UNIT 13 GRAPH COLOURINGS AND PLANAR GRAPHS

Structure **Page** No.

13.1 INTRODUCTION

You must have seen political maps of India with different states coloured differently to distinguish between them. Have you ever wondered what is the minimum number of colours required to colour the map so that any two states with a common boundary are given two different colours? 'This problem of finding the minimum number of colours needed to colour a given map is called the map colouring problem.

We can formulate the problem in terms of graph theory. We can construct a graph in such a way that each state of India corresponds to a vertex of India and if two states are adjacent, the corresponding vertices are also adjacent. So, we have to colour the vertices of the graph in such a way that any pair of adjacent vertices have different colours. In the map colouring: problem, we ask for the minimum number of colours needed to carry out such a colouring.

Note that the construction mentioned above leads to a special class of graphs called planar graphs. If we are interested in map colouring problem alone, it is enough to restrict ourselves to such graphs. However, the general vertex colouring problem, which asks for the minimum number of colours needed to colour the vertices of a given graph, not necessarily planar, is interesting in itself. So, we start our unit by discussing this problem in Sec.13.2.

In Sec.13.3, as a preparation for our study of map colouring problem, wc study planar graphs. In this section we will prove some basic results about planar graphs. We will also prove a characterisation of planar graphs due to Kuratowski.

In Sec,13.4, we study the map colouring problem. We give a brief history of the four colour theorem, which says that any map can be coloured with four colours. The proof of this theorem is beyond the scope of this course. However, we will prove the weaker result that any map can be coloured with five colours.

In Sec.13.5, we end our unit with a brief discussion of edge colourings. We restrict ourselves to the definition of edge colouring, some examples of edge colouring and statements of some of the well known results in this field.

Objectives

After reading this unit, you should be able to

- compute the chromatic number of some simple graphs;
- compute some upper and lower bounds for the vertex chromatic number $\chi(G)$ of a graph G;
- Verify whether a given graph is planar or not using Kuratowski's theorem in simple cases;
- give an edge-colouring with $\chi'(G)$ colours for some simple graphs, where $\chi'(G)$ is the edge chromatic number of a graph G.

13.2 VERTEX COLOURINGS

In this section we start our study of colourings with vertex colouring. In the Subsection 13.2.1, we define vertex colouring and give some examples. In the Subsection 13.2.2, we will prove some simple bounds on the minimum number of colours needed to colour the vertices of a given graph. Let us now start our study of vertex colouring with the definition and some examples of colourings.

13.2.1 Definition and Examples

Look at the graph in Fig.1. We have given a colouring of K_3 using three colours, namely red, green and blue.

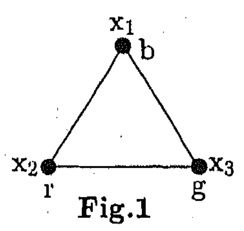

Fig.1

Why have we used three colours? It is because we want the adjacent vertices to have different colours. In K_3, any two vertices are adjacent so we need to colour each of the vertices with different colours. Keep this example in mind when you read the definition of vertex colouring given below.

Definition : A k-vertex colouring of a graph *G* is an assignment of k colours to each of the vertices of *G* in such a way that no two adjacent vertices have the same colour. A graph is k-vertex colourable if there is a k-vertex colouring. The minimum number of colours required to colour a graph G is called the vertex chromatic number of ***G***, usually denoted as $\chi(G)$.

In this section, we will be discussing only vertex colouring. So, we will use the terms 'k-colouring', 'k-colourable' and 'chromatic number', respectively. We will say that a graph is k-chromatic if it has chromatic number k.

In Fig.1, we were able to use the names of the colours, red, green and blue, because we needed only three colours. Suppose we need, say, 20 colours, can we still use the names to refer to the colours? We may not remember the names of so many colours and may probably decide to call them colour 1, colour 2, etc. This will do just as well, because the names of the colours are not important as long as you can distinguish between the different colours. We will also use $1, 2, 3, \ldots$ to denote our colours. However, to distinguish them from usual numbers, we will denote them as $\boxed{1}, \boxed{2}, \ldots$.

Let us now look at some examples.

Example 1: Colour the graphs in Fig.2 with the minimum possible number of colours. Also, find the chromatic numbers of the graphs.

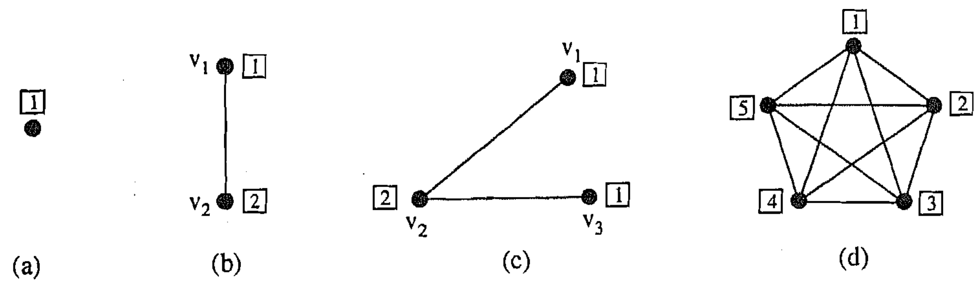

Fig.2 Some examples of colouring.

Solution: In Fig.2(a), K_1, has just one vertex. Let us colour this with $\boxed{1}$. Thus, this graph is 1-colourable and its chromatic number is 1.

In Fig.2(b), K_2 has two adjacent vertices. We assign $\square$ to the vertex v_1 and $\square$ to the vertex v_2. Thus, we have a 2-colouring. Is there a 1-colouring? No! The two vertices are adjacent and so we need at least two colours. In other words, the chromatic number $\chi(K_2) = 2$.

In Fig. 2(c), we have three vertices and we can colour them with three different colours. But, can we also have a two colouring? Notice that, v_1 and v_3 are not adjacent. So, we can colour them with the same colour, say, $\boxed{1}$. v_2 is adjacent to both v_1 and v_3. So, we cannot assign $\boxed{1}$ to this. Let us assign $\boxed{2}$ to v_2. So, we have a 2-colouring. As we cannot have a 1-colouring, this graph has chromatic number 2.

In Fig. 2(d), we have K_5. In this any two vertices arc adjacent, so we need as many colours as there are vertices, that is, we need five colours. So, K_5 has chromatic number 5.

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Remark: In the above example, we saw that the chromatic number of K_1 is 1. More generally, if a graph consists of isolated vertices, its chromatic number is 1. Conversely, if the chromatic number of a graph is 1, it consists of isolated vertices.

Also, we saw that the chromatic number of K_5 is 5. More generally, the chromatic number of K_n is n, because any pair of vertices are adjacent in K_n.

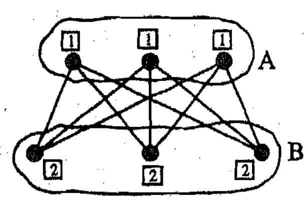

Fig.3

Example 2: Find the chromatic number of a bipartite graph with edge set non empty.

Solution: From unit 11, you may recall that a graph G is bipartite if the vertex set of G can be partitioned into two non empty disjoint subsets A and B such that any two vertices in a given set are non-adjacent. We get a 2-colouring of G by assigning $\boxed{1}$ to the vertices in A and $\boxed{2}$ to all the vertices in B.(This is illustrated in a particular case in Fig.3). Further, note that, since A and B are non empty and since the edge set of G is non empty, at least one vertex in A is adjacent to a vertex in B and these two vertices must have different colours. So, we cannot manage with less than two colours. So,

$\chi(G) = 2$ if G is a bipartite graph with non empty edge set.

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Remark: We saw in example 2 that the chromatic number of a bipartite graph with non empty edge set is 2. The converse is also true. Given a graph G and a 2-colouring of G, we can partition the edge set of G into two non empty sets **A** arid B defined as follows:

$$\mathbf{A} = \{v \in V(G) \mid v \text{ is assigned the colour } \boxed{1}\}$$

$$B = \{v \in V(G) \mid v \text{ is assigned the colour } \boxed{2}\}$$

By the definition of colouring no two vertices in **A** are adjacent and similarly for B. Since **A** and B are disjoint, G is bipartite by definition.

Here are some exercises to test your understanding of the above examples.

El) What is the chromatic number of a tree with at least two vertices?

E2) What is the chromatic number of an even cycle C_{2n}, $n \geq 2$?

E3) Is an odd cycle C_{2n+1}, $n \geq 1$, 2-colourable? What is its chromatic number?

If a graph is k-colourable, are all its subgraphs k-colourable? Let us see. Let G'be a k-colourable graph and H be its subgraph. We assign to each vertex of H the same colour that we assigned to it, considered as a vertex of G. If two vertices are non-adjacent in G, they are non-adjacent in H and therefore this gives a colouring of H. In other words, $\chi(H) \leq k = \chi(G)$ for every subgraph H of G. We can also recast this statement in the following form. If a graph *G* has a **subgraph** H with chromatic number k, the chromatic number of *G* must be at least k. This fact helps us in finding the chromatic number of a graph sometimes. We illustrate this in the next example.

Example 3: Find the chromatic number of G'rotzsch graph.(See Fig.4.)

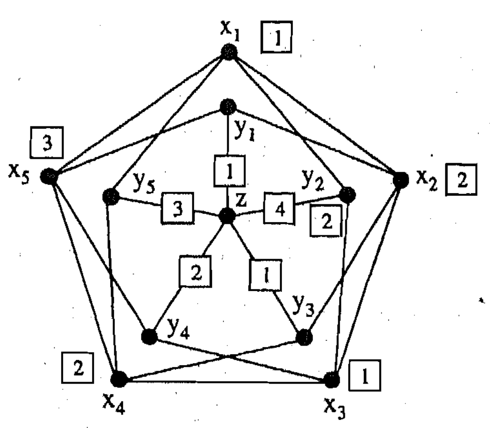

Fig.4 **Grötzsch** graph

Solution: The figure above gives a 4—colouring of this graph. Can this graph have a three colouring? Let us try to find one. Since the outer 5-cycle is an odd ,cycle, it needs three colours. So, we need at least three colours. Let us suppose the colours of $x_1, \cdots, x_5$ are as shown in Fig.4. Since y_1 is adjacent to x_2 and x_5 we have to give it a colour different from $\boxed{2}$ and $\boxed{3}$. So, we assign $\square$ to it. Similarly, the colours of y_4 and y_5 must be $\square$ and $\boxed{3}$ respectively. Since the vertex z is adjacent to vertices to which the colours $\boxed{1}$, 2 and ☐ have been allotted, we have to use a fourth colour for this vertex. So, this graph is not 3—colourable. Therefore, this has chromatic number 4.

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Try the exercises given below to test your understanding of the example given above.

E4) Show that the chromatic number of the Petersen graph, given in Fig. 5, is 3.

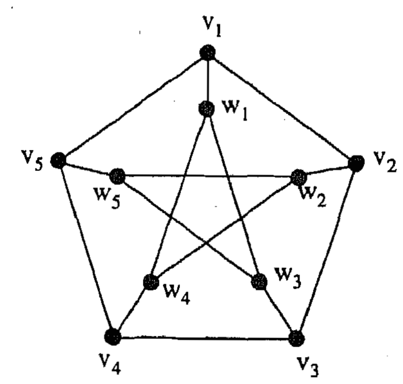

Fig.5 Petersen graph.

