with the form **(1.15)**, we have

$$M(x) = -\frac{5}{x},\ N(y) = \frac{y}{y^2+2}$$

So, we can rewrite it in the form (1.17) as

$$5\int \frac{dx}{x} + \int \frac{y\,dy}{y^2+2} = C$$

or

$$5\ \ln|x| + \frac{1}{2}\ln|y^2+2| = C$$

$$\therefore\ \ln|x|^5\,|y^2+2|^{1/2} = C$$

or

$$x^5(y^2+2)^{1/2} = C_1,\ \text{where } C_1 = \exp(C)$$

is the required solution.

Checking the solution

Let us quickly check the solution of the ODE in Example 1. Differentiating the solution we get

$5x^4(y^2+2)^{1/2} + \frac{x^5}{2}(y^2+2)^{-1/2}\,2yy' = 0$

or $y' = -\frac{5x^4(y^2+2)^{1/2}}{x^5y(y^2+2)^{-1/2}}$

$= -\frac{5}{xy}(y^2+2)$

which is the given ODE. Hence, the solution is correct.

Note : An important step in solving an ODE is to **check the solution.** You should always substitute the solution back into the ODE and check whether you get an identity. **Sometimes, you** get the ODE by simply differentiating the **solution** as in the case of Example 1.

Thus, we see that the method of separation of variables essentially consists of the **two** steps summarised below.

The method of separation of variables

Step 1: Write the **first** order ODE in the form $y' = \frac{M(x)}{N(y)}$ or

$M(x)\,dx - N(y)\,dy = 0$

Step 2: Integrate to obtain the solution.

Spend 5 min

SAQ 4

(a) Find the general solution of the ODE $(y + 1)\,y' + x = 0$.

(b) Solve the IVP $y' = -2xy,\ y(0) = 3$.

Remember to check the solutions,

Some ordinary differential equations may look non-separable. But on making some substitution they become separable.

Solution by the method of substitution

First we shall take up the case where substitution can be **done by** mere inspection of the equation. For example, let us consider the ODE, $\frac{dy}{dx} = \cos(x + y)$. The given equation is non-separable because of the factor $(x + y)$. So we put,

$$u = x + y$$

$$\therefore\ \frac{du}{dx} = 1 + \frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{du}{dx} - 1$$

Hence,

$$\frac{du}{dx} = 1 + \cos u = 2\cos^2\frac{u}{2}$$

$$\therefore \quad \frac{du}{2\cos^2 \frac{u}{2}} = dx$$

Thus, we have separated the variables u and x. Now, the above equation may be rewritten as

$$\frac{1}{2} \sec^2 \frac{u}{2} du - dx = 0$$

or
$$\frac{1}{2} \int \sec^2 \frac{u}{2} du - \int dx = C$$

or
$$\tan \frac{u}{2} - x = C$$

or
$$\tan \frac{x+y}{2} - x = C$$

This is the required solution. Now how about trying an **SAQ?**

Checking the solution
Differentiating the solution w.r.t. x, we have

$$\sec^2 \left(\frac{x+y}{2}\right) \left(\frac{1+y'}{2}\right) = 1$$

or $y' = 2\cos^2 \left(\frac{x+y}{2}\right) - 1$

$= \cos(x+y)$

which is the ODE being solved.

SAQ 5

Spend 10 min

Solve the ODEs:

(a) $(x - 2y - 1) = (x - 2y + 7)y'$

(b) $(1 + \cos\theta)\, dr = r \sin\theta\, d\theta$

Remember to check the solutions.

Let us now study a very typical case of substitution suitable for ODEs of the form $y' = f(y/x)$, where f is a function of y/x, e.g., $(y/x)^3$, sin (y/x), etc. Let us consider an example.

Example 2

Solve the differential equation

$$(x^2 + y^2)\, dx - xy\, dy = 0$$

Solution

We can rearrange the equation as

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy} = \frac{x}{y} + \frac{y}{x}$$

Now, this form suggests a substitution, $\frac{y}{x} = v$ where v is a function of x, Thus, we get

$$y = vx \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{1}{v} + v$$

or
$$v\,dv - \frac{dx}{x} = 0$$

Thus v and x are separated. On integrating, we get

$$\frac{v^2}{2} - \ln|x| = C$$

or
$$x = C_1 \exp(v^2/2) = C_1 \exp(y^2/2x^2)$$

Checking the solution
Differentiating the solution w.r.t. x, we gat

$$1 = C_1 \exp(y^2/2x^2) \times \left[\frac{2yy'}{2x^2} - \frac{2y^2}{2x^3}\right]$$

or $1 = \frac{xy}{x^2}\left[y' - \frac{y}{x}\right]$

or $y' = \frac{x}{y} + \frac{y}{x}$

which is the **ODE** being solved.

Having solved Example 2, you may well ask : How can we find out whether an **ODE** can be made separable by the substitution $y = vx$? We will answer this question in the following section.

Remember that we have defined higher order homogeneous ODEs in a different manner in Sec. 1.3.

