$$u_o \cos\theta = u_o \frac{x}{\sqrt{x^2 + 20^2}} = (20\ \text{ms}^{-1}) \frac{x}{\sqrt{x^2 + 20^2}}$$

Then the space dependence Doppler-shifted frequency is given by

$$\nu'(x) = \nu_o \frac{v + u_o \cos\theta}{v}$$

$$= 500\ \text{Hz.} \left(1 + 0.06 \frac{x}{\sqrt{x^2 + 20^2}} \right)$$

You can plot this as a function of x for $-100\ \text{m} \leq x \leq 100\ \text{m}$. At $x = 0$, the car is moving perpendicular to the wave and at the instant when the car passes this point, the driver hears the true frequency, 500 Hz.

5. The particle displacement for the incident, reflected and transmitted waves are

$$\psi_i(x, t) = a_i \sin(\omega_o t - k_1 x) \quad \text{(i)}$$

$$\psi_r(x, t) = a_r \sin(\omega_o t + k_1 x) \quad \text{(ii)}$$

and

$$\psi_t(x, t) = a_t \sin(\omega_o t - k_2 x) \quad \text{(iii)}$$

The boundary conditions in this case are :

1. The particle displacement $\psi(x, t)$ is continuous at the boundary. That is, it has the same value immediately to the left and the right of the boundary at $x = 0$.
2. The excess pressure is same on the two sides of the boundary.

The first condition implies that

$$a_i + a_r = a_t \quad \text{(iv)}$$

For a longitudinal wave, $\Delta p = -E \frac{\partial \psi}{\partial x}$ where E is elasticity. Since $E = \gamma p_o$, where $\gamma = \frac{C_p}{C_v}$ and p_o is equilibrium pressure, we find that p_o cancels out on both sides and the second condition implies that

$$\frac{\partial \psi_i}{\partial x} + \frac{\partial \psi_r}{\partial x} = \frac{\partial \psi_t}{\partial x} \quad \text{(v)}$$

Eq. (v) gives :

$$-a_i k_1 \cos\omega_o t + a_r k_1 \cos\omega_o t = -a_t k_2 \cos\omega_o t$$

giving

$$k_1 (a_i - a_r) = k_2 a_t \quad \text{(vi)}$$

We know that

$$k_1 = \frac{\omega_o}{v_1}$$

Multiplying by $\rho_1 v_1$, we get

$$k_1 = \frac{\omega_o}{\rho_1 v_1^2} \rho_1 v_1 = \frac{\omega_o Z_1}{\gamma p_o}$$

as $Z_1 = \rho_1 v_1$ and $v_1 = \sqrt{\frac{\gamma p_o}{\rho_1}}$

Similarly, you can show that

$$k_2 = \frac{\omega_o Z_2}{\gamma p_o}$$

Using these results in (vi), we find that

$$\frac{\omega_o Z_1}{\gamma p_o}(a_i - a_r) = \frac{\omega_o Z_2}{\gamma p_o} a_t$$

or

$$Z_1 (a_i - a_r) = Z_2 a_t \quad \text{(vii)}$$

Since relations (vi) and (vii) connecting the incident, reflected and transmitted amplitudes are exactly the same as in the transverse case, the reflection and transmission amplitude coefficients are also given by the same relations.

UNIT 8 SUPERPOSITION OF WAVES-I

Structure

8.1 INTRODUCTION

You have studied in Unit 2 of Block 1, how a particle acted upon simultaneously by two simple harmonic oscillations gives rise to the formation of Lissajous figures.

You have also read about the general characteristics of waves in Unit 6 of this block; and of their behaviour at the interface of two media in Unit 7. In this unit you will study about the principle of superposition of waves. Under certain conditions, the superposition of waves leads to some interesting phenomena like the formation of stationary waves, beats, wave groups, interference, diffraction etc. In the present unit you will study the phenomena of stationary waves, wave group and beats. The other two topics, viz. Interference and Diffraction, will be discussed in Unit 9 of this Block.

In the present unit you will study the basic features, ***especially the sound producing part of the woodwine instruments.*** There are two basic types of pipes, viz. flute pipe and reed pipe, which you will study in this unit. Stationary waves are formed when two waves of the same angular frequency (i.e., same ω), same wavelength (i.e., of the same wave vector or propagation constant k) and of same amplitude, travelling on opposite directions superpose on each other. On the other hand, if two sound waves of slightly different frequencies are superposed, they produce beats.

Wave groups, sometimes also called the wave packets, are the result of superposition of waves of slightly different frequencies, The concept of wave packet is of great importance in the study of quantum mechanics, which we consider later.

In the next Unit you will study the superposition of two waves, which leads to the phenomena of interference. There you will also study about the necessary conditions for the interference of two waves. Towards the end, you will learn about diffraction of waves and some typical cases of diffraction phenomena.

Objectives

After going through this unit, you will be able to .

- Describe the principle of superposition of waves
- Explain the ideas underlying the formation of stationary waves
- Identify the positions of nodes and antinodes on a stationary wave
  List the characteristics of stationary waves
- Describe the formation of wave groups
- Compute the value of group velocity knowing the dependence of wave velocity on wavelength
- Calculate the number of beats produced if the frequencies of two superposing notes are known.

8.2 PRINCIPLE OF SUPERPOSITION OF WAVES

In Unit 2 of Block 1 you have studied the superposition of simple harmonic motions. You saw that when two or more simple harmonic motions act simultaneously on a particle, the resultant displacement of the particle at any instant of time is simply given by the algebraic sum of the individual displacements. This can also be extended to the case of waves.

Two or more waves can traverse the same path in a given space, independent of one another. This means that the resultant displacement of a particle at a given time is simply the algebraic sum of the displacements that are given to the particle by the individual waves. In other words. we can say that the resultant displacement of the particle is found simply by adding algebraically the displacements due to the individual waves. This is known as superposition of waves.

An interesting case of superposition of waves is that of radio waves. You know that radio waves of different frequencies are transmitted by different radio stations to broadcast their programmes. When they fall on the receiving antenna, the resultant electric current set up in the antenna is quite complex because of the superposition of different waves. Nevertheless, we find that we can still tune to a particular station. That is, out of the many, we can still choose and pick up the particular wave we want. In other words, if we have a wave group obtained by the superposition of a large number of individual waves, one can still separate the different waves that were superposed. This is indicative of the individual behaviour of waves, which is the basis of the superposition principle in waves.