In the above examples and exercises, we saw that if a graph G has a subgraph H with chromatic number $\chi(H) = n$, $\chi(G) \geq n$. In particular, if a graph G has a subgraph H which is isomorphic to K_n (such a subgraph H is known as a **clique** of size n), the chromatic number of G is at least n. However, the converse is not true, i.e. if a graph has chromatic number $\geq n$, it need not have' a clique of size n. Petersen graph provides a counter example for this. As we have seen, the chromatic number of Petersen graph is **3.** Convince yourself—you need not prove it—that it does not contain a clique of size **3,** i.e., a subgraph isomorphic to K_3. More generally, in 1955, Mycielski proved that, for any integer k, there exists a k-chromatic graph without triangles. The proof of this result is beyond the scope of this course. However, it is not difficult to prove the much weaker result that if the chromatic number of a connected graph is greater than 2, it contains an odd cycle. We leave this as an exercise for you, along with some more exercises, to test your understanding of the material we have covered so far.

E5) Show that if $\chi(G) \geq 3$ for a graph G, it contains an odd cycle.

E6) (a) Find a 3-colouring of the figure in Fig.6.

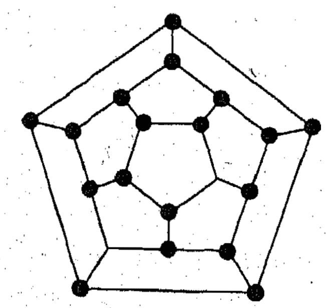

Fig.6

(b) What is the chromatic number of the graph in Fig.6?

E7) .Find the chromatic number of the following graph.

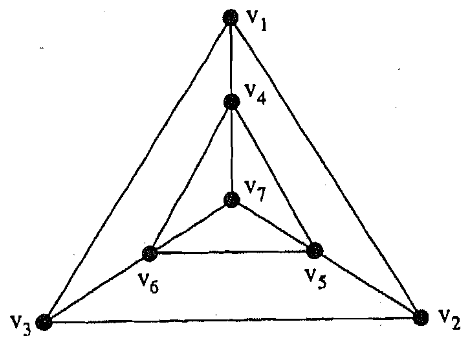

Fig.7

E8) Construct a graph with chromatic number 5.

Recall that, we have shown that any 2-colourable graph is bipartite. How was this done? We had put all the vertices having the same colour in a single set. There were two colours and so we got two subsets. They were disjoint. because no vertex can be assigned two colours.

We are going to extehd the ideas to n-colourable graphs. We do this through the concept of colour classes. First, let us define the colour classes of a colouring.

Definition : For a k-colouring of a graph G, consider the set $C_i = \{x \in V(G) \mid x \text{ is assigned the colour } \boxed{i}\}$, for $1 \leq i \leq k$. Clearly, $C_i \cap C_j = \phi$, for every $i \neq j$, and $V(G) = C_1 \cup \cdots \cup C_k$. If $\chi(G) = k$, each of the k colours is assigned to at least one vertex.(Why?) So none of these subsets is empty. Therefore, we get a partition of the vertex set V(G) into k mutually disjoint non empty subsets. The subsets $C_1, \cdots, C_k$ are called the **colour** classes of G given by the colouring.

So, the colour classes of a 2-colourable graph gives a bipartition of the vertex set of tlie graph, making it bipartite.

Let us now look at some examples of colour classes.

The colour classes **can** be **defined** for any colouring of a graph G, not **just** for a $\chi(G)$-colouring.

Example 4: Find the colour classes in the two different colourings of the same graph. The colour classes given by the colouring in Fig.8(a) are $C_1 = \{x_1\}, C_2 = \{x_4, x_6, x_8, x_{10}, x_{15}, x_{16}\}, C_3 = \{x_3, x_{12}, x_{13}, x_{14}\}$ and $C_4 = \{x_2, x_5, x_7, x_9, x_{11}\}$.
You can check that $C_1 = \{x_7, x_9, x_{11}, x_{15}\}, C_2 = \{x_1, x_5, x_8, x_{12}, x_{14}\}, C_3 = \{x_4, x_{10}, x_{16}\}, C_4 = \{x_2, x_6\}$, and $C_5 = \{x_3, x_{13}\}$ are the colour classes corresponding to the colouring in Fig.8(b).

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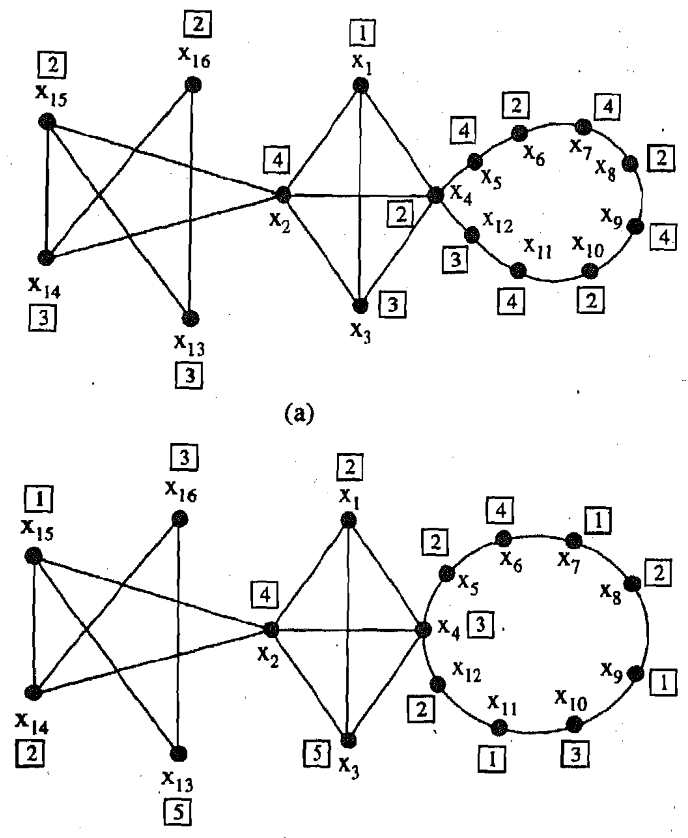

Fig.8

Try the next exercise to test your understanding of the above example.

E9) Colour the following graph in two different ways and give the colour classes in each of the cases.

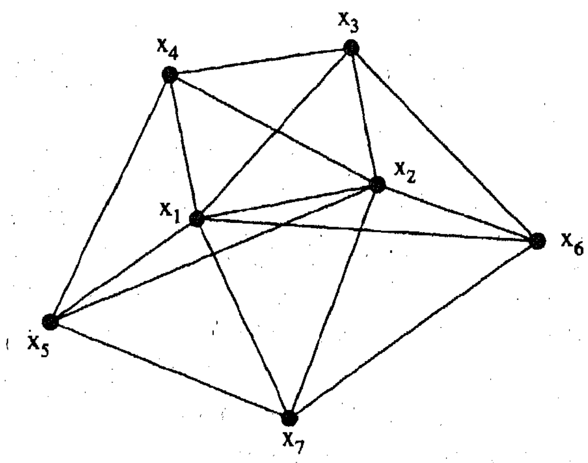

Fig.9

We have seen that any colouring of a graph G gives rise to the colour classes. You know that, if x, y are two vertices in a colour class C_i, then $xy \notin E(G)$. So, each colour class consists of mutually **non-adjacent** vertices. We now give a name to those subsets of the vertex set of a graph with this property.

Definition : A subset S of the vertex set V(G) of a graph G, is said to be an independent set if any two vertices in S are non-adjacent. An independent set is called maximal if it is not contained in any other independent set. The number of vertices in a largest independent set of G, is called the independence number of the graph G and it is denoted by $\alpha(G)$.

Example 5: Find three different maximal independent sets in the graph given Fig.10.

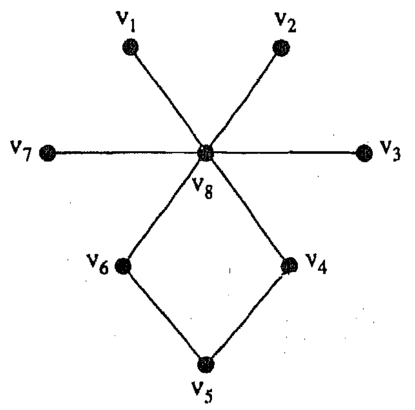

Fig.10

Solution: In Fig.10 we have the following maximal independence sets:

$$\{v_8, v_5\}, \{v_5, v_1, v_2, v_3, v_7\}, \{v_1, v_2, v_3, v_4, v_6, v_7\}$$

We check that $\{v_8, v_5\}$ is a maximal independent set. This is easy to see because all the other remaining vertices are adjacent to one of these two vertices. So, if any more vertices are added, the resulting set will no longer be an independent set. You can check that the other two sets are also maximal independent sets in the same way.

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Now test your understanding of independent set by trying the following exercise:

E10) Find an independent set of cardinality **4** in the graph given below:

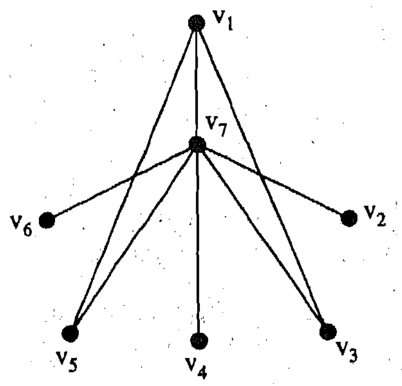

Fig.11

E11) Find out $\alpha(G)$ for the graphs given in Fig.7 and.Fig.8

Remark: We saw that both a colour class of a colouring and independent sets have the property that any two vertices in it are linearly independent, However, while colour classes depend on a particular colouring, independent set does not, This is the difference between these two concepts.

13.2.2 Bounds for **Chromatic** Numbers

In this section we will prove some bounds for the chromatic number of a graph in terms of $\Delta(G)$. For this, we need the concept of k-critical graphs.

Recall that,
$\delta(G) = \min\{d_G(v) \mid v \in E(G)\}$
$\Delta(G) = \max\{d_G(v) \mid v \in E(G)\}$

We will introduce you to this concept through an example. Consider the graph K_4. If we remove a vertex, we get a graph isomorphic to the graph in Fig.12(a).

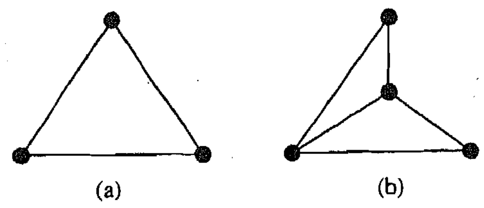

Fig.12 Graphs obtained by removing a vertex or an edge from K_4

If we remove an edge, we get a graph isomorphic to the graph Fig.12(b). Both the graphs have chromatic number three, as you can easily verify. Also, any other proper subgraph of K_4 is contained in one of these graphs. So, the chromatic number of any subgraph of K_4 is strictly less than that of K_4 which is 4. This shows that K_4 shows that K_4 is 4-critical, as the following definition shows.