1.5.2 Homogeneous Differential Equations of the First Order

The substitution $y = vx$ can be made to separate variables in **homogeneous first order ODEs.** What is a homogeneous first order ODE? A first order ODE of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{1.18}$$

is also called **homogeneous if M and N are homogeneous functions of the same degree.** Now, what is a homogeneous function? A function $f(x, y)$ is said to be **homogeneous** of degree **n** in x and y, if, for every k,

$$f(kx, ky) = k^n f(x, y),$$

where k is a real parameter. For example,

a) $f(x, y) = x^2 + xy + y^2$ is a homogeneous function of degree 2 since

$$f(kx, ky) = (kx)^2 + (kx)(ky) + (ky)^2$$
$$= k^2(x^2 + xy + y^2) = k^2 f(x, y)$$

b) $f(x, y) = \sqrt{x-y}$ is homogeneous of degree $\frac{1}{2}$ since

$$f(kx, ky) = \sqrt{(kx-ky)} = k^{1/2}\sqrt{x-y} = k^{1/2} f(x, y)$$

c) $f(x, y) = e^{x/y} + \tan\frac{y}{x}$ is homogeneous of degree zero since

$$f(x, y) = e^{kx/ky} + \tan\frac{ky}{kx} = e^{x/y} + \tan\frac{y}{x} = k^0 f(x, y)$$

d) $f(x, y) = x^3 + y^3 + 4$ is not homogeneous since

$$f(kx, ky) = k^3x^3 + k^3y^3 + 4.$$

Since M and N are homogeneous functions of the same degree, say n, $\frac{M}{N}$ is a homogeneous function of degree zero, as

$$\frac{M(kx, ky)}{N(kx, ky)} = \frac{k^n M(x,y)}{k^n N(x,y)} = \frac{M(x,y)}{N(x,y)}$$

Thus, we can say for Eq. (1.18) that $\frac{dy}{dx} =$ a homogeneous function of degree zero, and it is a homogeneous first order ODE; which can be solved by making the substitution $y = vx$. For example,

(a) the differential equation $y\,dx + (x+y)\,dy = 0$ is homogeneous as $M(x, y)\ \{= y\}$ and $N(x, y)\ \{= x + y\}$ are homogeneous functions of degree 1,

(b) the differential equation $y' = e^{x/y} + \cos(y/x)$ is homogeneous as y' has been expressed as a homogeneous function of degree zero.

To sum up, this method consists of the following steps.

> **Method of solving a homogeneous first order ODE**
>
> Step 1: Write the ODE in the form
>
> $$M(x,y)\,dx + N(x,y)\,dy = 0$$
>
> Step 2: Determine whether $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree,
>
> Step 3: Separate variables by making the **substitution** $y = vx$.

Indeed, you can now see that the ODE of Example 2 was a first order homogeneous **ODE.** This method of solving a first order homogeneous differential equation can also be applied to ODEs having linear coefficients. Such ODEs can be made homogeneous by a typical substitution called **linear substitution.**

ODEs with linear coefficients

Suppose the functions M and N in Eq. (1.18) are linear functions of x and y, **i.e.**

$$M(x, y) = a_1 x + b_1 y + c_1 \quad \text{and} \quad N(x, y) = a_2 x + b_2 y + c_2, \qquad (1.19)$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are constants. We can make such an equation homogeneous by making a linear substitution: $x = x' + h$, $y = y' + k$, where h and k are constants to be determined. In this case x' and y' are variables, and not derivatives. Let us illustrate this with the help of an example.

You can see that if $c_1 = c_2 = 0$ in Eq. (1.19), Eq. (1.18) is homogeneous of degree zero. Again, if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, you can make the substitution $u = a_1 x + b_1 y$, just by inspection, as you have done in SAQ 5(a).

Example 3

Solve the differential equation

$$(y - x - 2)\,dx + (x + y + 1)\,dy = 0$$

Solution

We put $x = x' + h$ and $y = y' + k$, so that we have

$$dx = dx' \text{ and } dy = dy'$$

The equation takes the form

$$(y' - x' + C_1)\,dx' + (x' + y' + C_2)\,dy' = 0$$

where $\quad C_1 = -h + k - 2 \qquad C_2 = h + k + 1$

We take the values of h and k to be such as to simultaneously satisfy the equations $C_1 = 0$ and $C_2 = 0$, i.e.,

$$-h + k - 2 = 0 \quad \text{and } h + k + 1 = 0$$

The solution to this system of equations is $h = -\frac{3}{2}, k = \frac{1}{2}$. And we have

$$(y' - x')\,dx' + (x' + y')\,dy' = 0$$

or

$$\frac{dy'}{dx'} = \frac{x' - y'}{x' + y'}$$

This equation is homogeneous in x' and y'.

You may now complete this example by putting $y' = vx'$. Do not forget to express the final solution in terms of x and y using the values of h and k.

SAQ 6

Spend 10 min

(a) Complete the solution to the differential equation given in Example 3. Do not forget to check your solution.

(b) Identify the homogeneous first order ODEs from the following:

i) $x^2 \frac{dy}{dx} = y^2 - 3xy + 5x^2$

ii) $(x^2 + y^2)\,dx + (x + y)\,dy = 0$

iii) $\{y + x \sin(y/x)\}\,dx - x\,dy = 0$

iv) $xy\,dx + (x^2 + 4)\,dy = 0$

v) $x\,dx + (y - 2x)\,dy = 0$

vi) $x\,dy - (y + \sqrt{x^2 - y^2})\,dx = 0$

So far we have discussed quite a few methods of solving first order differential equations. In all cases, we had taken the general form of the equation as

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

Now if the left-hand side is such that it can be expressed as $d[z(x, y)]$, then we get

$$d[z(x, y)] = 0 \qquad (1.20)$$

and the solution is

$$z(x, y) = C = \text{a constant.}$$

A first order ODE which can be expressed in the form (1.20) is called an **exact equation.** Let us study such equations.