Now you can demonstrate the principle of superposition by considering two pulses travelling on a rope in opposite direction as shown in Fig. 8.1. Before and after crossing each other, they act completely independently. At the time of crossing the resultant displacement is the algebraic sum of the individual displacements.

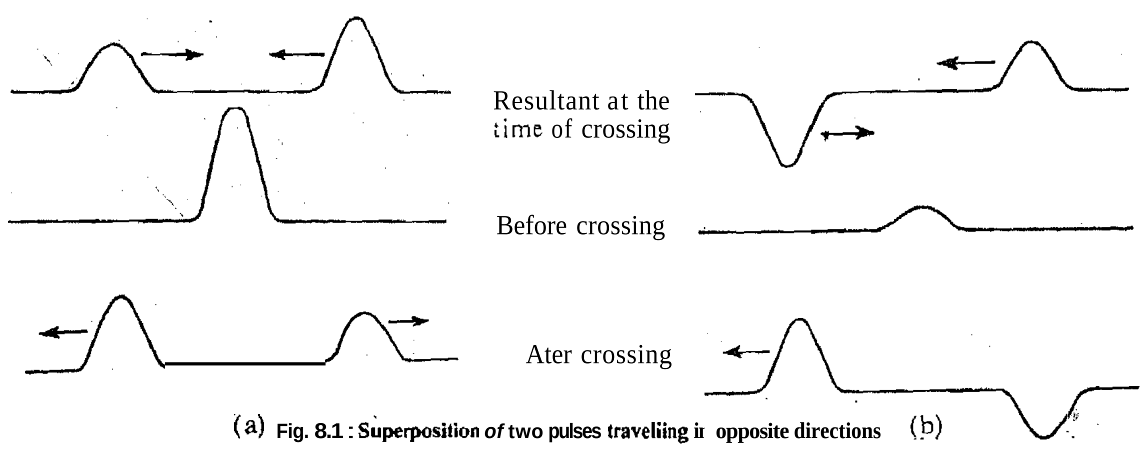


Fig. 8.1 : Superposition of two pulses travelling in opposite directions

You have also studied in Unit 2 of Block 1, the mathematical basis for the superposition of oscillations. It lies in the linearity of the equation. Consider two waves acting independently on a particle at any position x. Let $y_1(x, t)$ and $y_2(x, t)$ be the displacements of the particle at the instant of time t due to the two waves. Then the resultant displacement $Y(x, t)$ of the particle is mathematically written as :

$$Y(x, t) = y_1(x, t) + y_2(x, t) \qquad (8.1)$$

You have studied in Unit 6 of the present block, that a wave is essentially characterised by its amplitude, angular frequency, wave vector and phase. Depending on which of these components are same or different, you will study the various phenomena in Physics due to the superposition of waves. Let us consider some of these phenomena. For this, you consider the superposition of the following pair of waves.

I) $y_1 = a_1 \sin(\omega t - kx)$ and $y_2 = a_2 \sin(\omega t - kx)$

II) $y_1 = a \sin(\omega t - kx)$ and $y_2 = a \sin(\omega t - kx + \phi)$

III) $y_1 = a \sin(\omega_1 t - k_1 x)$ and $y_2 = a \sin(\omega_2 t - k_2 x)$

IV) $y_1 = a \sin(\omega t - kx)$ and $y_2 = a \sin(\omega t + kx)$ (8.2)

You will draw the following conclusion from above combination of waves.

(a) In case (I) only the amplitude of two waves differ.
Now let us consider the superposition of two waves of same angular frequency, wave vector and phase but different amplitude. These two waves are shown in case (I). Now applying Eq. (8.1) we can calculate that the resultant wave is given by

$$Y(x, t) = a_1 \sin(\omega t - kx) + a_2 \sin(\omega t - kx)$$
$$= (a_1 + a_2) \sin(\omega t - kx) \qquad (8.3)$$

Eq. (8.3) implies that the resultant wave has same frequency and phase and the resultant amplitude is $(a_1 + a_2)$. It is shown in Fig. 8.2.

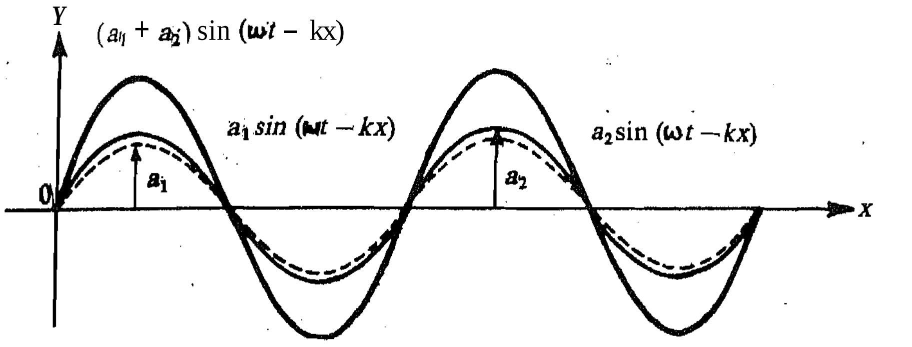


Fig. 8.2 : Superposition of two waves of same Frequency, wave vector and phase, but differing amplitudes a_1 and a_2.

(b) In case (II) only the phase of two waves differ.
Now you will consider the superposition of two waves which have same amplitude, frequency and wave vector but differin phase. When such waves superpose, you will find that the phenomenon of interference will occur. You will study this phenomenon in detail in Unit 9 of this block,

(c) In case (III), the frequency ω and wave vector k of the two waves differ.
Now let us consider the case when the frequencies and wave vectors of two waves differ slightly. In such a case, irrespective of phase difference the superposition results in an interesting phenomenon of 'Beats'. If however many waves of slightly different frequencies superpose, then they form waves groups or the wave packets. These give rise to group velocity, quite distinct from the wave velocity. You will study group velocity in detail in Section 8.4.