Definition : **A** graph G is said to be critical or k−critical or critically k−chromatic if $\chi(G) = k$ and $\chi(H) < k$ for every proper subgraph H of the graph G.

Thus, in the discussion before the definition of k-critical graphs, we have shown that K_4 is 4-critical. Let us look at one more example.

Example 6: Show that Grotxsch graph is 4-critical.

Solution: Refer to Fig.4 for Grotzsch graph. Let us remove a vertex of this graph, Depending on the vertex, we get a subgraph isomorphic to one of the three following graphs:

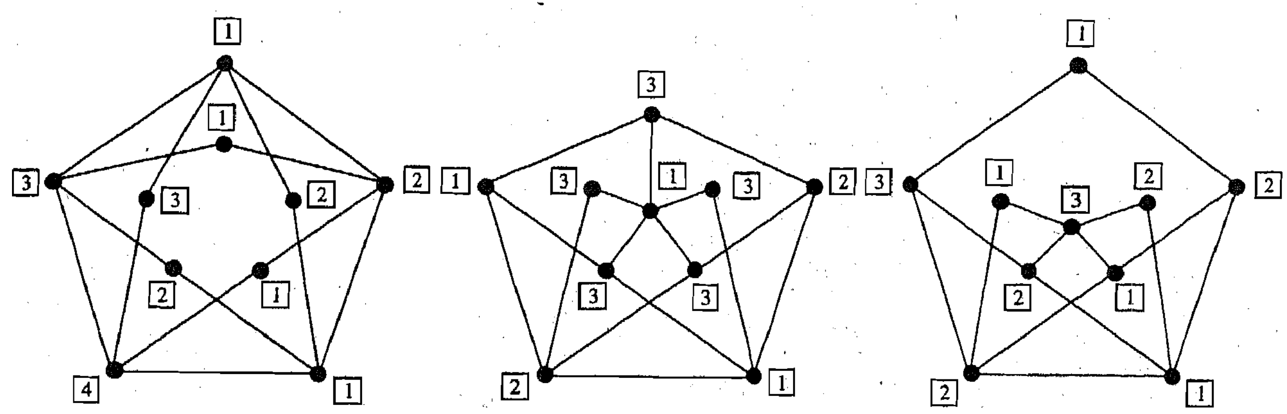

Fig.13 Graphs obtained by removing a vertex **from Grötzsch** graph

From the colouring given in these figures, it is clear that these graphs are 3−colourable. Moreover, all of them contain 5−cycles, that is, they are not 2−colourable. Thus, their chromatic number is $3 < 4 = \chi(G)$. This means that $\chi(G - v) < \chi(G)$, for every vertex v of the graph G.

Now, if we remove just one edge of G, without removing any vertex, we get a subgraph isoinorphic to one of the graphs in Fig.14. The dotted lines indicate the edges we have deleted.

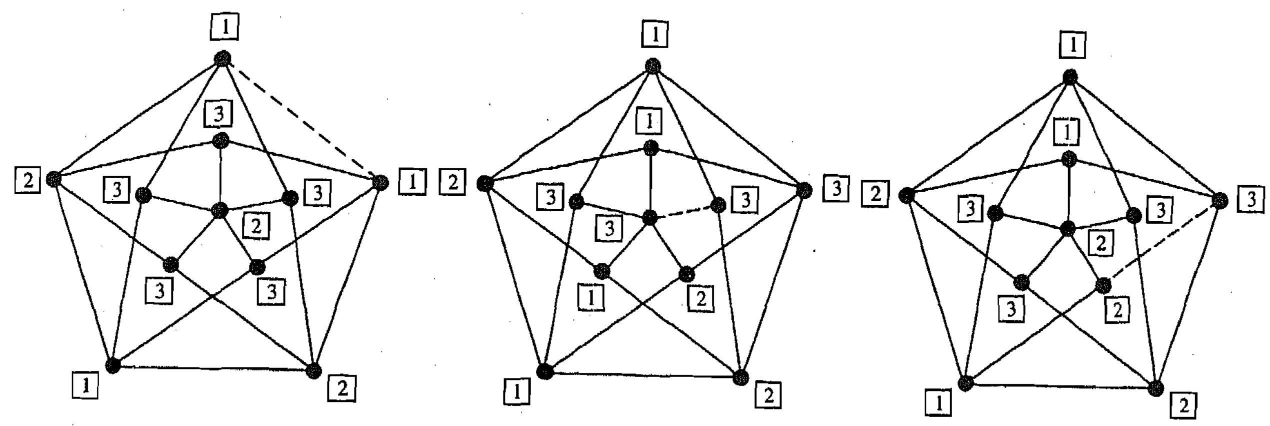

Fig.14 Graphs obtained by removing **an** edge from tho Grotzsch graph.

The 3—colouring of the subgraphs that we get is also given in the figure. Moreover, these graphs also contain 5—cycles and hence have chromatic number 3. Thus, $\chi(G-e) < \chi(G)$, for every edge e of G. But then, every proper subgraph of G is in fact subgraph of one of the six graphs in Fig.13 and Fig.14. Thus, $\chi(H) < \chi(G)$, for every proper subgraph H of the graph G. So, the Grotzsch graph is 4-critical.

* * *

Here is an exercise to test your understanding of the definition of k-critical graphs.

E12) Show that K_n is an n-critical graph.

E13) Check whether the Petersen graph is 3-critical.

Now, consider a graph with chromatic number k. Need it be k-critical? The following example will help you answer this question.

Example 7: Show that the graph given in Fig.15 is 3-chromatic; but it is not 3-critical.

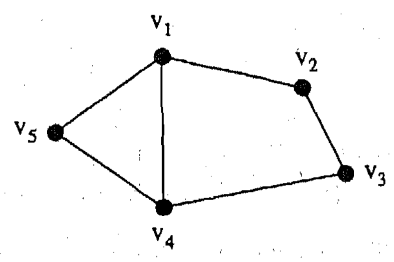

Fig.15

'Solution: We can assign [1] to v_1 and v_3, [2] to v_2 and v_4, [3] to v_5. This gives us a three colouring of the graph. This graph is 3-chromatic because it contains a 3-cycle $\{v_1, v_4, v_5\}$. If we remove the vertex v_2, the resulting graph still has chromatic number 3 because it still contains the 3-cycle v_1, vq, v_5. So, G has a subgraph which has the same chromatic number as G. So, it cannot be 3-critical.

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Now that we know that a graph with chromatic number k needn't be

3-critical, you may wonder if it will contain a k-critical subgraph. The following result tells you that this is true.

Theorem **1:** Let G be a graph with chromatic number k. Then, it has a subgraph which is k-critical.

Proof: Consider a graph G with $\chi(G) = k$. If it is k-critical, we are done. If it is not, it has a vertex v such that $\chi(G - v) = k$. If $G - v$ is k-critical, we are done. Otherwise, we can remove another vertex and get a subgraph with chromatic number k. We repeat the process. In the worst case, we will be left with a k-chromatic subgraph of G on k vertices. If we remove any vertex from this graph, we will get a graph on $k - 1$ vertices which is (k-1)-colourable. So, the k-chromatic subgraph on k vertices that we have obtained is k-chromatic.

We now discuss an example that illustrates Theorem 1.

Example 8: Find a 3-critical subgraph of the graph given in Fig.15.

Solution: On removing the vertices v_2 and v_3, we get **a** graph isomorphic to K_3. This is 3-critical.

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Here is a related exercise for you to try!

E14) Find a 3-critical subgraph of the Petersen graph.

Let now make table of the values of $\chi(G)$ and $\Delta(G)$ for some of the examples we have discussed so far to see if we can find a relationship between these two quantities.

G	$\chi(G)$	$\Delta(G)$
Grötzsch graph.	4	5
Petersen graph	3	3
Dodecahedron	3	3
C_5	3	2
K_4	4	3
K_5	5	4

Observe that, except for C_5, K_5 and K_4, all the other graphs satisfy the relation $\chi(G) \leq \Delta(G)$. In 1941, R. I. Brooks proved the following result.

Theorem 2: Let G be a connected graph which is neither an odd cycle nor a complete graph. Then,

$$\chi(G) \leq \Delta(G)$$

We will not prove theorem 2 in this course. However, we will prove the following weaker result.

Theorem 3: For every k−chromatic graph G,

$$\chi(G) \leq \Delta(G) + 1$$

Let us now illustrate the application of Brooks' theorem through an example.

Example 9: Find the chromatic number of the graph in Fig.16.

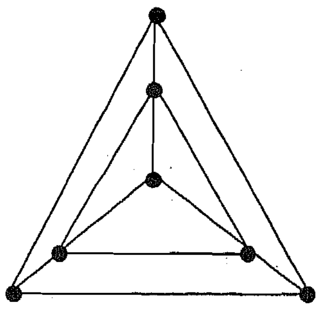

Fig.16

Solution: The maximum degree $\Delta(G)$ is 4 for this graph. So, by Brooks' theorem, the chromatic number at most 4. But, it has a subgraph isomorphic to K_4,(the subgraph formed by the inner triangle and the vertex in the centre.). So, its chromatic number is at least 4. Therefore its chromatic number is exactly 4.

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Remark: The bound given by Brooks' theorem may not be as good as it was in example 9. For example, in the case of $K_{1,n}$, $\Delta(K_{1,n}) = n$, $\chi(K_{1,n})$ is 2. So the difference $\chi(K_{1,n}) - \Delta(K_{1,n}) = n - 2$, is large when n is large.

We now prove a lemma that will be used in the proof of theorem **2.**

Lemma 1 If G is a k–critical graph with minimum degree $\delta(G)$, then $(k-1) \leq \delta(G)$.

Proof: If possible, let G be a k–critical graph with $\delta(G) < (k-1)$. Let $v \in V(G)$ such that $\delta(G) = d_G(v)$. Since G is k–critical, $\chi(G - v) \leq (k-1)$, that is $G - v$ has a $(k-1)$–colouring. Since $d_G(v) < (k-1)$, v is adjacent to fewer than $k-1$ vertices. So, there is at least one colour i, among the $k-1$ colours, that is not assigned to any of the $k-1$ vertices adjacent to v. We can assign this colour to v to get a $k-1$ colouring of G. This contradicts the fact that $\chi(G) = k$. Thus our assumption is wrong, that is, $\delta(G) \geq (k-1)$.

Corollary 1 Every k–chromatic graph G has at least k vertices of degree $\geq (k-1)$.

Proof: Let G be a k–chromatic graph, that is, $\chi(G) = k$. Let H be a k–critical subgraph of G. Thus, $|V(H)| \geq k$ and $\delta(H) \geq (k-1)$. This means that every vertex x of H satisfy the property that $d_G(x) \geq d_H(x) \geq \delta(H) \geq (k-1)$. There are at least k such vertices. This proves the result.

We now prove theorem **2.**

Proof of Theorem 2: Using Corollary 1 to Lemma 1, choose a vertex $x \in V(G)$, such that $d_G(x) \geq (k-1)$. But then, $\Delta(G) \geq d_G(x) \geq (k-1)$, that is $\chi(G) = k \leq \Delta(G) + 1$.