1.6 EXACT EQUATIONS

From the definition of an exact equation, we can see that an equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

will be **exact** if there exists a function $z(x, y)$ for which

$$\frac{\partial z(x, y)}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial z(x, y)}{\partial y} = N(x, y) \qquad (1.21)$$

Then we can express the ODE as

$$\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}\partial y = 0 \qquad (1.22a)$$

or

$$d[z(x, y)] = 0 \qquad (1.22b)$$

You must have studied about partial derivatives and total differential of a function in Unit 2, Block 1 of Mathematical Methods in Physics-I (PHE-04). Recall that the total differential of a function $f(x, y)$ is $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$

and the required solution of the ODE is

$$z(x, y) = C = \text{a constant.}$$

Now, how do we find out whether a given ODE is exact or not? From Eq. (1.21), we have

$$M = \frac{\partial z}{\partial x} \quad \text{and} \quad N = \frac{\partial z}{\partial y}$$

or

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$

and

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

Now we know that if $z = z(x, y)$, then $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$. In other words.

$M\,dx + N\,dy = 0$ is an **exact** ODE if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad (1.23)$$

Eq. (1.23) is a necessary and sufficient condition for $M\,dx + N\,dy$ to be exact, provided M and N are continuous and have continuous partial derivatives.

The question now is how to solve an exact equation.

The method of solving exact equations

We can solve an exact equation in the following way. Integrating the first of Eq. (1.21) with respect to x while holding y constant we have

$$z(x, y) = \int M(x, y)\,dx + f(y) \qquad (1.24a)$$

Here the arbitrary function $f(y)$ is the 'constant' of integration. To determine $f(y)$, we differentiate Eq. (1.24a) w.r.t. y and use the second of Eq. (1.21).

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\int M(x, y)\,dx + \frac{df}{dy} = N(x, y)$$

This gives $\frac{df}{dy} = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\,dx \qquad (1.24b)$

Finally, we integrate Eq. (1.24b) w.r.t. y and substitute the result in Eq. (1.24a). The solution of the ODE is $z(x, y) = C$.

Note : **In** the above method we could as well have started **from** the second of Eq. (1.21): $\frac{\partial z}{\partial y} = N(x, y)$. The analogues of Eq. (1.24a) and (1.24b) would be, respectively,

$$z(x,y) = \int N(x,y)\,dy + g(x) \qquad (1.24c)$$

and

$$\frac{dg}{dx} = M(x,y) - \frac{\partial}{\partial x}\int N(x,y)\,dy \qquad (1.24d)$$

Let us now illustrate this method with an example.

Example 4

Show that the differential equation

$3x(xy-2)\,dx + (x^3+2y)\,dy = 0$ is exact.

Hence, solve it.

Solution

Here, $M = 3x^2y - 6x, \quad N = x^3 + 2y$

$$\therefore \frac{\partial M}{\partial y} = 3x^2, \ \frac{\partial N}{\partial x} = 3x^2, \text{ i.e. } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So the equation is exact.

Now, we have to solve the **ODE**. Since the **ODE** is exact, there exists a function $z(x, y)$, such that $dz(x, y) = M\,dx + N\,dy = 0$. We can now use either Eqs. (1.24a) and (1.24b) or Eqs. (1.24c) and (1.24d). From Eq. (1.24a) we get

$$z = \int M(x,y)\,dx + f(y) = \int (3x^2y - 6x)\,dx + f(y) = x^3y - 3x^2 + f(y)$$

Since $\frac{\partial z}{\partial y} = N(x,y)$, we have

$$x^3 + \frac{df}{dy} = x^3 + 2y$$

$$\therefore \frac{df}{dy} = 2y \text{ or } f(y) = y^2 + k,$$

where k is an arbitrary constant.

Thus, $z = x^3y - 3x^2 + y^2 + k$

So the required solution is $x^3y - 3x^2 + y^2 + k = C$ = a constant

or $x^3y - 3x^2 + y^2$ = a constant.

Checking the solution
Differentiating it w.r.t. x, we get
$3x^2y + x^3y' - 6x + 2yy' = 0$
or $3x(xy-2) + (x^3+2y)y' = 0$
or $3x(xy-2)\,dx + (x^3+2y)\,dy = 0$
which is the **ODE** being solved.

Let us summarise the method of solving an exact equation.

> **The method of solving an exact equation**
>
> **Step 1 :** Write the differential equation in the form
> $M(x,y)\,dx + N(x,y)\,dy = 0$ and check to make sure that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
>
> **Step 2 :** Evaluate (i) $z(x,y) = \int M(x,y)\,dx + f(y)$ or (ii) $z(x,y) = \int N(x,y)\,dy + g(x)$ (treating y and x, as constants in the integration processes (i) and (ii), respectively.).
>
> **Step 3 :** Evaluate the arbitrary functions f (y) or $g(x)$ that occur in Step 2 by putting $\frac{\partial z}{\partial y} = N(x,y)$ or $\frac{\partial z}{\partial x} = M(x,y)$.
>
> **Step 4 :** Write your solution in the form $z(x,y) = C$.