(d) In case (IV), the waves equations have differtnt signs before the wave vector (k). In this case, the first wave, $y_1(x, t)$ is propagating along the positive direction of x-axis, while the other wave, $y_2(x, t)$, is propagating in negative direction along x-axis. This implies that they are propagating in opposite directions. When such kind of waves superpose then stationary or standing waves are produced. You will study stationary waves in Section 8.3.

8.3 STATIONARY WAVES

You have just learnt in above section that stationary waves result if two waves of same angular frequency (i.e., ω) and wavelength (i.e. of same wave vector k), and of same amplitude travelling in opposite directions superpose on each other. To realize waves of exactly the same amplitude and wavelength, it is easier to consider one wave as incident wave, and the other as reflected wave from a *rigid boundary.*

The reflection of the incident wave can take place at a fixed boundary (like that of a string fixed to a wall, or the closed end of an organ pipe) or at a free boundary (like the free end of a string, or the open end of an organ pipe). We have learnt in the last Unit at a fixed boundary, the displacement $y(x, t)$ stays zero, and the reflected wave changes its sign. At a free boundary, however, the reflected wave has the same sign as the incident wave. In other words, at a fixed boundary, a phase change of π takes place, while at a free boundary, no such change of phase takes place.

Let us consider the case where the reflection is taking place at a free boundary. In this case, the resultant displacement is given by :

$$Y(x, t) = a \sin(\omega t - kx) + a \sin(\omega t + kx)$$
$$= 2a \sin \omega t \cos kx \qquad (8.4)$$

This can be written as :

$$Y(x, t) = (2a \cos kx) \sin \omega t \tag{8.5}$$

From Eq. (8.5) you see that the amplitude is given by ($2a \cos kx$) which is not fixed. It is dependent (or varies harmonically) on the position x of the particle. Further, the resultant motion has the same frequency and the wavelength as the individual waves.

Looking at equation (8.4) we note that the particles distributed along the x-axis execute vibrations perpendicular to the x-axis. The amplitudes with which they execute these vibration. are different at different positions along the x-axis. However, the time period of vibrations of all the particles is same.

We note that Eq. (8.5) does not represent a travelling wave since the argument of the sine . function is independent of the space variable x. We thus see that although we started with two waves propagating in opposite directions, we have ended up with something that does not propagate in space. The wave that does not travel (or propagate) is called a stationary (or a standing) wave. Since it does not propagate, it transports no energy ***along with*** if.

From equation (8.5) it is clear that the displacement $Y(x, t)$ is maximum when

$$\cos kx = \cos \frac{2\pi}{\lambda} x = \pm 1 \tag{8.6}$$

and minimum when

$$\cos kx = \cos \frac{2\pi}{\lambda} x = 0 \tag{8.7}$$

To satisfy **Eq. (8.6)** we require, $\frac{2\pi}{\lambda} x = m\pi$. Similarly **Eq.** (8.7) requires $\frac{2\pi}{A} = (2m + 1)$ $\pi/2$, with $m = 0, 1, 2, ...$ These give the points of maximum displacement at $x = 0, \lambda/2, \lambda, ..., m\lambda/2$; and minimum displacement at

$x = \lambda/4, 3\lambda/4, ..., (2m + 1)\lambda/4$

The points of maximum displacement are called **'Antinodes'**, while those of minimum displacement are called 'Nodes'. **The** distance between any two consecutive nodes or antinodes is $\lambda/2$, while that between a node and an **antinode** is $\lambda/4$ (Fig. 8.3).

From the above discussion you have learnt that a stationary wave results due to the **superposition** of **two** identical **progressive** waves travelling in opposite directions. The result is a **non-progressive** wave in which the disturbance is not handed over from one particle to the next. **The** space **(or** the region) where the two waves superpose gets divided into compartments or segments (Fig. 8.3). Each segment ends **with** points called the nodes where the displacement **of** the particles is always zero.

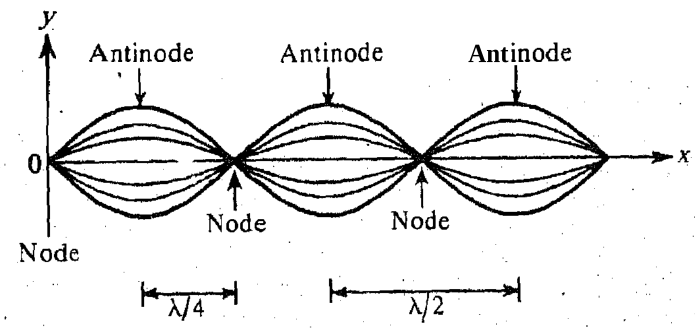


Fig 8.3 : The envelope of a standing wave showing the pattern of nodes and antinodes.

The **particles** at the **central** points of **these** segments **(called** the **antinodes)** execute vibrations with **maximum** amplitude. The particles lying in between the nodes and the antinodes execute vibrations with amplitudes lying in between zero and the maximum amplitude. This'is shown in Fig. 8.4.

The particle a, for example, is always at rest. The particle b always executes vibration with maximum amplitude, and the **particle** c always with intermediate amplitude **as** shown in **Fig.** 8.4.

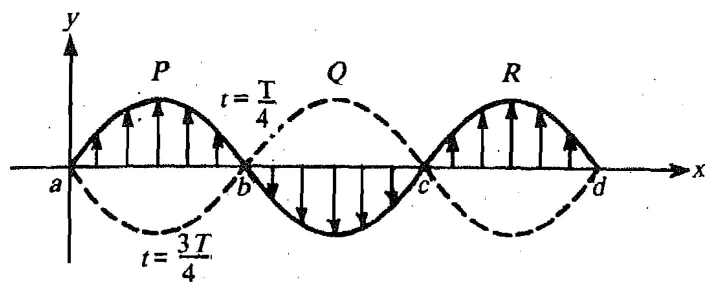


Fig 8.4 : Stationary wave, with mow-heads indicating the amplitudes with which various particles vibrate

SAQ 1

Derive equation for the displacement of a particle lying on a standing wave on a string fixed at both ends. Will the fixed end of a string be a node or an antinode? If the standing wave is in an open ended air pipe, will there be a node or an antinode at the end? How do you explain the absence of energy flow in a standing wave?