In the introduction, we mentioned that the map colouring problem can be reduced to finding the minimum colours needed to colour a special class of graphs called planar graphs. In the next section, we define planar graphs and prove some basic results that will be useful in the study of map colouring problem.

13.3 PLANAR GRAPHS

In transistor radios and television sets, you must have seen printed circuit boards. These boards have slots for various components and these slots are connected to each other. The connections between these slots must be made in such a way that no two connections cross each other. Given an electronic circuit, is it always possible to design a printed circuit board corresponding to it?

This can be formulated as a problem in graph theory. We replace the electronic components by vertices and the connections between them by edges. If the resulting graph can be drawn in such a way that no two of the edges cross each other except at the vertices, then we can design a printed circuit board for the given circuit. Graphs that can be drawn this way are called planar graphs.

We begin our study of planar graphs in this section. We begin this section by defining planar graphs. After giving some examples, we will prove some basic results on planar graphs like Euler's formula. From this, we will derive some necessary conditions for a graph to be planar. Using these conditions, we will show that $K_{3,3}$ and K_5 are non-planar.

Let's start by seeing what a planar graph is.

Definition : **A** graph G is called planar if it can be drawn on the plane in such a way that no two edges cross each other at any point except possibly at the common end vertex. Such a drawing is called a plane drawing.

Here are some examples of planar graphs. In the first row of Fig.17 we have given the five regular solids called Platonic solids. In the second row, we have given the corresponding graphs. In each of these graphs, the vertices correspond to the vertices of the corresponding solid and the edges correspond the edges of the solid.

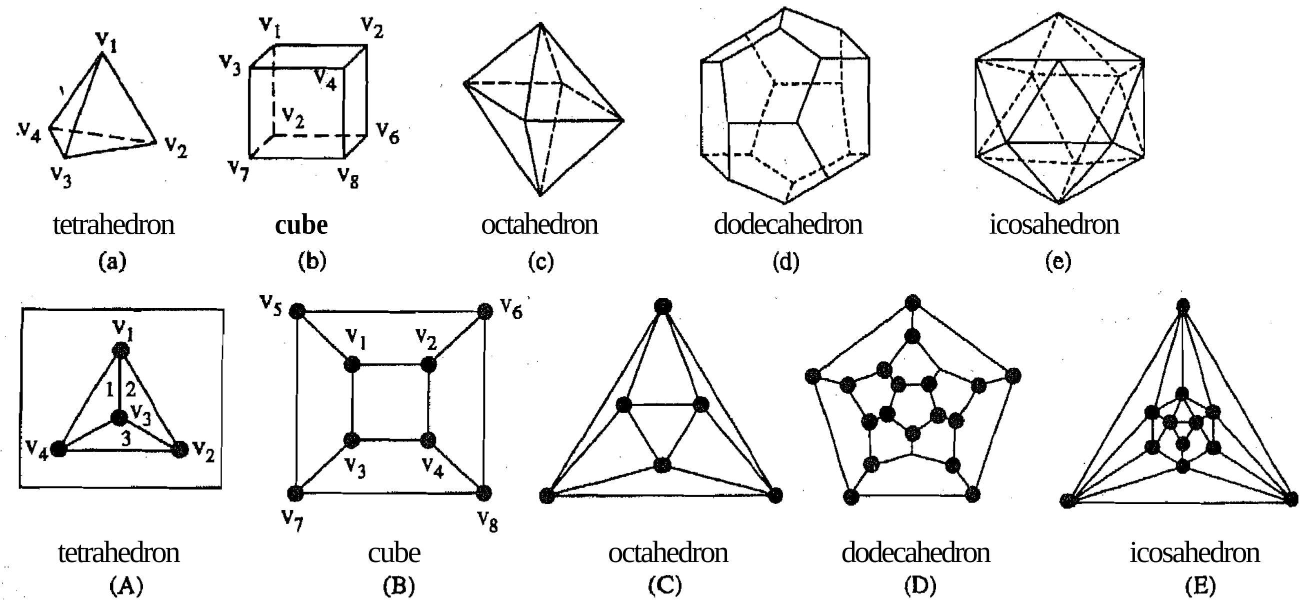


Fig.17 Regular solids and their graphs.

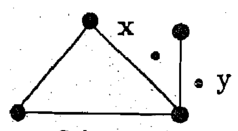


Fig.18

Note that x and y are just points in the plane, they are not vertices.

Next, we introduce the concept of a region. Look at the tetrahedron in Fig.17(a). It has four faces. The graph corresponding to it is given in Fig.17(A). It divides the plane into four faces or regions. Similarly, the cube, given in Fig.17(b) divides the plane into six regions.

In all the above cases, it is very clear what the different regions are. But, look at the graph in Fig.18. Into how many regions does it divide the plane? Two or three? Do the points x and y lie in the same region or in different region? To avoid such confusion we need to define the concept of a region rigorously. Here is the definition of a region.

Definition : Given a plane drawing of a planar graph G, by a region or face of G, we mean a maximal portion of the plane for which any two points a, b in it can be joined by a simple curve in such a way that, neither does the curve have any point in common with the curve representing any edge nor does any vertex lies on that curve, that is, the curve lies completely in that portion of the plane. If R is a region of a planar graph G, by the boundary of R, we mean all those points x in the plane corresponding to the vertices and edges of G having the property that x can be joined to any point in that region by a simple curve all whose points, except x, are in that region. There is always one unbounded region and it is called the exterior region of G. Any other region is called an interior region.

Let us go back to Fig.18 again. Armed with this definition, we can answer the question we raised. As you can see in Fig.19, the points x and y can be joined by a curve that does not cross any of the edges. So, there are only two regions, the region inside the triangle and the region outside it. Both the points lie in the exterior region of the triangle.

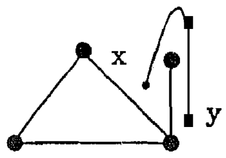
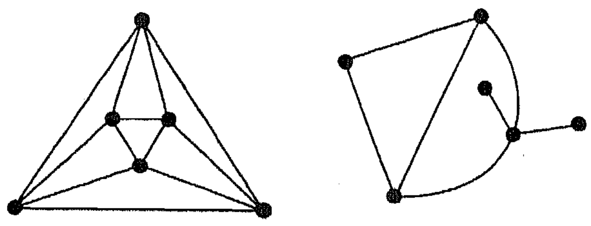


Fig.19

Let us now look at an example to understand these concepts better.

Example 10: Find the number of regions in the graphs given in 20, including the exterior region.



(a) (b)

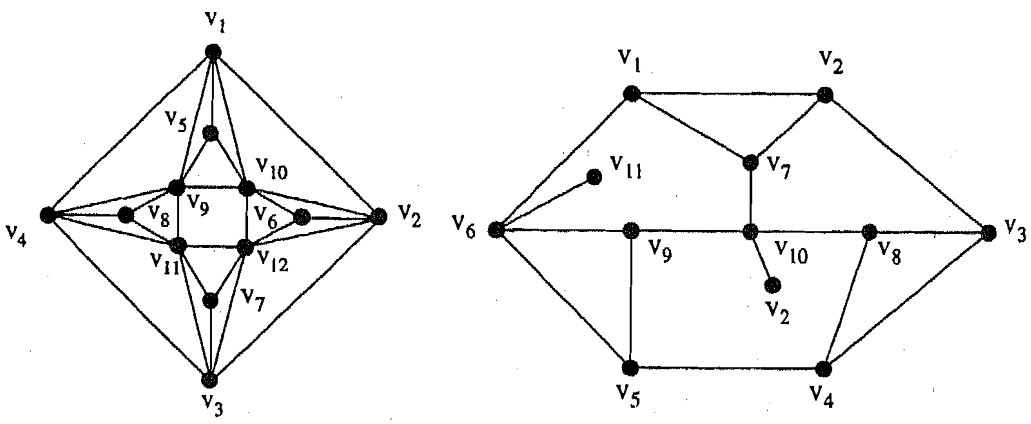
Fig.20

Solution: The graph in Fig.20(a) has 8 regions. In the graph in Fig.20(b), there are **3** regions.

***

Check your understanding now. Try the following exercise.

E15) Find the number of regions in each of the graphs in Fig.21.



(a) (b)

Fig.21

Let us now calculate the quantity $p - q + r$ for all the planar graphs in Fig.17 and for the graph in Fig.20(b):

	p	q	r	p − q + r
Fig.20(b)	6	7	3	2
K_4	4	6	4	2
Tetrahedron	4	6	4	2
Cube	8	12	6	2
Octahedron	6	12	8	2
Dodecahedron	20	30	12	2
Icosahedron	12	24	18	2

As you can see, $p - q + r$ is always **2** for all these planar graphs.

Tbe following theorem, proved by Euler in 1736, proves our observation.

Theorem 4: If G is a connected planar (p,q)−graph, then for any plane drawing of G, the number r of the regions of G is constant and

$$p - q + r = 2 \qquad (1)$$

Proof: We apply induction(See Unit 2) on the number q of the edges of G. For our convenience let us write down the equation in words also.

$$\text{Number of vertices} - \text{Number of edges} + \text{Number of regions} = 2 \qquad (2)$$

for any planar graph G.

If $q = 0$, then G just consists of p isolated vertices. Hence, $r = 1$ and the formula holds. Now, by induction, assume that the formula holds for plane drawing of a (p, t)−graph for every $t \le (q - 1)$, and suppose G is a (p,q)−graph. If G is a tree, then $p = q + 1$ and $r = 1$ so that the formula, holds. If G is not a tree, let e be an edge that lies on a cycle of G and consider the subgraph $G - e$ of G.

When we remove an edge e, we join exactly two regions to make one region out of them, that is $G - e$ has p vertices, $(q - 1)$ edges and $(r - 1)$ regions. This is illustrated in a particular case in Fig.22.

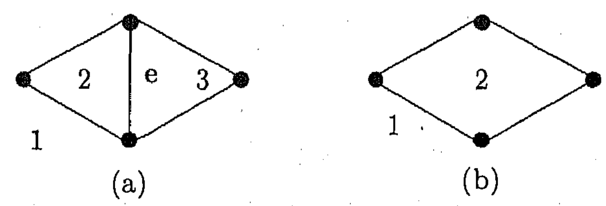


Fig.22

In Fig.22(a), there are 4 vertices, 5 edges and **3** regions. After removing the edge labelled e, the regions 2 and **3** merge and become a single region. The new graph in Fig.22(b) has 4 vertices, 4 edges and 2 regions.

Now, by induction assumption, the relation in Eqn.(1) holds for this $G - e$. Using the form given in Eqn.(2) for $G - e$, we get,

$$\begin{aligned} 2 &= \text{Number of vertices} - \text{Number of edges} + \text{Number of regions} \\ &= p - (q - 1) + (r - 1) \\ &= p - q + r \end{aligned}$$

i.e., $p - q + r = 2$. But, p, q and r are, respectively, the number of vertices, the number of edges and the number of regions in G. This proves the result for the graph G.