You may now like to work out an **SAQ** on solving exact equations.

Spend 10 min

SAQ 7

Check each of the following ODEs for exactness and solve the one that is exact

(a) $(x\cos y - y)\,dx + (x\sin y + x)\,dy = 0$

(b) $(e^x + y - 1)dx + (3e^y + x - 7)\,dy = 0$

Remember to check the solution.

You have now learnt to solve an exact equation. But what to do when $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, **i.e.**, the equation is inexact?

In fact, an inexact ODE can be made exact by multiplying it by a suitable function $F(x, y)$ ($\neq 0$). Such a function is called an integrating factor of the **ODE**. We can determine integrating factors for linear first order ODEs in a systematic way.

1.6.1 First Order Linear Differential Equations

You may recall from **Sec.** 1.3.2 that a first order linear nonhomogeneous differential equation defined on an interval in x has the form

$$a_1(x)\,y' + a_0(x)\,y = f(x), \qquad (1.25)$$

where $a_1(x) \neq 0$. On dividing both sides of Eq. (1.25) by $a_1(x)$, we get

$$y' + p(x)y = q(x) \qquad (1.26)$$

This is the **standard** form of a **first** order linear nonhomogeneous differential equation. We can now show that Eq. (1.26) can be solved by **obtaining** an integrating **factor** $v(x)$ which **depends only** on x. Now, if such a factor exists, **then** on multiplying Eq. (1.26) by $v(x)$, we should get an exact equation. In other words

$$v(x)y' + v(x)p(x)y = v(x)q(x)$$

must be an exact equation. We rewrite it as

$$[v(x)p(x)y - v(x)q(x)]\,dx + v(x)dy = 0$$

From the condition of exactness [Eq. (**1.23**)], we get

$$\frac{\partial}{\partial y}[v(x)p(x)y - v(x)q(x)] = \frac{\partial}{\partial x}[v(x)] = \frac{dv(x)}{dx} \qquad (1.27)$$

Hence, **from** Eq. (1.27), we get

$$\frac{dv(x)}{dx} = v(x)p(x)$$

or

$$\frac{d[v(x)]}{v(x)} = p(x)dx$$

Integrating **both** sides, we get

$$\ln|v(x)| = \int p(x)\,dx$$

$$\therefore\ v(x) = \exp[h(x)], \quad \text{where } h(x) = \int p(x)\,dx \qquad (1.28)$$

We have deliberately left out the constant of integration as we wish to have only one, the integrating factor $v(x)$. Now multiplying Eq. (1.26) by the integrating factor we get

$$e^h[y' + py] = e^h q$$

Since from Eq. (1.28), $h' = p$, we can write this equation as

$$\frac{d}{dx}[y e^h] = e^h q$$

Integrating both sides of the equation and dividing by e^h, we obtain

$$y = e^{-h}\left[\int e^h q\,dx + C\right], \quad h = \int p(x)dx \qquad (4.29)$$

This is the general solution of a first order linear nonhomogeneous ODE of the form (1.26). Let us apply this method to an example.

Example 5

Solve the equation

$$L\frac{di}{dt} + Ri = E_0 \sin \omega t$$

Solution

We may rewrite the equation as

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E_0}{L} \sin \omega t$$

$\therefore$ Integrating factor $= \exp\left[\int \frac{R}{L} dt\right] = e^{Rt/L}$

On multiplying the ODE by this factor, we get

$$e^{Rt/L}\left[\frac{di}{dt} + \frac{Ri}{L}\right] = \frac{E_0}{L} e^{Rt/L} \sin \omega t$$

$$\therefore \frac{d}{dt}\left(i\, e^{Rt/L}\right) = \frac{E_0}{L} e^{Rt/L} \sin \omega t$$

$$\therefore i\, e^{Rt/L} = \frac{E_0}{L} \int e^{Rt/L} \sin \omega t \, dt + C$$

where C is an arbitrary constant.

$\therefore$ The required solution is

$$i = \frac{E_0 \sin(\omega t - \theta)}{\sqrt{R^2 + \omega^2 L^2}}$$

Let $I = \int e^{Rt/L} \sin \omega t \, dt$

Integrating by parts, we get

$$I = e^{Rt/L}\left(-\frac{\cos \omega t}{\omega}\right) + \frac{R}{L\omega}\int e^{Rt/L} \cos \omega t \, dt$$

$$= -e^{Rt/L}\frac{\cos \omega t}{\omega} + \frac{R}{L\omega}\left[e^{Rt/L}\frac{\sin \omega t}{\omega} - \frac{R}{L\omega}\int e^{Rt/L} \sin \omega t \, dt\right]$$

or $I = \frac{e^{Rt/L}}{\omega^2 L}[R \sin \omega t - \omega L \cos \omega t] - R^2 I/\omega^2 L^2$

or $I = \frac{L\, e^{Rt/L}}{(R^2 + \omega^2 L^2)}(R \sin \omega t - \omega L \cos \omega t)$

Putting $\cos\theta = \frac{R}{\sqrt{R^2 + \omega^2 L^2}}$

and $\sin\theta = \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}}$

$$I = \frac{L\, e^{Rt/L} \sin(\omega t - \theta)}{(R^2 + \omega^2 L^2)^{1/2}}$$

where $\theta = \tan^{-1}(\omega L/R)$

The procedure for solving a first order linear non-homogeneous ODE can be **summarised** as follows.