8.3.1 Velocity of a Particle and Strain at any Point in a Stationary Wave

You know that the velocity of a particle is defined as the rate of change of displacement with respect to time. The velocity of a particle in a stationary wave is calculated by differentiating the resultant displacement $Y(x, t)$ with respect to time keeping x as constant. If we differentiate Eq. (8.5) w.r.t. time, we get

$$\therefore \text{ Velocity } = \frac{dY}{dt} = 2a\omega \cos kx \cos \omega t \qquad (8.8)$$

The velocity is maximum when $\cos kx = \pm 1$, i.e. at points where $x = 0, \lambda/2, \lambda, \ldots.., \frac{m\lambda}{2}$ (see Eq. (8.6) and the discussion that follows). The velocity is minimum (zero) when $\cos kx = 0$, i.e. at points where $x = \lambda/4, 3\lambda/4, \ldots.., (2m + 1)\lambda/4$. It means that the velocity is maximum at the antinodes where the displacement is also maximum. The velocity is zero at the nodes where the displacement is zero. At points in between the antinodes and nodes, the velocity gradually decreases from maximum at the antinodes to zero at the nodes. The lengths of the arrow heads in Fig. 8.4 may also be taken to represent the Velocities of the particles in a stationary wave.

The strain on a particle in a stationary wave can be calculated by differentiating the resultant amplitude i.e. $Y(x, t)$ w.r.t. x keeping t constant. If we differentiate Eq. (8.5) w.r.t. x we get strain

$$\frac{\partial y}{\partial x} = - 2ak \sin kx \sin \omega t. \qquad (8.9)$$

You can show that the strain is maximum at the particle at the nodes where the displacement and the velocity are zero. This can also be visualised from Fig. 8.4. The particles at the nodes are stretched by particles moving in opposite directions. The strain is minimum at the antinodes where the displacement and velocity are maximum. Again refemng to Fig. 8.4, we can see that the particles at the antinodes always move along with the particles at their sides, not causing much strain on particles at the antinodes.

In case of stationary waves, the particles get divided into segments like the P, Q, and R in as shown in Fig. 8.4. Particles in one segment always move along in the same direction, When particles in segment P move up, those in Q move down, When those in Q move up, the ones in Pmove down. That is, in any two adjacent segments, particles move in opposite directions.

All particles in a particular segment reach the extreme positions at the same time, and also pass through the mean positions at the same time. This is shown in Fig. 8.5. All this is possible since all particles have the same time period T but have different velocities. These particles which have to cover larger distances have greater velocities. Those which have to cover the' smaller distances, have smaller velocities. .

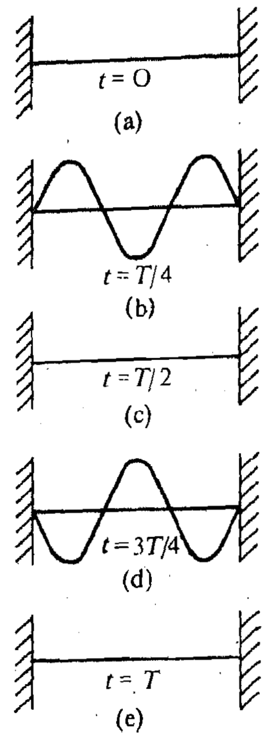


Fig 8.5 : Stationary waves on a string fixed at both ends. Shape of the string at different times during a time period is shown.

Now coming to the individual particle, we can see when its velocity is maximum, and **when** it is zero. Writing Eq. (8.8) as :

$$\frac{\partial Y}{\partial t} = 4\pi a\nu \cos \frac{2\pi}{\lambda} x \cos 2\pi\nu t$$

$$= \frac{4\pi a}{T} \cos \frac{2\pi}{\lambda} x \cos \frac{2\pi}{T} t$$

You can see that the particle velocity is zero for $t = T/4$ and $3T/4$, and is maximum for $t = 0, T/2$ and T. ***Thus*** during each time period, the particles of the medium have their **maximum** velocity when they pass through the mean position, and have zero velocity when they are at the extreme positions. Now in the next section you will study the conditions for producing different harmonics in stationary waves.

8.3.2 Harmonics in Stationary Waves

All musical instruments **based** on strings **utilise** the stationary wave phenomena. A string clamped at both ends allows stationary waves with some fixed wavelengths.

If the length of the string is L, the wavelength of the possible stationary waves on this string, starting from the longest wavelength are :

$= 2L, L, 2/34\ L/2$, ... etc. (**See** Fig. 8.6)

These wavelengths determine the frequencies of oscillation of the string through the relation $\lambda\nu = v$. **Here** v the velocity of the transverse **wave** on the string. It is given **by the** relation

$$v = \sqrt{\frac{T}{\mu}}$$

where T is the **tension** in the string, and μ is the linear **mass** density (**mass** per unit length) of the string.

The lowest frequency ν_0 of vibration is called the **fundamental** frequency. It is given by :

$$\nu_0 = \frac{v}{\lambda} = \frac{1}{2L}\sqrt{\frac{T}{\mu}} \qquad (8.10)$$

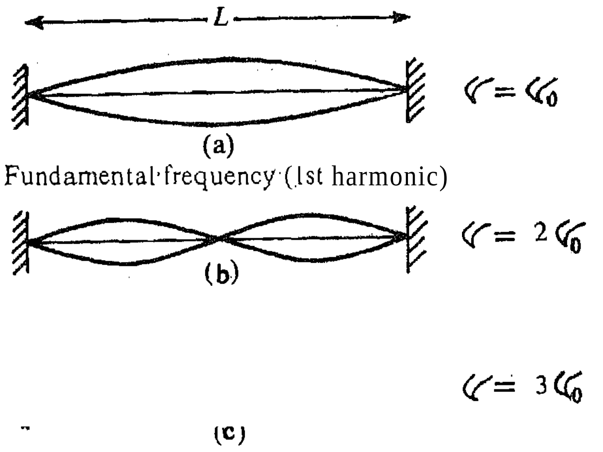


Fig. 8.6 : Allowed stationary waves on a *string* of length L clamped at its ends

The other frequencies are called the overtones, ana are integral multiples of the fundamental frequency ν_o (See Fig. 8.6).

The fundamental frequency is also called the first harmonic. The first overtone, with frequency $\nu = 2\nu_o$, is called the second harmonic. The second overtone, with frequency $\nu = 3\nu_o$, is called the third harmonic, and so on.

