UNIT 11 SPECIAL GRAPHS

Structure **Page No.**

11.1 INTRODUCTION.

In the last unit you saw that graphs are often used to represent (that is, model) communication or transportation networks and several other systems such as representation of a molecule in a chemical compound and so on. In a transportation network, it is necessary to know which destinations are connected by a direct route. For example, if air travel is abolished then the people without any seaport cannot go to any other country unless their neighbours provide the initial road passage through their territory. When we use a graph to model this situation, it is important that there be a way to connect from any vertex to any other vertex. Such graphs are called connected graphs. In Sec.2.2 we will define connected graphs and we will show that any graph can be partitioned into connected graphs.

In Sec.2.3, we will familiarise you with a type of graph which is useful in electronics and other areas. These graphs are called bipartite graphs. Such graphs are very useful in studying real-life problems, for example in moclelling neural network.

In Sec. 2.3 we have considered another type of graphs called 'Trees'. The graphs which represent chemical compounds butane and isobutane are trees. You are familiar with these graphs in Unit 10. Such graphs are of special interest to chemists. They want to find out whether any trec correspond to a chemical compound or not. Here we will show that a tree has got several interesting properties and these properties are used in studying some real-life situations.

Objectives

After reading this unit, you should be able to

- distinguish between walks, paths, circuits and cycles in a graph

identify

1) connected graphs
2) bipartite graphs
3) trees

11.2 CONNECTED GRAPHS

From Unit 10, you know that graphs are model for different real life situations especially situations involving routes; the vertices represent towns or junctions and each edge represents a road or some other form of communication link. This kind of a picture is very helpful in understanding connected graphs that we introduce in this section. To understand such graphs we need some definitions which describe ways of "going from one vertex to another". We shall first give these definitions in the following subsection.

11.2.1 Paths, Circuits And Cycles

Consider the graph in Fig.1. Imagine yourself walking along its the edges, going from vertex to vertex.

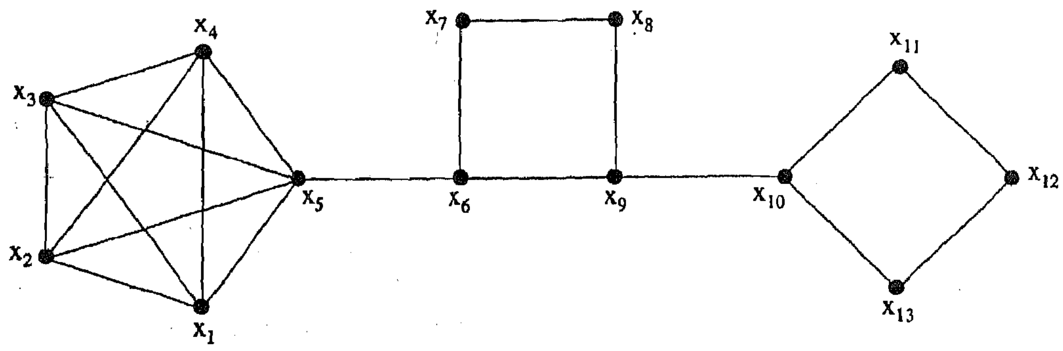

Fig.1

Suppose we want to start at the vertex x_1 and reach the vertex x_{12}. Is this possible? Onepossible way is to start from vertex x_1, walk along the edge x_1x_2, reach x_2, walk along the edge x_2x_3, reach x_3, walk along x_3x_4 reach x_4, continue this till we reach x_{12}. Suppose we denote the edge joining x_{i-1} and x_i as $(x_{i-1}x_i)$. Then we can describe this walk in an alternating sequence of vertices and edges as x_1, (x_1x_2), x_2, (x_2x_3), x_3, (x_3x_4), x_4, (x_4x_5), x_5, (x_5x_6), x_6, (x_6x_9), x_9, (x_9x_{10}), x_{10}, $(x_{10}\,x_{11})$, x_{11}, $(x_{11}x_{10})$, x_{10}, $(x_{10}x_{13})$, x_{13}, $(x_{13}x_{12})$, x_{12}. What does this represent? You recall from Unit 10 that this represents a walk. This is by no means the shortest way to reach x_{12} from x_1. We could have gone from x_1 to x_5 directly. Moreover, we passed through the vertex x_{10} twice. This is not necessary. So the above walk can be described as a leisurely walk. If we have more time at our disposal, we can trace and retrace more edges. For example, we could have gone from x_6 to x_9 and again back to x_6.

So what are we doing when choosing a walk? We are, in fact, choosing a sequence whose elements are vertices and edges, alternately.

Now we forinally define a walk.

Definition: A walk in a graph G is a finite sequence $W = \{v_0, e_1, v_1, e_2, \ldots, e_k, v_k\}$, where $v_0, v_1, \ldots, v_k$ are vertices and $e_1, e_2, \ldots, e_k$ are edges joining the vertices v_{i-1} and v_i, $1 \le i \le k$. Note that all the v_is or e_is may not be distinct. There may be repetition.