> **The method of solving first order linear ODEs**
>
> **Step 1 :** Put the equation into the standard form $y' + p(x)y = q(x)$.
> (Note: The coefficient of y' must be 1).
>
> **Step 2 :** Identify $p(x)$ and compute $v(x) = \exp[\int p(x)\, dx]$
>
> **Step 3 :** Multiply the standard form of the equation by $v(x)$. The LHS of the equation will always be an ordinary derivative of the product $[y\, v(x)]$, w.r.t. the independent variable.
>
> **Step 4 :** Integrate both sides of the modified equation and solve for y.

With some experience you can also find the integrating factors by inspection for first order ODEs of other types. However, ***we will not test you on this count.***

You may now like to work out an SAQ on the above method.

SAQ 8

Spend 10 min

Solve $\quad xy' + 2y = x^3$

Note: Check the solution you obtain.

Thus, you have learnt some commonly used methods of solving first order ODEs. There can be higher order ODEs which can be reduced to first order and solved by applying any of these methods. We shall discuss some of them before ending this discussion.

1.7 EQUATIONS REDUCIBLE TO FIRST ORDER

Here, we shall consider two cases, each corresponding to a second order ODE.

i) If a second order ODE in x and y is devoid of y, then it can be expressed as

$$F(y'', y', x) = 0 \tag{1.30}$$

We make the substitution $w = y' = \frac{dy}{dx}$. Thus, Eq. (1.30) takes the form of a first order ODE

$$F(w', w, x) = 0 \tag{1.31}$$

To illustrate this technique, we consider the ODE

$$y'' + 2y' = 0 \tag{1.32}$$

We put w = y', so that

$$\frac{dw}{dx} + 2w = 0 \tag{1.33}$$

You can solve this equation by the method of separation of variables. Let us now consider the second case.

ii) If a second order ODE in x and y is devoid of x, then it can be expressed as

$$F(y'', y', y) = 0 \tag{1.34}$$

We again make the substitution w = y'. Then we express y'' as follows:

$$y'' = \frac{dy'}{dx} = \frac{dw}{dx} = \frac{dw}{dy}\frac{dy}{dx} = w\frac{dw}{dy}$$

Thus, Eq. (1.34) becomes

$$F\left(w\frac{dw}{dy}, w, y\right) \tag{1.35}$$

which is a first order ODE in w with y as the independent variable. To illustrate the method, we consider the following **ODE**

$$yy'' + (y')^2 = 0 \tag{1.36}$$

We put w = y', so that $y'' = w\frac{dw}{dy}$

So, we get

$$y\,w\frac{dw}{dy} + w^2 = 0 \tag{1.37}$$

Now this can be solved by the method of separation of variables.

Spend 5 min

SAQ 9

Complete the solutions of the ODEs (1.33) and (1.37).

Note: **Check** the solutions.

In this unit you have studied various methods of solving first order ODEs. Henceforth, whenever you come across a first order ODE, begin by classifying it. Then you could consider the following questions:

- Do the variables separate?
- Is there an obvious substitution which simplifies the ODE?
- Is the ODE exact?
- Is the equation linear, nonhomogeneous?

More often than not, the answers to these questions will tell you what method to use for solving the given ODE. Of course, some ODEs may be solved by more than one method. Then you could opt for the easiest one!

Let us now sum up what we have learnt in this unit.

1.8 SUMMARY

- Equations that contain ordinary or partial derivatives or differentials of one or more dependent variables **w.r.t.** independent variables are called differential equations.

- We classify a DE by its type: ordinary or partial; by its order and degree, and by whether it is linear or nonlinear. A linear ODE may be homogeneous or nonhomogeneous.

- A function $y = \phi(x)$ is a solution of a differential equation on some interval if $\phi(x)$ is defined and differentiable throughout that interval and is such that the DE becomes an identity when y is replaced by $\phi(x)$ in the DE. A solution involving arbitrary constant(s) is called a general solution. If definite value(s) can be assigned to the arbitrary constant(s) in a general solution by specifying certain conditions then it becomes a particular solution. Depending on the way the conditions are specified we get an initial value problem or a **boundary** value problem.

- The method of solution for a first order ODE depends on an appropriate classification of the equation. We summarise four methods.

  - An equation is separable if it can be put into the form $N(y)dy = M(x)dx$. The solution is obtained by integrating both sides of the equation.
  - The differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be **homogeneous** of first order if $M(x,y)$ and $N(x,y)$ are **homogeneous** functions of the same degree. It can be made separable by making the substitution $y = vx$. Further, if M and N are linear functions of x and y, the ODE can also be solved by making the substitutions $x = x' + h$, $y = y' + k$, $y' = vx'$, where x' and y' are variables and not derivatives.
  - The differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact if $M(x,y)\,dx + N(x,y)\,dy$ is an exact differential $[dz(x,y)]$. When M and N are continuous and have continuous partial derivatives, then $\partial M/\partial y = \partial N/\partial x$ is a necessary and sufficient condition for $M\,dx + N\,dy$ to be exact. Then there exists some function z for which $M(x,y) = \partial z/\partial x$ and $N(x,y) = \partial z/\partial y$. The method of solution of an exact ODE starts by integrating either of these latter expressions.
  - If a first order linear ODE can be put in the form

    $$y' + p(x)y = q(x),$$

    it can be reduced to the exact form by multiplying it with the integrating factor $\exp\left[\int p(x)dx\right]$. We can solve this equation by integrating both sides of the equation $\frac{d}{dx}\left[\left\{\exp\left(\int p(x)dx\right)\right\}y\right] = \left\{\exp\left(\int p(x)dx\right)\right\}q(x).$

- An ODE may be reduced to one of the familiar Forms by an appropriate substitution or change of variables.