The musical instruments based on the principle of standing wave are flute and reed etc. The primary elements, which determine the tone quality and overall sound are (1) the **source** of noise or vibration (2) the size and shape of the bore, and (3) the size and positions of the finger holes. The quality of woodwind tones depends on the combination of physical and ***musical*** experience. From the physics point of view, the air is stored under pressure in the wind chest. A large reservoir is required to keep the pressure steady, while the various combinations of notes are played with fingers. In the above instruments one end is open, making them open ended organ pipes. The closed end of an organ pipe acts as a fixed boundary, while the open end as a free boundary. At the closed end there is always a node, and at the open end there is always an antinode.

For a pipe having one end closed, the fundamental wavelength is $\lambda = 4L$. This gives the fundamental frequency $\nu_o = \frac{v}{4L}$. In such a pipe, the even-numbered harmonics are absent (See Fig. 8.7). For a both ended open pipe, the **fundamental** wavelength is $\lambda = 2L$, giving **fundamental** frequency $\nu_o = \frac{v}{2L}$.

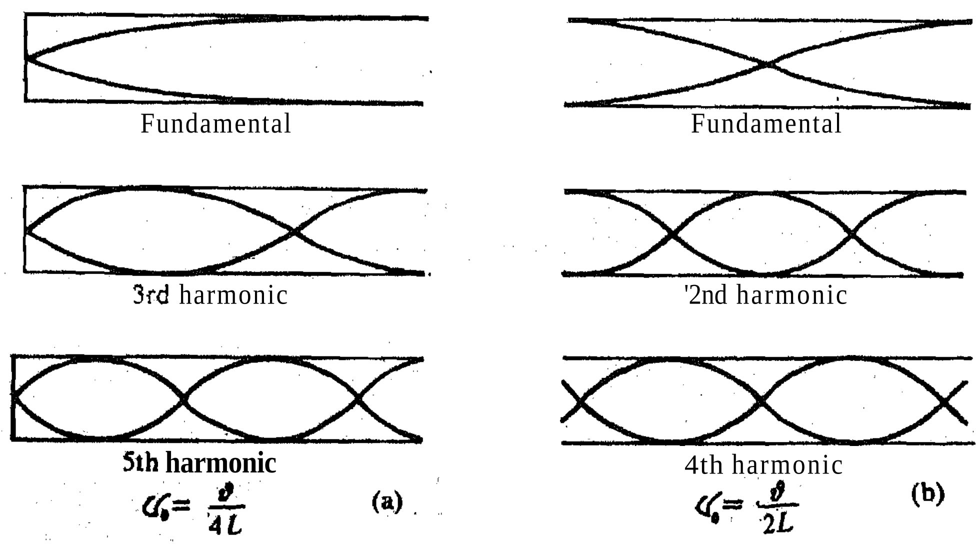


Fig. 8.7 : Modes of vibrations of longitudinal stationary waves in organ pipes with (a) one end closed, and (b) both ends open

Waves

SAQ 2

(a) A piano string of length 1m is fixed at both ends. Its mass per unit length is 0.015 kg m^{-1} is used to strike a fundamental note of frequency $\nu = 220$ Hz. Find the tension to be applied to the string.

(b) Estimate the frequency of the fundamental mode in a one end closed organ pipe of length 1.0 m. Use velocity of sound $v = 350$ m/s. What happens to frequency if the pipe is overblown?

8.3.3 Properties of Stationary Waves

The properties of the stationary waves that distinguish them from progressive waves have been highlighted in the forgoing discussion. Can you now write down the various points which characterise the stationary waves. After doing this, compare your points with the ones listed below :

i) Stationary waves are not progressive. In these the disturbance is not handed over from one particle to the next.

ii) The amplitude of each particle is not the same. It is maximum at the antinodes and zero at the nodes. In between it gradually decreases from that at the antinode to the one at the node, i.e. zero.

iii) The distance between two consecutive nodes or two consecutive antinodes is half the wavelength of the stationary wave. The medium splits into segments, with length of each segment equal to half the wavelength.

iv) All the particles between, two consecutive nodes are in the same phase, i.e. they reach their maximum and minimum displacement positions (mean positions) at the same time. The phase of particles, in one segment is opposite to that of particles in the adjoining segment.

v) The velocity of particles at the nodes is zero. The velocity of particles at the antinodes is maximum. For particles in between, the velocity gradually decreases from that at the antinodes to the one at the nodes (i.e. zero).

8.4 WAVE GROUPS AND GROUP VELOCITY

So far we have considered the superposition of two identical waves travelling in opposite directions to give rise to stationary waves. Now let us see what happens when two waves of slightly different angular frequencies ω_1 and ω_2, travelling in the same direction, superpose on each other (Case III). To avoid unnecessary mathematical complexities, we take the amplitudes of the two waves to be equal. The superposition of such two waves is given by

$$Y(x, t) = a \sin(\omega_1 t - k_1 x) + a \sin(\omega_2 t - k_2 x)$$

$$= 2a \sin\left[\frac{(\omega_1 + \omega_2)t - (k_1 + k_2)x}{2}\right] \cos\left[\frac{(\omega_1 - \omega_2)t - (k_1 - k_2)x}{2}\right] \qquad (8.11)$$

If ω_1 and ω_2, and similarly k_1 and k_2, are only slightly different, we can write

$\omega_1 - \omega_2 = \Delta\omega$ and $k_1 - k_2 = \Delta k$

Further, writing

$$\omega_{av} = \frac{\omega_1 + \omega_2}{2} \text{ and } k_{av} = \frac{k_1 + k_2}{2},$$

Eq. (8.11) becomes :

$$Y(x, t) = 2a \sin(\omega_{av} t - k_{av} x) \cos\left(\frac{\Delta\omega}{2} t - \frac{\Delta k}{2} x\right) \qquad (8.12)$$

Now let us see what the new wave form represented by Eq. (8.12), looks like. Firstly, its amplitude is twice that of the amplitude of either wave. Secondly, it is made up of two parts. The faster varying part (i.e. sine part) has a frequency which is the mean of the frequencies of. the two component waves. The slowly varying part (i.e. cosine part) has a frequency which is half of the difference of the two frequencies. The propagation vector of the slowly varying part of the superposed wave is $\Delta k/2$. It acts as an envelope over the faster varying part as shown in Fig. 8.8.