From the formula in Eqn.(1), we have $r = q - p + 2$. Since p and q are fixed once **we fix** a graph, it also follows that the number of regions in a plane drawing of a planar graph is independent of the plane drawing.

Recall that a graph on p-vertices can have up to $\frac{p(p-1)}{2}$ edges. In the case of planar graphs, there is a much better bound. We give this bound(without proof) in tlie next theorem.

Theorem 5: If G is a planar (p,q)−graph, with $p \geq 3$, then $q \leq 3p - 6$. Further, if G is also bipartite, we have $q \leq 2p - 4$

So far, we have given many examples of planar graphs. But, we haven't given any example of non-planar graphs. We now make use of the bound given in theorem 5 in the next example to give such an example.

Example 11: Show that K_5 is planar.

Solution: Suppose tliat K_5 is planar. Then tlie number of edges and vertices in K_5 satisfy the relation $q \leq 3p - 6$ given in theorem 5. K_5 has 5 vertices and 10 edges, so $10 \leq 3 \times 5 - 6$, i.e. $10 \leq 9$, a contradiction.

* * *

Try the next exercise to check your understanding of theorem 5.

E16) Verify tliat $K_{3,3}$ is non-planar using theorem 5.

You must have noticed that, so far, we have given only necessary conditions for planarity. In the next subsection we will give a necessary and sufficient condition.

13.3.1 When is a graph planar?

We have already seen that K_5 and $K_{3,3}$ are not planar. To prove this we used a necessary condition derived from Euler's formula. However, the condition is not sufficient. For example, in Grotzsch graph, $p = 11$, $q = 20$ arid $20 \leq 33 - 6 = 27$. So, the condition in Theorem 5 is satisfied. But, as we shall show later, Grötzsch graph is riot planar. Is there a necessary and sufficient condition for a graph to be planar'?

Yes! In 1930, K. Kuratowski, a Polish mathematician, proved a necessary and sufficient condition for a grapli to be planar. We will state this theorem arid illustrate its application through an example. To understand the statement, let us first consider Fig.23 below.

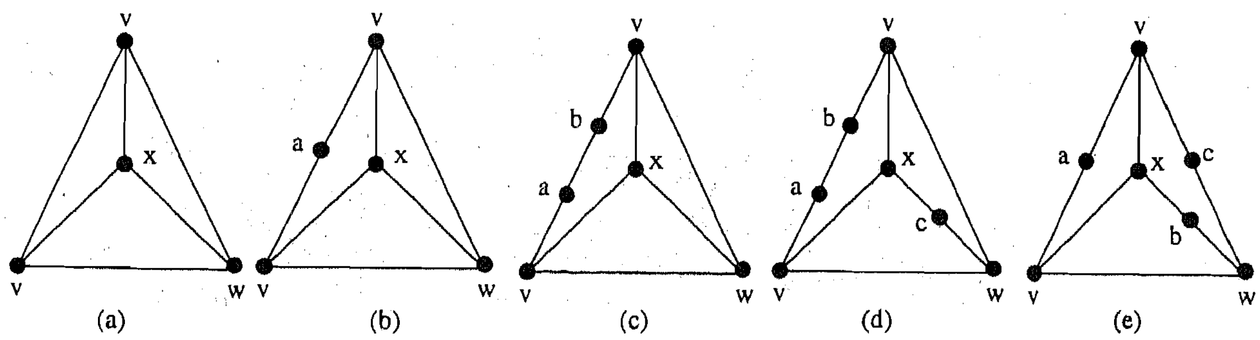


Fig.23 Subdivision of a graph

In this figure, wc have started with K_4 and inserted vertices of degree 2 at some of the existing edges. For example, in Fig.23(b), we have removed the edge uv, added a new vertex a and two more new edges va and au. We have similarly altered the graphs in Fig.23(b), Fig.23(c), Fig.23(d) and Fig.23(e). In this way we have got subdivisions of the graph in Fig.23(a), as you shall now see.

Definition : A graph G' is a subdivision of a graph G if it can be obtained by adding one or more new vertices of degree 2 on the existing edges of G.

In other words, we 'subdivide' some of the existing edges.

Note that, if a graph is planar, all its subgraphs are planar. Equivalently, if a subgraph of a graph is non-planar, the graph itself is non-planar. Also, if a graph G' is the subdivision of a planar graph G, then G' is also planar. If a graph G is non-planar, any subdivision of G is also non-planar. So, if a graph contains a non-planar subgraph or a subgraph which is a subdivision of a non-planar graph, it is non-planar. For example, the graph in Fig.24(a) is non-planar since it contains as a subgraph a subdivision of K_5, (shown by dotted lines)which is a non-planar graph.

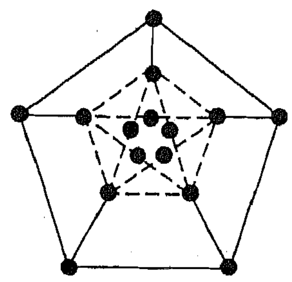


Fig.24

In proving the non-planarity of the graph in Fig.24, is it just a coincidence that it had a subdivision of K_5 as a subgraph? No! Kuratowski theorem(stated below) says that a non-planar graph has to contain a subgraph which is a subdivision of K_5 or $K_{3,3}$. So, we need to restrict our search for non-planar subgraphs(or their subdivisions) to only these two 'graphs.

We now state Kuratowski's theorem.

Theorem **6:** A graph G is non-planar if and only if it contains a subdivision of K_5 or $K_{3,3}$ as a subgraph.

Let us now look at an example to see how this theorem can be used to prove non-planarity.

Example 12: Show that the Grotzsch graph(See Fig.4) is non-planar.

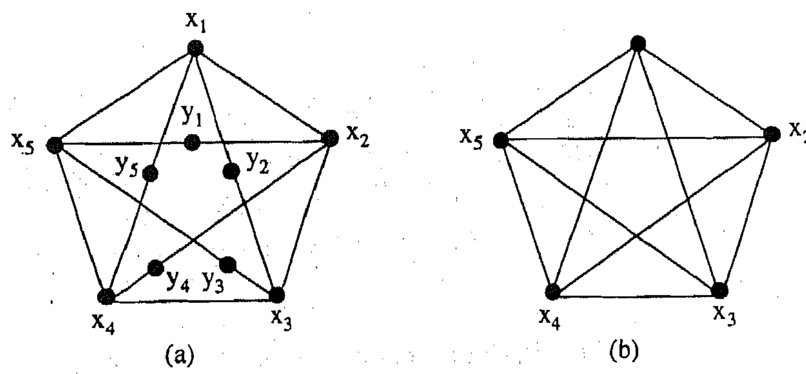


Fig.25 Non-planarity of Grotzsch graph.

Solution: From Kuratowski's thearem we know that we have to look far a subgraph which is a subdivision of K_5 or K_4. But, in this case, which of these two should we look for? Note that subdivision of a graph does not affect the degree of any of the vertices of a graph; it only introduces new vertices of degree 2.

So, if our graph contains a subdivision of K_5, it will contain at least 5 vertices of degree 4. If it contains a subdivision of $K_{3,3}$ it will have at least six vertices of degree 3. Let us first check if our graph contains a subdivision of $K_{3,3}$. But, the Grotzsch graph contains only five vertices of degree 3, namely, y_1, y_2, y_3, y_4 and y_5. So, it cannot contain a subdivision of $K_{3,3}$. So, let us check if it contains a subdivision of K_5. K_5 contains 5 vertices of degree 4. In Grotzsch graph also there are vertices of degree 4, namely x_1, x_2, x_3, x_4 and x_5. Let us remove the middle vertex, labelled as z. We get the graph given in Fig.25(a). As you can see, it can be obtained from K_5 in Fig.25(b) by adding degree two vertices to x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_3x_5 and x_1x_5. So, it is non-planar.

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Now, an exercise for you to try!

E17) Show that the Petersen graph is non-planar.
(Hint: Consider the graph obtained by removing the two horizontal edges.)

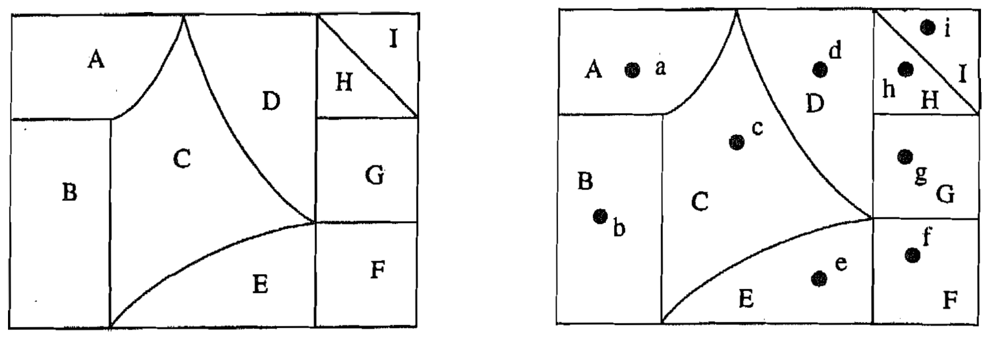
In the next section we will discuss the map colouring problem. We will show that this can be reduced to colouring of planar maps. We will also show that any planar map can be coloured with five colours.

13.4 MAP COLOURING PROBLEM

The four colour problem asks whether any map can be coloured with 4 colours. We begin this section with a brief discussion of the history of the four colour problem. We then show how to construct a planar graph corresponding to a given map in such a way that colouring the graph is equivalent to colouring the map. So, if we can prove that any planar map can be colomed with four colours, we would have proved that any map can coloured with four colours. Appel and Haken proved that four colours are enough to colour planar graphs in 1979, so'the four colour problem is now solved. They used nearly 1200 hours of computer time on some of the fastest computers available at that time. This gives an idea about the complexity of the proof and we will not be giving the proof in this course. However, we will prove the weaker result that five colours are always enough to colour any planar graph. Now, for some history!

Francis Guthrie communicated the four colour problem to De Morgan through his brother Fredrick Guthrie, who was a student at the University College, London at that time. It appeared in print for the first time when Cayley published a paper on this problem in Royal Geographical Society in 1879. In this paper, he outlines where the difficulties lie in this problem. In the same year, A. B. Kempe published a proof of the theorem in American Journal of Mathematics. However, in 1890, P. J. Heawood pointed out a mistake in Kempe's proof. He also showed that the proof can be modified to show that five colours are enough to colour any map. Since then, many mathematicians, G. D. Birkhoff, Veblen, Ore, Franklin among others, contributed to the solution of the problem. Appel and Haken finally solved the problem in 1979, We now show how to construct a planar graph corresponding to a given map in such a way that colouring vertices of the graph is equivalent to colouring the map.

Consider the map given in Fig.26(a) below. There are 10 regions in the map,, A, B, C, D, E, F, G, H, I and J, including the exterior region. In this map we add a vertex corresponding to each region of the map. (See Fig.26(b).) Note that we have added a vertex corresponding to the exterior region, namely, J.



(a) (b)

Fig.26

We join two vertices if the corresponding regions have an edge in common. For example, we have connected a and c because they have a common boundary(see Fig.27 below).