  Second order ODEs of the forms $F(y'', y', x) = 0$ and $F(y'', y', y) = 0$ may be reduced to first order and hence solved by making the substitution $y' = w$.

1.9 TERMINAL QUESTIONS

Spend 30 min

1) Obtain the general solution of the first order ODE

$$2y' - 4y = 16e^x.$$

2) Obtain the particular solution of the differential equation

$$x\,dy - (y - \sqrt{x^2 + y^2})\,dx = 0 \text{ when } y = 4 \text{ for } x = 3$$

3) Reduce Eq. (1.2) to a first order ODE and hence solve it. It is given that $\frac{dx}{dt} = 0$ when $x = a$ and $\omega^2 = \frac{k}{m}$.

1.10 SOLUTIONS AND ANSWERS

SAQs (Self-assessment **questions**)

1) **ODEs:** (i), (iii), (v), (viii), **(ix). PDEs:** (iv), (vi), (x).

2)

Eqs.	Order	Degree	L/NL	NH/H
(2)	2	1	L	NH
(5)	2	3	NL	–
(6)	3	1	L	
(8)	2	1	L	H

3) a) Differentiating both sides of the equation

$$x^2 + y^2 - 1 = 0$$

with respect to x, we get

$$2x + 2yy' = 0$$

or $yy' + x = 0$, which is the given differential equation.

So $x^2 + y^2 - 1 = 0$ is a solution. It is implicit.

b) It is given that

$$y = Ax + \cos A$$

$$\therefore\; y' = \frac{dy}{dx} = A$$

Thus, on replacing A by y', the given equation becomes

$y = y'x + \cos y'$, which is the given ordinary differential equation.

So, $y = Ax + \cos A$ is a solution of the given ODE. As A is an arbitrary constant, it is a general solution.

4) a) $(y+1)y' + x = 0$

This is a separable equation. Using the method of separation of variables, we have

$$\int (y+1)\,dy = -\int x\,dx$$

or $$\frac{1}{2}y^2 + y = -\frac{x^2}{2} + C$$

or $$y^2 + x^2 + 2y = 2C$$

or $$(y+1)^2 + x^2 = 2C + 1$$

This is the equation of a family of concentric circles centred at (0, –1) and of radii $\sqrt{2C+1}$ (Fig. 1.3a).

(b) IVP : $y' = -2xy$, $y(0) = 3$ is also a separable ODE.

$$\int \frac{dy}{y} = -2\int x\,dx + C$$

or $$\ln|y| = -x^2 + C$$

or $$y = C_1 e^{-x^2} \text{ where } C_1 = \ln|C|$$

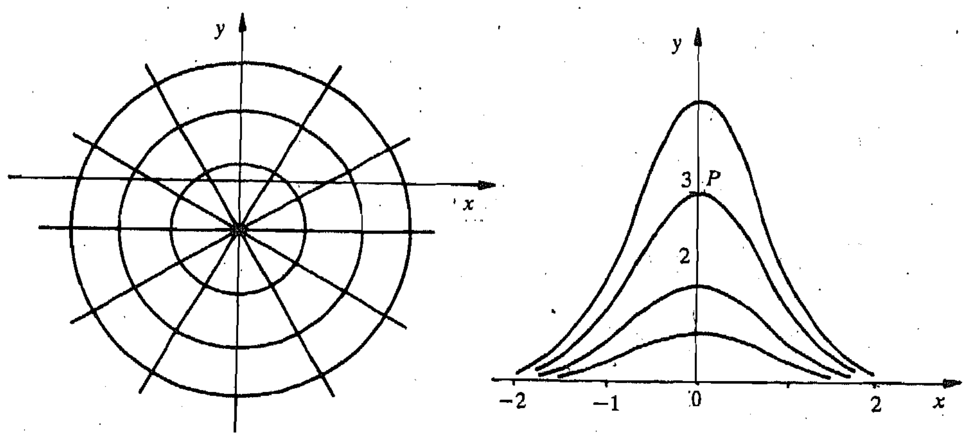

Fig. 1.3(a) : Concentric circles centred at (0, −1) ; (b) solutions of $y' = -2xy$ ('bell shaped' curves) in the upper half plane ($y > 0$). The curve P represents the particular solution.

From the initial condition we have

$$C_1 = 3$$

And the particular solution is

$$y = 3e^{-x^2}$$

The general and the particular solution is shown in Fig. 1.3b. You will come across such curves quite often in physics.