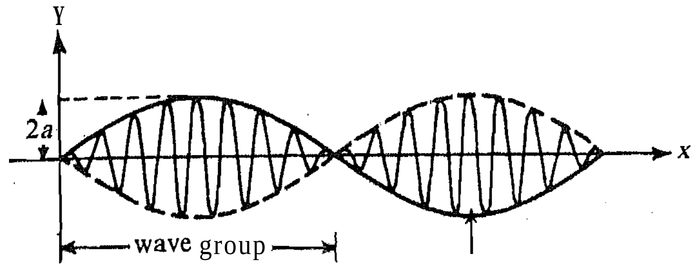


Fig. 8.8 : Superposition of two waves of slightly different frequencies ω_1 & ω_2

The superposition, as you can see in Fig. 8.8, results in the formation of groups (or segments) called the wave groups (or the wave packets). A wave group can travel with a velocity which may be different from that of the individual waves, or of the resultant wave. The velocity of the wave group is called the group velocity. The ratio of angular frequency and wave vector of the slowly moving part of the superposed wave is called group velocity. It is given by the following relation :

$$v_g = \frac{\Delta\omega/2}{\Delta k/2} = \frac{\Delta\omega}{\Delta k} \quad (8.13)$$

If a group consists of a number of component waves with angular frequencies lying between ω_1 and ω_2 (with $\omega_1 \approx \omega_2$), and similarly in wave vector k_1 and k_2 (with $|k_1| \approx |k_2|$), the group velocity V_g is then written as :

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk} \quad (8.14)$$

Here $d\omega$ and dk represent the spreads (gaps between the maximum and the minimum) in angular frequencies and propagation constants of the component waves that go to make a wave group.

The velocity of the resultant superposed wave is called the phase velocity. You can obtain this using Eq. (8.12), i.e.

$$v_p = \frac{\omega_{av}}{k_{av}}$$

If, however, the individual wave velocities are equal, i.e.

$$\frac{\omega_1}{k_1} = \frac{\omega_2}{k_2} = v \text{ (say)}$$

then $v_p = \frac{\omega_1 + \omega_2}{k_1 + k_2} = \frac{k_1 v + k_2 v}{k_1 + k_2} = v$

and $v_g = \frac{\omega_1 - \omega_2/2}{k_1 - k_2/2} = \frac{vk_1 - vk_2}{k_1 - k_2} = v$

i.e. the group velocity is equal to the phase velocity.

The group velocity is a more fundamental quantity in physics as the energy is transferred by the wave with the group velocity. The relation between the phase and the group velocities is given by :

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(kv)$$

which on simplification gives

$$v_g = v + k\frac{dv}{dk} \quad (8.15)$$

If we write,

$k = 2\pi/\lambda$

Then $dk = -\frac{2\pi}{A'} dA$

Waves

Inserting this in Eq. (8.15), we get

$$v_g = v + \frac{2\pi}{\lambda}\frac{dv}{(-2\pi/\lambda^2)(d\lambda)}$$

$$= v - \lambda\frac{dv}{d\lambda} \qquad (8.16)$$

This gives another relation connecting the phase and the group velocities.

The wavelength of the resultant wave is given by

$$\lambda = \frac{2\pi}{k}$$

and that of the enveloping wave by :

$$\lambda_e = \frac{2\pi}{\Delta k/2} = \frac{4\pi}{\Delta k}$$

since Δk is very small compared to k, $\lambda_e >> \lambda$.

If λ_1 and λ_2 represent the wavelengths of the component waves, it can be easily shown that

$$\frac{\lambda_e}{2} = \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1} \qquad (8.17)$$

This gives the length (or the extent) of the wave group. We can see from Fig. 8.9 that the length of the wave group is half of the wavelength of the enveloping wave, i.e., it is **equal** to $\lambda_e/2$.

To illustrate the difference between phase and group velocities, we consider the striking example of waves in deep water-called "gravity waves". These waves are strongly dispersive. For them the phase velocity is found to be proportional to the square root of the wavelength i.e.,

$$v_p = C\lambda^{1/2}$$

or

$$v_p = C_1 K^{-1/2} \text{ (Since } K = \frac{2\pi}{\lambda}\text{)}$$

Here, the new constant $C_1 = C\sqrt{2\pi}$

$v_p = \frac{\omega}{K}$, therefore $\omega = C_1 K^{1/2}$

Differentiating ω with respect to k, we get

$$v_g = \frac{d\omega}{dk} = \frac{1}{2}C_1 K^{-1/2} = \frac{1}{2}v_p$$

That is the group velocity for gravity waves is just half of the phase velocity. In other **words,** for these waves, the component wave crests move faster through the group as a whole.

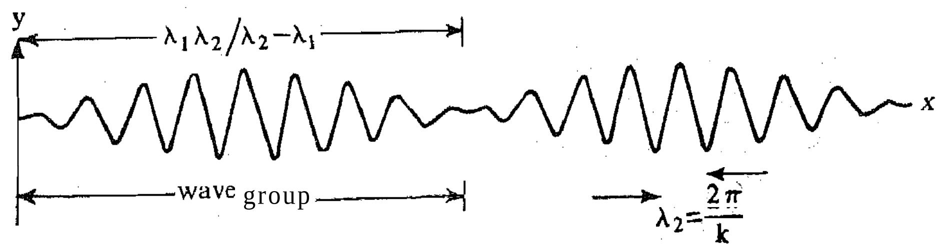


Fig. 8.9 : Wave group and its extent

SAQ 3

The phase velocity of a wave in a certain medium is represented by :

$$v = C_1 + C_2\lambda$$

where C_1 and C_2 are constants. What is its group velocity?

8.5 BEATS

We have **seen** above that the superposition of two **waves** of slightly different **angular** frequencies ω_1 and ω_2 leads to the **formation** of wave groups. You may have **noticed from.**

Figs. *8.8* and Fig. *8.9* that we have plotted the resultant displacement $Y(x, t)$ against distance x. This may be called the superposition in space. For this we kept the time t as constant, We may now consider another type of superposition, where we may plot $Y(x, t)$ against t, and call it as superposition in time. For this we may keep x as constant.