In this case we say that W is a walk from v_0 to v_k or W is a v_0-v_k **walk** or W is a walk **joining** v_0 **and** v_k. The vertex v_0 is called the initial vertex and the vertex v_k is called the **end vertex** of the walk W. The integer k which is the number of **edges** contained in a **walk** is called the length of the walk W and is denoted by $l(W)$. Since the vertices as well as the edges can be repeated, the length can very well be greater than the number of the edges of the graph G.

Note: As you have seen, in a walk the vertices as well as edges can be repeated. So we cannot view this as a subgraph unless all the vertices as well as the edges in the walk are distinct.

Let's consider an example .

Example 1: Consider the graph on 5 vertices and 7 edges given in Fig. 2. Find a x_1-x_5 wallc of length 8.

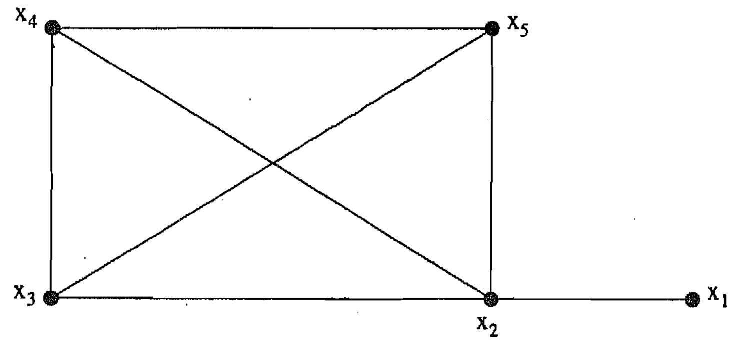

Fig.2

Solution: Consider the walk $W = \{x_1, x_1x_2, x_2, x_2x_3, x_3, x_3x_4, x_4, x_4x_2, x_2, x_2x_5, x_5, x_5x_3, x_3, x_3x_4, x_4, x_4x_5, x_5\}$. Then W is x_1-x_5 walk of length 8.

* * *

Another possible walk for the same graph could be $\{x_1, x_1x_2, x_2, x_2x_4, x_4, x_4x_3, x_3, x_3x_5\}$. Its length is $l(W) = 4$.

Why don't you try this exercise now?

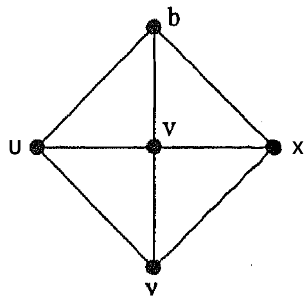

Fig.3

E1) For the graph given in Fig.3, find a u-v walk of length 7.

Since we are considering only graphs which do not have multiple edges or loops, we often write a walk W as $\{v_0, v_1 \ldots, v_k\}$. While doing so we assume that two consecutive vertices'in a walk are joined by an edge in the graph and that edge is included in the walk. For example, the walk corresponding to Fig.1 can be written as

$$W = \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{11}, x_{10}, x_{13}, x_{12}\}.$$

The concept of a walk is too general for our purposes, so we shall impose some further restrictions. Before that let us consider the graph given in Fig.4.

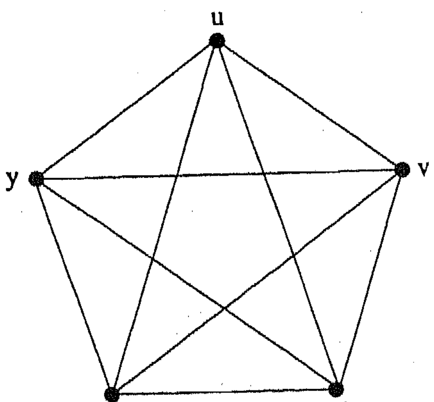

Fig.4

This is the complete graph K_5. Then

$W = \{u, v, x, w, v, x, y\}$, $W_1 = \{u, x, w, v, x, y\}$ and $W_2 = \{u, x, w, v, y\}$ are three walks joining u and y having lengths 6, 5 and 4, respectively. Also note that

i) in the walk W, vertices v and x as well as the edge v x are repeated,

ii) in the walk W_1, only the vertex x is repeated but no edge is repeated and

iii) in the walk W_2, neither a vertex nor an edge is repeated.

The walks W, W_1 and W_2, corresponding to Fig.4, are given special names according to the definition given below.

Definition: A walk is called a **trail** if all the edges in it are distinct. For example W_1 corresponding to Fig. 4 is a trail. Note that in a trail vertices can be repeated. A walk W is called a **path** if all the vertices .are distinct. For example W_2 corresponding to Fig. 4 is a path.