5) a) By inspection, we can suggest the substitution

$$x - 2y = v$$

Differentiating w.r.t. x we get

$$1 - 2\frac{dy}{dx} = \frac{dv}{dx} \quad \text{or} \quad y' = \frac{1}{2}(1 - v')$$

Substituting in the original ODE we get

$$(v-1) = \frac{(v+7)}{2}\left(1 - \frac{dv}{dx}\right)$$

$$\text{or} \quad \frac{dv}{dx} = 1 - \frac{2v-2}{v+7} = \frac{-v+9}{v+7}$$

Using the method of separation of variables we have

$$\int \frac{v+7}{v-9}\,dv = -\int dx + C$$

$$\text{or} \quad \int \left(1 + \frac{16}{v-9}\right)dv = -\int dx + C$$

$$\text{or} \quad v + 16\ln|v-9| = -x + C$$

Since $v = x - 2y$, we get the general solution in the form

$$x - 2y + 16\ln|x - 2y - 9| = -x + C$$

$$\text{or} \quad 2x - 2y + 16\ln|x - 2y - 9| = C$$

b) $(1 + \cos\theta)\,dr = r\sin\theta\,d\theta$

$$\therefore \frac{dr}{r} - \frac{\sin\theta\,d\theta}{1+\cos\theta} = 0$$

or $\int \frac{dr}{r} + \int \frac{(-\sin\theta)\,d\theta}{1+\cos\theta} = \ln|C|$

or $\ln|r| + \ln|1+\cos\theta| = \ln|C| \quad \left[\because \frac{d}{d\theta}(1+\cos\theta) = -\sin\theta\right]$

$\therefore \; r(1+\cos\theta) = C$

6) a) $\frac{dy'}{dx'} = \frac{x'-y'}{x'+y'} = \frac{1-y'/x'}{1+y'/x'}$

We put $y' = vx' \quad \therefore \frac{dy'}{dx'} = v + x'\frac{dv}{dx'}$

$\therefore v + x'\frac{dv}{dx'} = \frac{1-v}{1+v}$

or $x'\frac{dv}{dx'} = \frac{1-v}{1+v} - v = \frac{1-2v-v^2}{1+v}$

or $\frac{dx'}{x'} = \frac{(1+v)\,dv}{1-2v-v^2}$

or $\frac{dx'}{x'} + \frac{(1+v)\,dv}{v^2+2v-1} = 0$

or $\int \frac{dx'}{x'} + \int \frac{(1+v)\,dv}{v^2+2v-1} = \ln|C|$

Let $I = \int \frac{(1+v)\,dv}{v^2+2v-1}$

We put $u = v^2+2v-1$

$\therefore du = (2v+2)\,dv$

$= 2(v+1)\,dv$

or $I = \frac{1}{2}\int \frac{du}{u}$

$= \frac{1}{2}\ln|u|$

or $\ln|x'| + \frac{1}{2}\ln|u| = \ln|C|, \quad u = v^2+2v-1$

or $x' u^{1/2} = C_1$

$\therefore \; x'(v^2+2v-1)^{1/2} = C_1$

$\therefore \; (y'^2+2y'x'-x'^2)^{1/2} = C_1$

or $y'^2+2y'x'-x'^2 = C_1^2$

or $\left(y-\frac{1}{2}\right)^2 + 2\left(y-\frac{1}{2}\right)\left(x+\frac{3}{2}\right) - \left(x+\frac{3}{2}\right)^2 = C_1^2$

i.e., $x^2-y^2-2xy+4x-2y+A = 0$ is the required solution, where $A = C_1^2 + \frac{7}{2}$

b) (i), (iii), (v), (vi)

7) a) $M = x\cos y - y, \qquad N = x\sin y + x$

$\therefore \frac{\partial M}{\partial y} = -x\sin y - 1, \qquad \frac{\partial N}{\partial x} = \sin y + 1$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Hence, the equation is inexact.

b) $M = e^x + y - 1, N = 3e^y + x - 7$

$\therefore \frac{\partial M}{\partial y} = 1, \; \frac{\partial N}{\partial x} = 1$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and the equation is exact.

$\therefore \frac{\partial z}{\partial x} = e^x + y - 1$ and $\frac{\partial z}{\partial y} = 3e^y + x - 7$

From the former, we get

$z = e^x + xy - x + f(y)$

Hence, we have

$$\frac{\partial z}{\partial y} = 3e^y + x - 7 = x + \frac{df}{dy}$$

$$\therefore \quad \frac{df}{dy} = 3e^y - 7$$

or $\quad f(y) = 3e^y - 7y + C_1$

The required solution is $z(x, y) = C'$

or $\quad e^x + xy + 3e^y - x - 7y + C = 0.$

8) The given ODE may be expressed as

$$y' + \frac{2}{x} y = x^2$$

The integrating factor $= \exp\left(\int \frac{2}{x} dx\right) = \exp[2 \ln|x|] = \exp[\ln|x^2|] = x^2$

So, we have

$$\frac{d}{dx}(x^2 y) = x^4$$

or $\quad x^2 y = \int x^4 dx + C$

or $\quad x^2 y - \frac{x^5}{5} = C \quad$ is the required solution.

9) From Eq. (1.33), we get

$$\frac{dw}{2w} + dx = 0$$

or $\quad \frac{1}{2}\int \frac{dw}{w} + \int dx = C, \quad$ where C is an arbitrary constant.

or $\quad \frac{1}{2}\ln|w| + x = C$

$\therefore \quad w = Ae^{-2x}, \quad$ where $A = e^{2c}$

or $\quad \frac{dy}{dx} = Ae^{-2x}$. The required solution is $y = -\frac{A}{2}e^{-2x} + B,$

where B is an arbitrary constant.