The superposition in time for sound waves leads to the interesting phenomenon of beats. The beats are loud sounds which we hear at regular intervals of time depending on the difference in. frequencies of the two superposing waves. The beats are often used by musicians for tuning their instruments.

Let us consider two waves of slightly different angular frequencies ω_1 and ω_2, and of the same amplitude a, proceeding in the same direction, as we have done in the last section. Let us fix the spatial coordinate x in Eq. (8.10), say, at $x = 0$. This corresponds to an observer standing at $x = 0$, and watching the waves passing by. He will observe a resultant waveform given by :

$$Y(x, t) = Y(4\ t) = a \sin(\omega_1 t) + a \sin(\omega_2 t)$$

$$= 2\ a \sin \omega_{av} t \cos \frac{Ao}{2} t \qquad (8.18)$$

Like the earlier case discussed in Section *8.4*, Eq. *(8.18)* indicates that the amplitude of the resultant wave at a given point is not constant, but varies in time. This has a angular frequency $\omega_{av} = \frac{\omega_1 + \omega_2}{2}$. Its amplitude varies between 2a and zero, because of the presence of tlie $\cos(\frac{Ao}{2} t)$ term. This term acts as an envelope on the sine term.

If ω_1 and ω_2 are nearly equal, Ao is small. In that case the amplitude of the resultant wave varies slowly. The periodic rise and fall of this wave leads to the appearance of beats; or to the hearing of loud sounds at regular intervals of time.

Beat are heard at the maxima of amplitude (See Fig. *8.10)*. They occur whenever $\cos \frac{\Delta\omega}{2} t = \pm 1$. This is because the intensity of sound is directly proportional to the square of the amplitude. The maximum amplitude occurs twice in every time period associated with the angular frequency $\frac{Aw}{2}$. Thus the frequency of beats is simply the difference of the two component frequencies i.e. $(\omega_1 - \omega_2)$.

In terms of frequencies ν_1 and ν_2, the beat frequency is $\Delta\nu = \nu_1 - \nu_2 = \frac{Aw}{2\pi}$. The time elapse between any two consecutive beats, called the beat period $= \frac{1}{\Delta\nu}$ (see Fig. *8.10)*.

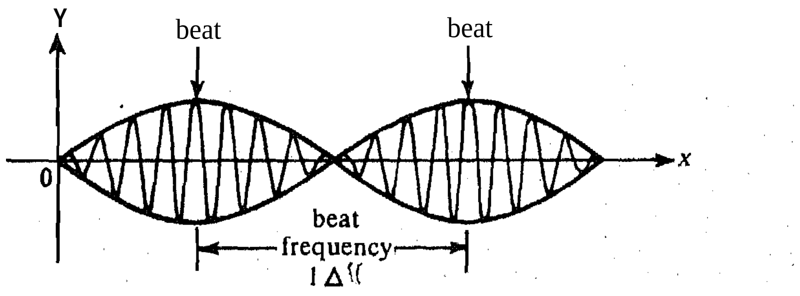


Fig. 8.10 : Formation of beats due to superposition of two waves of nearly the same frequency.

SAQ 4

When a certain note of a piano is sounded with a tuning fork of frequency 560 Hz, 6 beats are heard every second. Find the frequency of the note.

8.6 SUMMARY

'1. When two waves travelling through the same space superpose on each other, the resultant displacement at any point is given by the algebraic sum of the individual displacements,

2. Stationary waves result because of the superposition of two waves of same amplitude,

Waves

frequency and wavelength travelling in opposite directions and confined between two points.

3. On a stationary wave, nodes and antinodes are points of zero and maximum displacement, respectively. Distance between any two consecutive nodes or antinodes is half the wavelength of the stationary wave.

4. Both transverse and longitudinal waves can have different modes of vibration.

5. Superposition of two waves of slightly different frequencies travelling in the same direction gives rise to wave groups, and beats.

6. Number of beats produced per second is equal to the difference in the frequencies of the two waves.

7. The velocity with which a wave group travels is called the group velocity. It is equal to the wave velocity if the two component waves have the same velocity; otherwise, it is different from the wave velocity.

8. The smaller is the difference between the wavelengths of the component waves, the greater is the length of the wave group.

8.7 TERMINAL QUESTIONS

1. Two points on a string are observed as a travelling wave passes them. The points are at $x_1 = 0$ and $x_2 = 1$ m. The transverse motion of the two points are found to be as follows :

   $y_1 = 0.2 \sin 3\pi t$

   $y_2 = 0.2 \sin (3\pi t + \pi/8)$

   (a) What is the frequency in hertz?

   (b) What is the wavelength?

   (c) With what speed does the wave travel?

2. Fifty tuning forks are arranged in order of increasing frequency and any two successive forks gives 5 beats per second when sounded together. If the last fork gives the octave of the first, calculate the frequency of the latter. (A note is octave of another note if its frequency is double that of the other)

3. A closed pipe, 25cm long, resounds when full of oxygen, to a given tuning fork. Find the length of a closed pipe, full of hydrogen which will resound to the same tuning fork. (Velocity of sound in oxygen = 320 m/s and velocity of sound in hydrogen = 1280 m/s).

4. The phase velocity (v) of transverse wave in a crystal of atomic separation d is given by

   $$= C' \frac{\sin (kd/2)}{(kd/2)}$$

   where C is a constant. Show that the group velocity is C' cos $(kd/2)$.

8.8 SOLUTIONS

SAQ 1

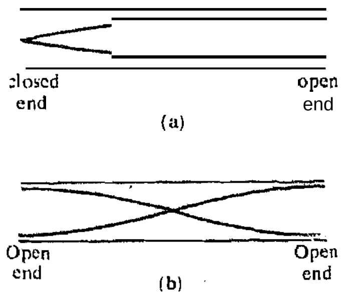
Since there is a phase change of π on reflection at the fixed end, the reflected wave is given by :

$y_2 = - a \sin (\omega t + kx)$

This leads to the resultant displacement $Y(x, t)$ as :

$$Y(x, t) = 4 \sin (\omega t - kx) - a \sin (wt + kx)$$
$$= - 2a \sin kx \cos \omega t$$
$$= A \cos \omega t$$

with $A = - 2a \sin kx$.