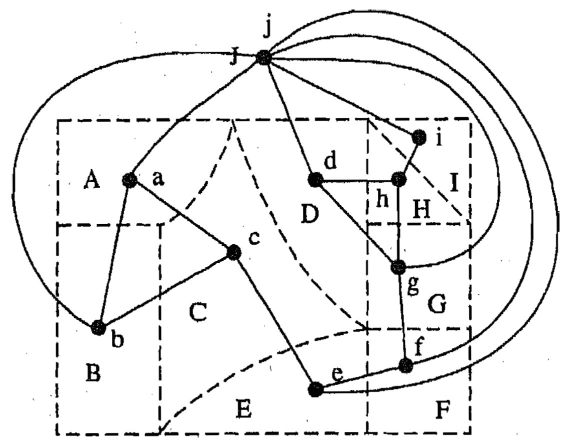


Fig.27

We have not connected the vertices a and e because they do not have a common boundary. We do not connect two vertices, if the corresponding regions share only a point and not a boundary. For example, we have not connected c and g by an edge for this reason. As you can see, we get a planar graph and colouring the graph is equivalent to colouring the map. (We assume that the exterior region of the map is coloured with a single colour.) So, the four colour problem can be stated as follows:

Is **it** possible **to colour** any planar graph with four **colours?** The following theorem answers this question.

Theorem 7 **[Appel-Haken](1979):** Any planar graph can be coloured with four colours.

As we mentioned in the introduction we will not be proving this theorem. Can we do with three colours always? No! As we have seen, K_4(it is the graph corresponding to a tetrahedron) is planar, but it cannot be coloured with three colours. So, we cannot improve the result in Theorem 7.

We now prove a result that will be used in the proof of the five colour theorem.

Theorem **8:** For every planar graph G, the minimum degree $\delta(G)$ is at most 5.

Proof: If possible, let G be a planar graph such that $\delta(G) \geq 6$. But then, by Theorem 5,

$$6p \leq \sum_x d_G(x) = 2q \leq 6p - 12.$$

This is impossible. Hence, $\delta(G) \leq 5$.

We can prove the five colour theorem now.

Theorem 9: Every planar graph is 5-colourable.

Proof: Let G be a planar graph on p vertices. We prove the theorem by induction on p. If $p \leq 5$, then the theorem is clearly true. Now, assume that every planar graph with $(p-1)$ vertices, $p > 1$, is 5-colourable. By Theorem 8, $\delta(G) \leq 5$. Let v be a vertex of G such that $\delta(G) = d_G(v)$. Consider $G - v$. By induction this is 5-colourable. Let us take a 5-colouring of $G - v$. In this colouring all the vertices other than v have received some colour. We have to get a 5-colouring of the graph G by changing the colours assigned to the vertices other than v, if necessary, and assigning some colour to v.
If $d_G(v) < 5$, then there are at most four vertices adjacent to v in G. Hence, there is at least one colour $\boxed{i}$ not assigned to any of the neighbours of v. By assigning $\boxed{i}$ to v and retaining the same colours for the other vertices, we get a 5-colouring of G.
If $d_G(v) = 5$ but the neighbours of v in G utilise only four or less colours then as before we can complete a 5-colouring of G.
Now suppose $d_G(v) = 5$ and the neighbours of v in G utilise all the five colours. Renumbering the vertices if necessary, we can suppose that neighbours v_1, v_2, v_3, v_4, v_5 of the vertex v are numbered in such a way that the colour $\boxed{i}$ is assigned to the vertex v_i and they are arranged around v in a plane drawing of G as shown in Fig.28:

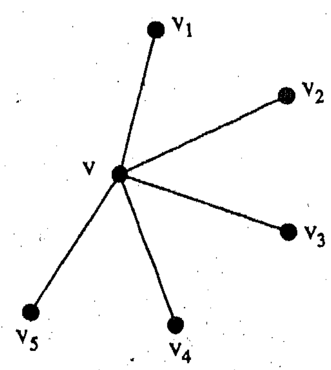


Fig.28

Let $S_i = \{x \in V(G) : \boxed{i}$ is assigned to x). Consider the vertex induced subgraph $H_{1,3}$ of G induced by $S_1 \bigcup S_3$.
Case 1: If the vertices v_1 and v_3 belong to two different components of $H_{1,3}$, then take the component(see unit 11 for the definition of component of a

graph) containing v_1.

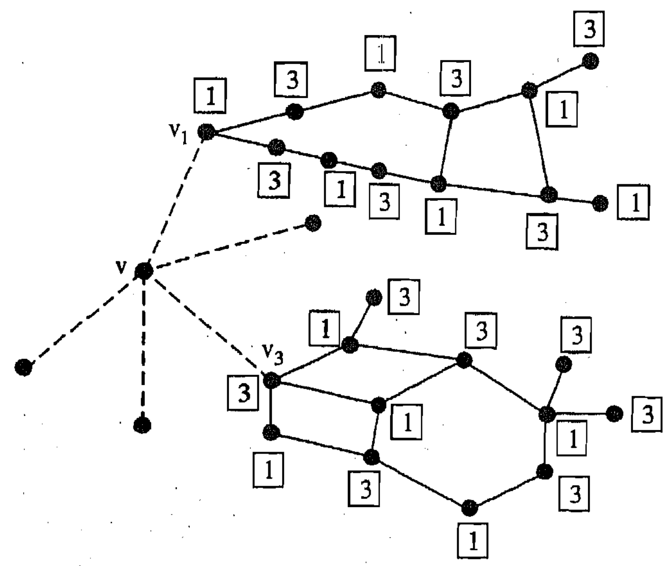


Fig.29

Interchange the colours only in this component. In other words, assign the colour □ to all the vertices that are assigned □ and assign [3] to all the vertices that are assigned [1]. In the modified colouring the vertices v_1, v_3 both receive the colour [3]. Now we can assign the colour [1] to the vertex v and get a 5−colouring of G.

Case 2: If v_1, v_3 belong to same component, then there is a path P joining them. Because of the colouring, the vertices of this path must have received colours □ and □ alternately, starting with colour 1 at v_1 and ending with colour □ at v_3.

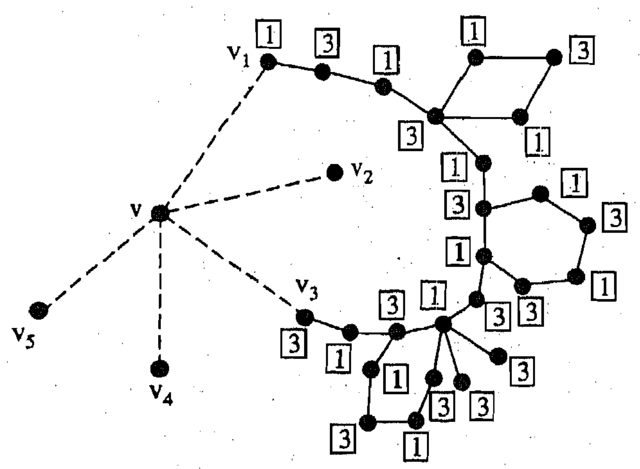


Fig.30

This means the union of P and $\{v_1v, v_3v\}$ is a cycle C (say). Moreover, the vertex v_2 belongs to the interior region created by this cycle and the vertex v_4 belongs to the exterior region created by this cycle. You must remember that the vertices of this cycle, other than v, have received colours [1],[3] only. Now, consider the vertex induced subgraph $H_{2,4}$ of G induced by $S_2 \bigcup S_4$. If there **is a** path joining v_2 and v_4 in this subgraph, then vertices on it will have colours [2], [4] only, But then it has to cross the barrier created by the cycle C. Where can it cross? **All** the vertices of C have colours 1, 3 only. So there cannot be a common vertex to use. This means there cannot be a path joining v_2 and v_4 in the subgraph $H_{2,4}$, that is the vertices v_2 and v_4 belong

to different components of $H_{2,4}$. Instead of taking $H_{1,3}$, we take $H_{2,4}$ and go back to the Case 1 and complete the 5–colouring of G.
Thus G is 5–colourable.

If we can colour the vertices of a graph, why can't we colour the edges of a graph? Is it interesting? In the next chapter, we will answer this question.

13.5 EDGE COLOURINGS.

In this section, we consider the problem of colouring the edges of a graph in such a way that no two adjacent edges receive the same colour. We will not prove any of the important results in this subject although we will state some of them. The purpose of section is to give a brief introduction to edge colouring. We begin by defining edge colouring.

Definition : **A** k-edge colouring of a graph G is an assignment of k colours to each of the. edges of G in such a way that no two edges incident with the same vertex have the same colour. **A** graph is k-edge colourable if there is a lc-edge colouring. The minimum number of colours required to colour a graph is called the edge chromatic number of ***G***, usually denoted as $\chi'(G)$.

Let us now look at some examples of edge colouring. The easiest case is the edge colouring of those graphs which have edge chromatic number 1.

Example **13:** Find all the graphs that have edge chromatic number 1.

Solution: Suppose a graph G has edge chromatic number 1. Since the edge chromatic number is one, the graph is 1-edge colourable and no two edges share an end vertex, that is, the graph must be union of some isolated vertices and some mutually disjoint edgos. Conversely, graph which are union of isolated vertices and mutually disjoint edges have edge chromatic number 1.

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Example 14: Colour the edges of the graphs K_3, K_4, K_5.

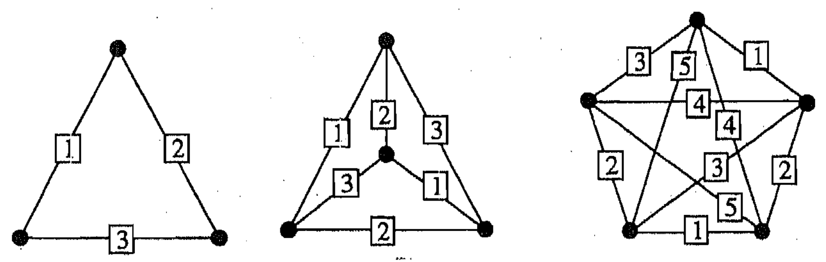

Fig.31

The colouring of K_3, K_4, K_5 is given in Fig.31. Here no two adjacent edges have received same colour. In all the cases, we have used least possible colours.

***

Example **15:** Give a edge colouring of Petersen graph.

Solution: Fig.32 gives a 4-edge colouring of the Petersen graph.

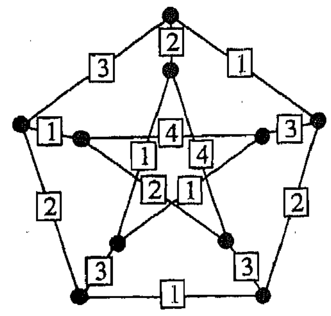

Fig.32

Again no two adjacent edges have received same colour. You can quickly check that three colours will not be enough.

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Example 16: Give edge colourings of all the trees on 5 vertices..