From Eq. (1.37), we get

$$\frac{dw}{w} + \frac{dy}{y} = 0$$

On integration, we get $\ln(wy) = C$ or $wy = e^C$

$\therefore \quad \frac{dy}{dx} = \frac{A}{y}, \quad$ where $A = e^C$

or $\quad \int y\,dy = \int A\,dx$

So, the required solution is $\frac{y^2}{2} = Ax + B.$

Terminal Questions

1) You can see that the given ODE is a linear non-homogeneous first order ODE. So we can use the method discussed in Sec. 1.6.1. Rewriting the equation in the standard form, we get

$$y' - 2y = 8e^x$$

We note that $p(x) = -2$. So the integrating factor is

$$v(x) = \exp\left[-\int 2\,dx\right] = \exp(-2x)$$

Multiplying the given ODE (in standard form) by e^{-2x}, we get

$$e^{-2x}y' - 2y\,e^{-2x} = 8\,e^{-x}$$

or $$\frac{d}{dx}(y\,e^{-2x}) = 8\,e^{-x}$$

or $$d\,[y\,e^{-2x}] = 8\,e^{-x}dx$$

Integrating both sides yields

$$y\,e^{-2x} = -8\,e^{-x} + C$$

So the general solution is

$$y = -8\,e^{x} + C\,e^{2x}$$

2) The equation may be written as

$$\frac{dy}{dx} = \frac{y - \sqrt{x^2+y^2}}{x}$$ which is a first order homogeneous ODE.

We put $y = v\,x$, so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

And the right hand side $= \frac{vx - x\sqrt{1+v^2}}{x} = v - \sqrt{1+v^2}$

So we get,

$$v + x\frac{dv}{dx} = v - \sqrt{1+v^2}$$

or $$\frac{dv}{\sqrt{1+v^2}} = -\frac{dx}{x}$$

$$\therefore \quad \int \frac{dv}{\sqrt{1+v^2}} + \int \frac{dx}{x} = \ln|C|$$

or $$\ln|v + \sqrt{v^2+1}| + \ln|x| = \ln|C|$$

or $$\ln|x(v + \sqrt{v^2+1})| = \ln|C|$$

$$\therefore \quad x[v + \sqrt{v^2+1}] = C$$

i.e., $y + \sqrt{y^2+x^2} = C$ is the required general solution.

It is given that $y = 4$ for $x = 3$

$$4 + \sqrt{4^2+3^2} = C \quad \text{or} \quad C = 9$$

Hence, the particular solution is

$$y + \sqrt{y^2+x^2} = 9$$

Checking the Solution

We can rewrite the general solution as $y + \sqrt{y^2+x^2} = C$ (i)

or $x^2 + y^2 = (C-y)^2$

$= C^2 + y^2 - 2Cy$

or $x^2 = C^2 - 2Cy$ (ii)

This gives us

$$2x\,dx = -2C\,dy$$

or $dy = -\frac{x}{C}dx$ (iii)

Substituting $\sqrt{x^2+y^2}$ from (i), and dy from (iii) in the ODE, we get

$$x\left(-\frac{x}{C}dx\right) - dx\,(y - C + y) = 0$$

or $dx\left[-\frac{x^2}{C} - 2y + C\right] = 0$ (iv)

Substituting (ii) in (iv) gives us an identity. Thus, (i) is a solution of the given ODE.

3) $$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dv}{dt} = \frac{dx}{dt}\left(\frac{dv}{dx}\right) = v\left(\frac{dv}{dx}\right) \text{ and } \frac{k}{m} = \omega^2$$

Eq. (1.2) may thus be written as

$$v\frac{dv}{dx} + \omega^2 x = 0$$

or $\quad v\,dv + \omega^2 x\,dx = 0$

On integrating, we get

$$\frac{v^2}{2} + \frac{\omega^2 x^2}{2} = C, \text{ where } C \text{ is an arbitrary constant, i.e.,}$$

$$v^2 + \omega^2 x^2 = C', \quad \text{where } C' = 2C$$

But $\quad \frac{dx}{dt} = v = 0$, when $x = a$

$\therefore \quad C' = \omega^2 a^2$

$\therefore \quad v^2 = \omega^2(a^2 - x^2)$

or $\quad v = \frac{dx}{dt} = \pm\omega\sqrt{a^2 - x^2}$

or $\quad \frac{dx}{\pm\sqrt{a^2 - x^2}} = \omega\,dt$

$\therefore \quad \int \frac{dx}{\pm\sqrt{a^2 - x^2}} = \omega t + \delta \quad$ where δ is an arbitrary constant.

$$\therefore \quad \left.\begin{array}{l} \sin^{-1}\frac{x}{a} \\ \text{or} \\ \cos^{-1}\frac{x}{a} \end{array}\right| = \omega t + \delta$$

Thus, $\quad \frac{x}{a} = \begin{array}{c} \sin(\omega t + \delta) \\ \text{or} \\ \cos(\omega t + \delta) \end{array}$

Thus, the required solutions are

$$x = a\sin(\omega t + \delta) \text{ and } x = a\cos(\omega t + \delta)$$