At the fixed end, there is always a node, as the displacement is zero. In open ended pipes, as shown, there is always an antinode at the end.

A standing wave is formed because of a positive x directed incident wave, and a negative x directed reflected wave. Each carry the same amount of energy in opposite directions. The net energy flow is thus always zero.

SAQ 2

(a) Wavelength of fundamental mode

$\lambda = 2L = 2 \times 1m = 2m$

Velocity v of wave = 220 Hz $\times$ 2m = 440 m/s

From $v = \sqrt{\frac{T}{\mu}}$, $T = v^2\mu$

$= (440\ m/s)^2 \times 0.015\ kg/m$

$= 2.9 \times 10^3$ N.

(b) Wavelength of fundamental mode

$\lambda = 4L = 4 \times 1m = 4m$

Frequency $\nu = \frac{v}{\lambda} = \frac{350\ m/s}{4m} = 87.5\ Hz \simeq 88\ Hz.$

By over-blowing the pipe, pitch jumps by a factor of 3 giving the next harmonic with frequency $\nu = 3 \times 88 = 264$ Hz.

SAQ 3

We know that

$v_g = v - \lambda \frac{dv}{d\lambda}$

For the wave in question, $\frac{dv}{d\lambda} = C_2$

Inserting in above equation,

$v_g = C_1 + C_2\lambda - \lambda C_2 = C_1$

SAQ 4

Let the frequency of the note be ν. Then

$6 = [560 - \nu]$

$\therefore$ ν = 554 Hz or 566 Hz

In this case, the frequency of the note cannot be found without ambiguity. However, it is either of the above two.

TQs

1. (a) $\nu = 1.5$ Hz

(b) $\lambda = \frac{16}{16n - 1}$ m, n = 1, 2, 3 ...., for +ve moving wave.

$= \frac{16}{16n+1}$ m, n = 1, 2, 3 ...., for +ve moving wave.

(c) $U = +8/5$ m/s etc.,

$V = -24$ m/s etc.

2. Let the frequency of the first note be n.

Then the frequency of the Second fork = $n + 5 = n + (2 - 1)5$

Frequency of the Third fork = $n + 5 + 5 = n + 10 = n + (3 - 1)5$

Frequency of the Fourth fork = $n + 5 + 5 + 5 = n + 15 = n + (4 - 1)5$

Frequency of the Fifth fork = $n + 20 = n + (5 - 1)5$

Therefore the frequency of the 50th fork

$= n + (50 - 1) \times 5 = n + 245$

Since the frequency of 50th fork is 2n then

$n + 245 = 2n$

So n = 245 Hz.

3. For the first pipe, the fundamental frequency

$\nu_1 = \frac{V_0}{4l_1}$

Waves

where V_o is the velocity of sound in oxygen and l_1 is the length of the 1st pipe. For the second pipe, the fundamental frequency is

$$\nu_2 = \frac{V_h}{4l_2}$$

Where V_h is the velocity of sound in hydrogen and l_2 is the length of the pipe.

Since both the pipes resound to the same frequency, therefore

$$\nu_1 = \nu_2 \text{ or } \frac{V_o}{4l_1} = \frac{V_h}{4l_2}$$

$$\therefore\ l_2 = \frac{V_h}{V_o} \times l_1 .$$

Substitutions the values of V_h, V_o & l_1, we get

$$l_2 = \frac{1280\ ms^{-1}}{320\ ms^{-1}} \times 25\ cm = 100\ cm$$

4. Group velocity

$$v_g = \frac{d\omega}{dk}$$

and

$\omega = kv$ we have

$$= C' \frac{\sin (kd/2)}{(kd/2)}$$

Now

$$= kv = k\,C' \frac{\sin (kd/2)}{(kd/2)}$$

$$= \frac{2C'}{d} \sin (kd/2)$$

or

$$v_g = \frac{d\omega}{dk} = \frac{2C'}{d} (\cos \frac{kd}{2}) \frac{d}{2}$$

or

$$v_g = C' \cos (kd/2)$$

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UNIT 9 SUPERPOSITION OF WAVES-PI

Structure

9.1 INTRODUCTION

In the last unit you have studied the principle of superposition of waves and employed it to study the phenomena of formation of stationary waves, wave groups and beats.

You have also learnt about the superposition of two waves which have the same amplitude and frequency but differ in phase. When such waves superimpose on each other, the phenomenon of interference is said to take place. For producing interference, the sources of waves must be coherent. That is they must emit waves with zero or constant difference of phase. In this unit you will study how coherent sources are produced, and how intensity varies in an interference pattern. You will also learn about the appearance of colours in thin films of oil spread over water.

The phenomenon of diffraction which results due to the superposition of many waves of same amplitude and frequency, but differing slightly in phase, is usually referred to as the bending of waves round the corners. Because of this phenomenon, we are at times inclined to think as if waves do not travel in straight lines. There are two classes of diffraction patterns, called Fresnel and Fraunhofer classes of diffraction. You will learn that the distinction between these two types of diffraction is related to the relative separations between the sources of waves and the obstacles (or the apertures) producing the diffraction patterns.

Both interference and diffraction are very important phenomena in physics. They have contributed immensely in justifying the wave nature of light. The difference between the two is quite subtle. Interference arises because of superposition of waves originating from two (or more) narrow sources, derived from the same source. Diffraction arises from superposition of wavelets from different numerous parts of the same wavefront, as will be discussed later in this unit.

In Unit 6, you have studied about different kinds of waves like sound and light waves. You have also studied that sound waves are longitudinal while light waves are transverse. Both give rise to the same phenomena when waves superpose on each other. Basically whatever is true for one kind of wave is also true for the other. If light waves show the phenomenon of interference and diffraction, so do the sound waves. Light wave effects have to be observed while sound wave effects have to be heard. Since the wavelength of sound waves is much greater than the wavelength of visible region light waves; the sound wave effects are in general on a larger scale compared to the effects of the light waves.

Objectives

After going through this unit, you will be able to :

- give examples of coherent sources
- derive the condition connecting the path difference between waves from two coherent sources and the wavelength of the waves used for getting maxima and minima of intensity on a screen placed in the path of waves
- outline the variation of intensity in an interference pattern