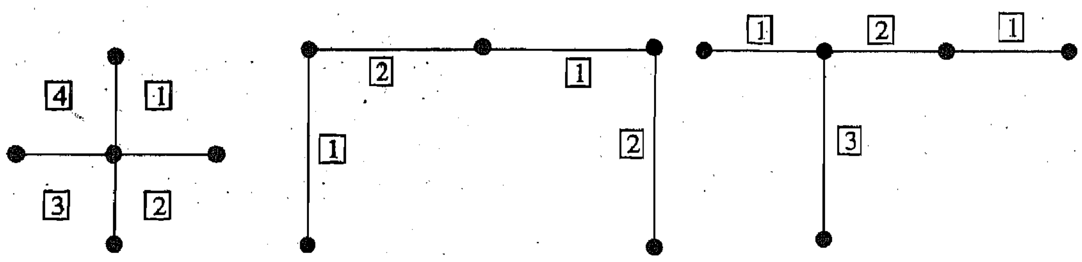

Fig.33

Again, we have used the least possible number of colours and no two adjacent vertices received same colour.

***

Example 17: Find the edge-chromatic number of C_n.

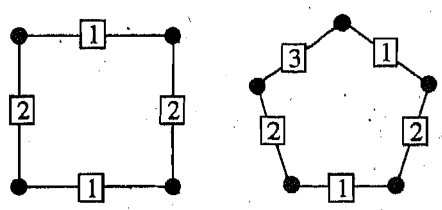

Fig.34

Solution: As in the case of vertex colouring, if n is even, the edge chromatic number is 2. We can colour the edges alternately with the two colours. If n is odd, the edge chromatic number is **3.** We have illustrated this in the case of C_4 and C_5 in Fig.34.

***

If G is a graph and $v \in V(G)$ such that $d_G(v) = \Delta(G)$, then all the edges incident on v must receive different colours. Hence, any edge colouring of G will need at least $\Delta(G)$ colours, that is,

$$\Delta(G) \leq \chi'(G) \qquad (3)$$

Regarding an upper bound for $\chi'(G)$, in 1964, Vizing proved the following result.

Theorem 10: For any graph G, we have

$$\chi'(G) \leq \Delta(G) + 1 \quad (4)$$

From (3) and (4) it follows that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) \quad (5)$$

From (5), it follows that there are only two possibilities for the edge chromatic number of a graph G, either $\Delta(G)$ or $\Delta(G) + 1$. Thus, this result divides the set of all graphs into two classes. A graph G is said to belong to class 1, if $\chi'(G) = \Delta(G)$ and it is said to belong to class 2, if $\chi'(G) = \Delta(G) + 1$. We often say that G is a class 1 graph or G is a class 2 graph. The problem of determining the class of a graph is called the classification problem. We now discuss some of the results known in this direction.

Theorem 11: The edge chromatic number of K_n is n if it is odd(# 1) and n – 1 if it is even.

Recall that, K_n is n – 1-regular. So, $\Delta(K_n) = n - 1$. So, K_n belongs to class 1 if n is even and it belongs to class 2 if it is odd.

Regarding bipartite graphs, in 1916, König proved that $\chi'(G) = \Delta(G)$, in other words, it is a class 1 graph.

In 1977, Erdos and Wilson proved that if p(n) is the probability that a graph on n vertices selected at random belongs to class 1, then $p(n) \longrightarrow 1$ as $n \longrightarrow \infty$, that is almost all graphs belong to class 1. However, large families of class 2 graphs are known.

E18) What is the edge chromatic number of $K_{m,n}$?

E19) Consider following tree T.

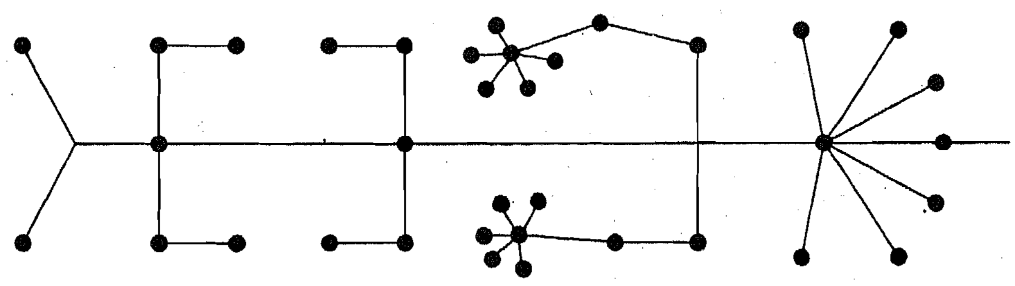

Fig.35

Give an explicit $\Delta(T)-$ colouring of T.

We have now reached the end of this unit. Let us now summarise briefly what we have learnt so far.

13.6 SUMMARY

In this unit we defined:

a) Vertex colouring of a graph: A vertex colouring of a graph is an assignment of colours to vertices in such a way that no two adjacent vertices receive the same colouring.

b) Vertex chromatic number of a graph: The chromatic number of a graph is the minimum number of colours required to colour the graph.

c) **A** colour class of a colouring: For each colour of a colouring, the set of all vertices that are coloured with that colour is the colour class of that' colour.

d) Independent set **A** subset of the vertex set is independent of any two vertices in the set are non adjacent.

e) Planar graph: **A** graph is planar if there is a plane drawing in which no two edges cross each other, except at vertices.

d) Subdivision of a graph: **A** graph G_2 is a subdivision of another graph G_1 if it can be obtained by from G_1 by adding vertices of degree two at the existing edges.

e) Edge colouring of a graph: An edge colouring of a graph is an assignment of colours to edges in such a way that no two edges incident at the same vertex are given the same colour.

f) Edge chromatic number of **a** graph: The edge chromatic number of a graph is the minimum number of colours needed to colour the edges of graph.

g) Class **1** and class 2 graphs: A graph is of class 1 if its edge chromatic number is $\Delta(G)$; it is of class 2 if it has chromatic number $\Delta(G)+1$.

In this unit, we studied:

a) some upper bounds for the chromatic number of a graph.

b) Euler's formula for planar graphs, which states that
Number of vertices − Number of edges + Number of regions = 2
for any planar graph.

c) Kuratowski's characterisation of planar graphs which says that a graph is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5.

d) the four colour theorem(without is proof) which says that any planar graph can be coloured with four colours.

e) the five colour theorem(with proof) which says that any planar graph can be coloured with five colours.

f) the Vizing's bound for the edge chromatic number of a graph, namely $\chi'(G) \leq \Delta(G)+1$.

13.7 SOLUTIONS / ANSWERS.

E1) Recall that bipartite graphs were characterised as graphs without odd cycles. Trees are acyclic graphs, i.e. they do not contain cycles as subgraphs and therefore they are bipartite. Since trees are connected and we have assumed it has atleast two vertices, it has chromatic number 2.

E2) Even cycles do not contain odd cycles as subgraphs. So, they are bipartite. Therefore, they have chromatic number 2.

E3) The chromatic number of an odd cycle is **3.** Since it is not bipartite, its chromatic number is atleast **3.** We get a 3-colouring of C_{2n+1} **as** follows: Let $\{v_1, v_2, \ldots, v_{2n+1}\}$ be the vertex set of C_{2n+1}. We assign $\boxed{1}$ to all the vertices in the set $\{v_i \in V(C_{2n+1}) \mid i \text{ odd}, 1 \leq i \leq 2n)$ and $\boxed{2}$, to all the vertices in the set $\{v_i \mid i \text{ even}, 2 \leq i \leq 2n)$. Now, v_{2n+1} is adjacent to both v_1 and v_{2n}. So, we cannot assign [1] or □ to this vertex. Therefore, we assign the third colour [3] to v_{2n+1}.

E4) A three colouring of Petersen graph is given below:

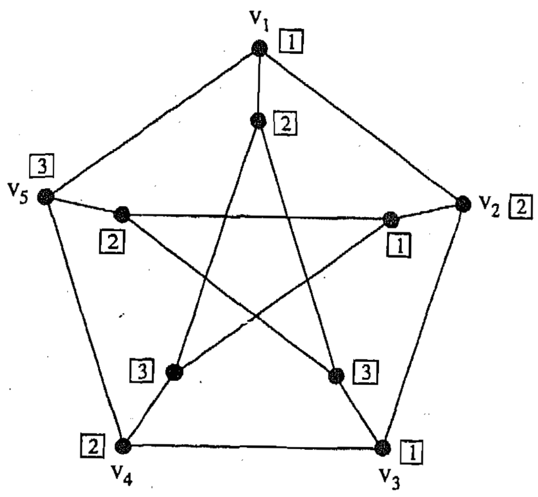

Fig.36

Further Petersen graph contains a 5–cycle which has chromatic number three. So, Petersen graph has chromatic number three.

E5) Since it has chromatic riumber greater than 2, it cannot be bipartite. So, it must coritain an odd cycle.

E6) (a) A 3-colouring of the graph is given below:

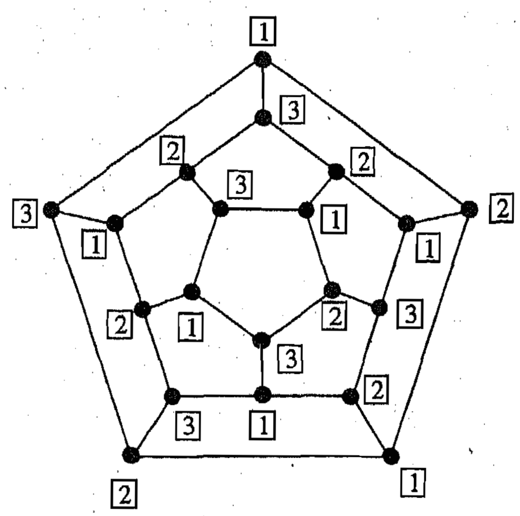

Fig.37

(b) The chromatic number of this graph is three. We have already seen that this graph has a 3-colouring. Further, it has cycles of length 5 as subgraphs and we have already seen that cycles of odd length have chromatic number **3.**

E7) In the graph in **Fig.7(a)**, the graph induced by v_4, v_5, v_6, v_7 is K_4. So, it has a clique of size **4** and therefore we need atleat **4** colours. We get a 4-colouring by assigning [1] to v_1, □ to v_2, [3] to v_3, [2] to v_4, □ to v_5, [1] to v_6 and [4] to v_7. So, the chromatic number is **4.**

E8) The figure given in Fig.38 is 5-chromatic.

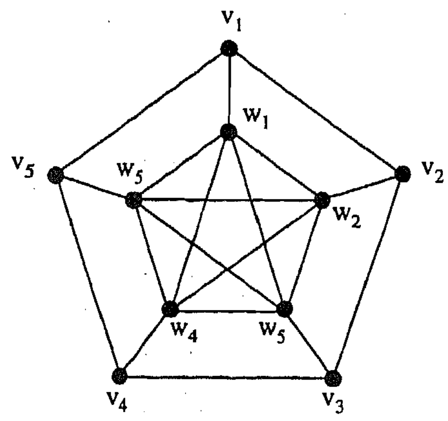

Fig.38

The graph in Fig.38 contains a clique of size 5, namely the subgraph induced by the vertices w_1, w_2, w_3, w_4, w_5. So, we need atleast 5 colours. We first give a 5-colouring to the subgraph isomorphic to K_5 by assigning $\boxed{i}$ to w_i, $1 \leq i \leq 5$. Next, we assign $\square$ to v_1, $\square$ to v_2, $\boxed{1}$ to v_3 $\boxed{2}$ to v_4, and $\square$ to v_5.

E9) Two different colourings are given in Fig.39

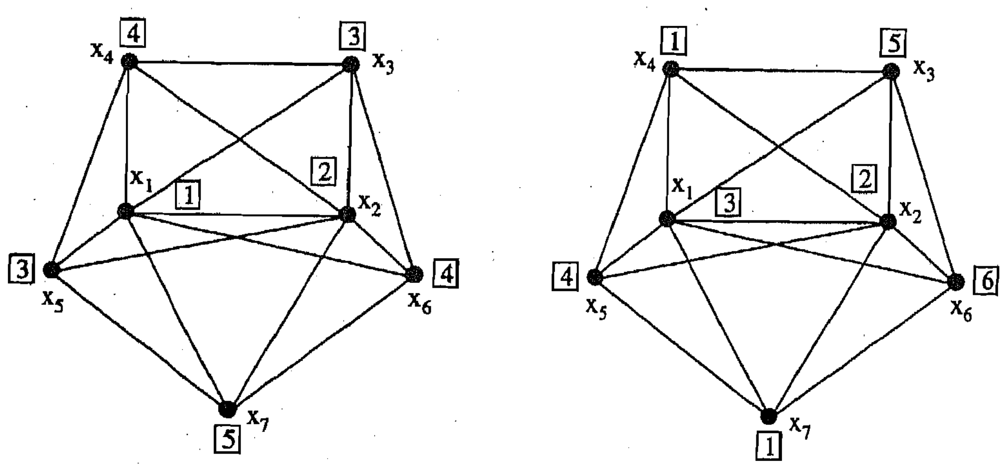

Fig.39

The colour classes for the colouring in Fig.39(a) are $\{x_1\}$, $\{x_2\}$, $\{x_7\}$, $\{x_3, x_5\}$, $\{x_4, x_6\}$. The colour classes for the colouring in Fig.39(b) are $\{x_1\}$, $\{x_2\}$, x_3, $\{x_4, x_7\}$ and $\{x_5, x_6\}$

E10) $\{v_1, v_2, v_4, v_5, \}$

E11) For Example 1, you can see that $\{x_{13}, x_{14}, x_3, x_5, x_7, x_9, x_{11}\}$ is an independent set and any other set has ≤ 7 elements. Thus $\alpha(G) = 7$.

For Example 2, you note that for every vertex x_i, there are precisely two vertices in G not adjacent to x_i. But those two are adjacent. Hence, $\alpha(G) = 2$.

E12) If we remove any vertex from K,, we get K_{n-1} which has chromatic number $n - 1$. Let $\{v_1, v_2, v_3, v_4, v_5\}$ denote the vertex set of K,. Let us remove an edge from K,. Renumbering the edges if necessary, we can assume that the edge we have removed is v_1v_n. Then, we get an $n - 1$ colouring as follows: Assign $\square$ to both v_1 and v,. For each v_i, $2 \leq i \leq n - 1$, assign $\boxed{i}$.

E13) If we remove any of the vertices v_1, v_2, v_3, v_4, v_5, (See Fig.40 below)

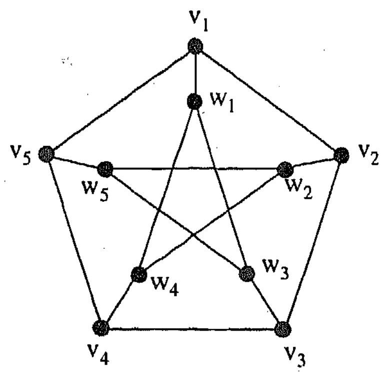

Fig.40

the odd cycle $\{w_1, w_2, w_3, w_4, w_5\}$ is unaffected and the resulting subgraph has chromatic number 3. Similarly, the graph obtained by deleting any of the vertices v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, we will get a graph which has chromatic number three since it will contain the odd cycle $\{w_1, w_2, w_3, w_4, w_5\}$.

E14) Refer to Fig.40. The odd cycle $\{w_1, w_2, w_3, w_4, w_5\}$ is a 3-critical subgraph.

E15) a) 18 b) 7

E16) Since $K_{3,3}$ is bipartite, we can apply Theorem 5. Here $p = 6$ and $q = 9$. But, $2p - 4 = 10 > 9 = q$. So, $K_{3,3}$ is not planar.

E17) The graph obtained by deleting the two horizontal edges is shown in Fig.41(a).

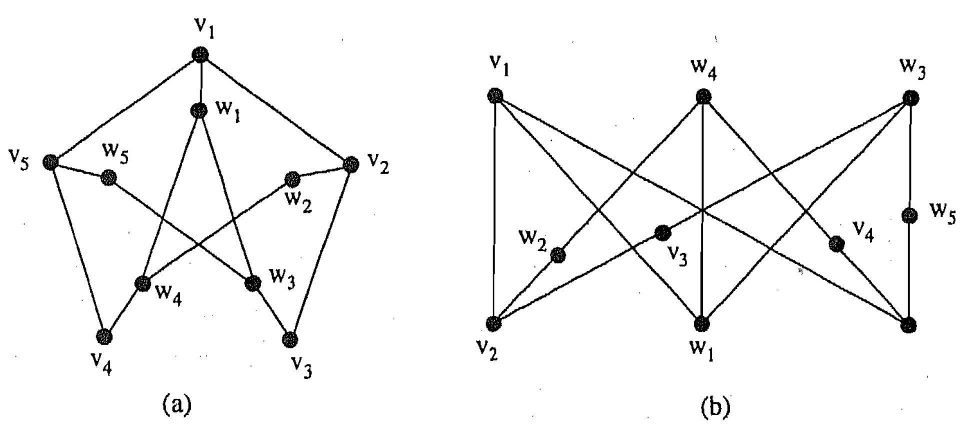

Fig.41

We have redrawn Fig.41(a) in Fig.41(b) so that you can clearly see that it is a subdivision of $K_{3,3}$.

E18) Since $K_{m,n}$ is bipartite graph, by König's result, $\chi'(K_{m,n}) = \Delta(K_{m,n}) = \text{Min}(m, n)$.

E19) The required $\Delta(T)-$ colouring is given Fig.42. You must remember that it is not unique.

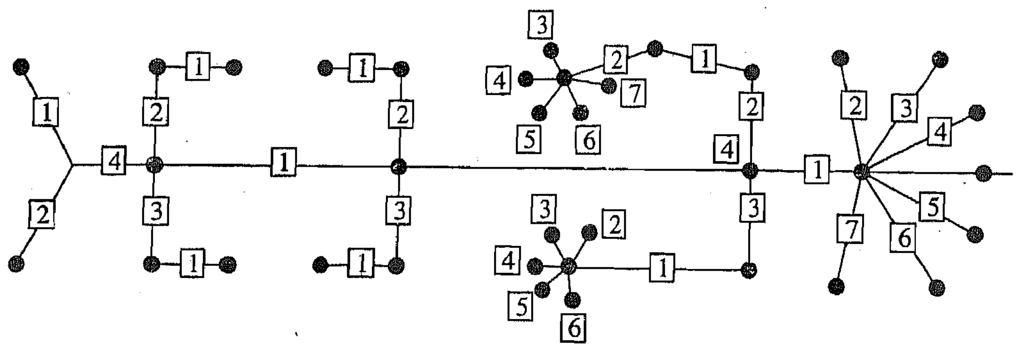

Fig.42

LIST OF STUDY CENTRES FOR B. Sc. PROGRAMME

Sl. No.	Centre Code	Centre Address
1.	0103	K.B.N. College, Kothapet, Vijayawada, Andhra Pradesh.
2.	0201	Government College, Port Blair, Andaman & Nicobar.
3.	0401	Gauhati University, Guwahati, Assam. (Hendique Girls College, Guwahati Assam)
4.	0501	Patna Science College, Patna, Bihar.
5.	0504	Bihar University Library, Muzaffarpur, Bihar. (L.S. College, Muzaffarpur, Bihar)
6.	0506	St. Xavier's College, Ranchi, Bihar.
7.	0701	Shivaji College, Raja Garden, Ring Road, New Delhi.
8.	0707	Jamia Millia Islamia, Jamia Nagar, New Delhi.
9.	0711	Gargi College, Siri Fort Road, New Delhi.
10.	0801	Shree Damodar College of Commerce and Economics, Margo, Goa. (S.P. Chowgule College of Arts and Science, Margao, Goa.)
11.	0902	M.S. University, Baroda, Gujarat.
12.	0906	J.B. Tacker Commerce College, Bhuj, Gujarat. (Lalan College, Bhuj, Gujarat)
13.	1001	Mukhannd Lal National College, Yamuna Nagar, Haryana.
14.	1005	Chhotu Ram College of Education, Rohtak, Haryana (All India Jat Heroes Memorial College, Rohtak, Haryana)
15.	1101	Government College, Sanjauli, Shimla, Himachal Pradesh.
16.	1105	Government College, Dharamshala, Himachal Pradesh.
17.	1201	University of Jammu, Jammu Tawl, Jammu & Kashmir. (Gandhi Memorial Science College, Jammu Tawi)
18.	1303	J.S.S. College, Dharwad, Karnataka.
19.	1309	Vivekanand College df Arts, Commerce & Economics, Rajaji Wagar II Stage, Bangalore, Karnataka.
20.	1401	Institute of Management in Govt., Vikas Bhawan, Trivandrum, Kerala. (University College, Trivandrum)
21.	1403	J.D.T. Islam, Calicut, Kerala.
22.	1501	Motilal Vigyan Mahavidyalaya, Bhopal, Madhya Pradesh.
23.	1509	Government P.G. College, Jagdalpur, Madhya Pradesh.
24.	1510	Ravi Shankar University, Raipur, Madhya Pradesh. (Science College, Raipur, Madhya Pradesh)
25.	1603	Parle College, Bombay, Maharashtra.
26.	1607	Lakshrni Narayana Institute of Technology, Amravati Road, Nagpur, Maharashtra. (Institute of Science, Nagpur, Maharashtra)
27.	1701	Manipur University, Imphal, Manipur.
28.	1802	Government College, Tura, Meghalaya.
29.	1901	Aizawl College, Aizawl, Mizoram. (P.G. College, Aizwal, Mizoram)
30.	2103	Government College, Rourkela, Orissa.
31.	2104	Khallikote College, Berhampur, Ganjam, Orissa.
32.	2201	D.A.V. College, Jalandhar, Punjab.
33.	2306	Dayanand College, Ajmer, Rajasthan.
34.	2504	Bishop Heber College, Tiruchirapalll, Tamil Nadu.
35.	2601	Tripura University, Agartala, Tripura.
36.	2720	Lucknow Christian College, Lucknow, U.P.
37.	2704	Bareilly College, Barelly, Uttar Pradesh.
38.	2708	Udai Pratap P.G. College, Varanasi, Uttar Pradesh.
39.	2810	Maulana Azad College, Calcutta, West Bengal.

Centres in parentheses indicate the places where science practicals may be organised for that Study Centre.