If all the vertices in a walk are distinct, can edges repeat'? Remember that an edge is traced only after tracing an end vertex. Therefore, all the edges of are also distinct. Hence, a path is always a trail. What about the converse? We leave it as an exercise for you. (see E2)

Next we shall give some more definitions.

Definitions: A walk u - v is **closed** if u = v and **open** if u # v.
A closed trail is called a **circuit.**
A circuit in which the only repeated vertex is the first vertex, this being the same as the last vertex is called a cycle.
Let us consider an example.

Example 2: Consider the following graph on 15 vertices.

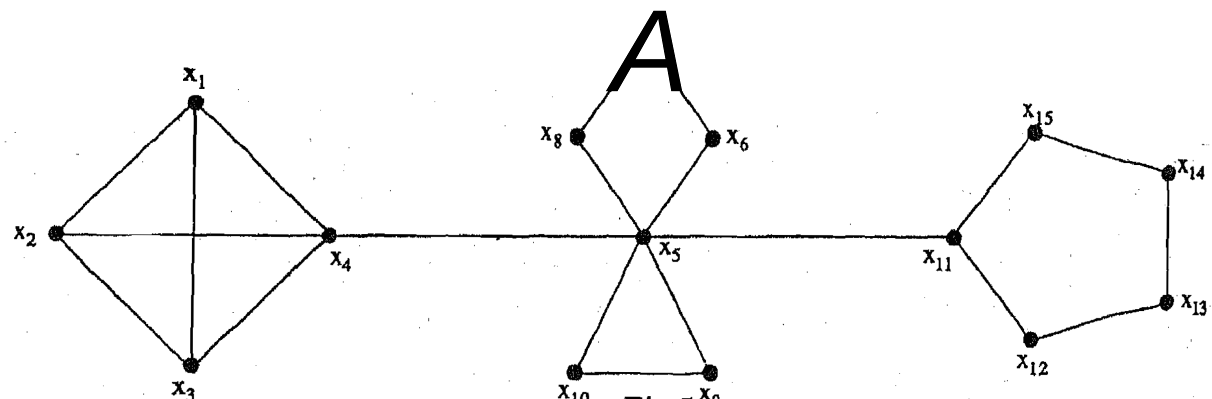

Fig.5

In this graph find the following:

i) a closed walk which is not a circuit,

ii) **a** circuit which is not a cycle,

iii) a cycle.

Solution: We shall find **(i),** (ii) and (iii) one by one.

i) There are several closed walks in it, which are not circuits.
$W = \{x_5, x_6, x_7, x_8, x_5, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{11}, x_5\}$ is a closed walk Here the edge $x_5 x_{11}$ is repeated. Hence it is not a circuit,

ii) $W_0 = \{x_5, x_6, x_7, x_8, x_5, x_9, x_{10}, x_5\}$ is a circuit. Here the vertex x_5 is repeated three times. Thus, this is not a cycle.

iii) $W' = \{x_5, x_6, x_7, x_8, x_5\}$ is a cycle.

* * *

Try these exercises now.

E2) i) Is every trail a path? Give reasons for your answer.

ii) If all edges are distinct, then all vertices are distinct. True or false? Why?

E3) Is a cycle a path? Give reasons for your answer.

E4) Let $G = (V,E)$ be a graph where
$V = \{t, u, v, w, x, y, z)$ and
$E = \{tu, tv, tw, ux, vw, vy, uz, wx, wz, xy, xz)$. In G, find

i) a u-v trail that is not a path,

ii) a (u-u) circuit that is not a cycle,

iii) a (v-v) cycle of minimum length.

E5) Let G be a graph such that $\delta(G) \geq k$. Use the principle of induction to to show that the graph G has a path of length k starting at any given vertex. (Recall that $\delta(G) = \min\{d_G(x) : x \in V(G)\}$.

Now let us go back to the graph given in Fig. 4 again. In this graph $W = \{u, v, x, w, v, x, y)$ is a walk. Suppose we omit the part $\{w, v)$ we get $P = \{u, v, x, y)$. You know that this object is a path. In the next theorem, we will prove that this phenomenon is true in general.

Theorem 1: If W is a u-v walk joining two distinct vertices u and v, then there is a path joining u and v contained in the walk.

Proof: Let W be a u-v walk given by
$W = \{u = u_0, e_1, u_1, \cdots, e_k, u_k = v)$
We will find a path joining u and v contained in W, using the principle of mathematical induction.

Let $p(k)$ denote the statement that if W is a u-v walk of length k, then there exists a path joining u and v contained in W.

If $k = 1$, then $p(1)$ is true since every walk of length 1 is a path.

Now we assume that the statement $p(k-1)$ is true for all walks of length $\leq k-1$. In other words, we assume that given any x-y walk of length $\leq k-1$, there exists a path joining x and y contained in the walk. Then we want to show that the statement $p(k)$ is true for W.

If W is already a path, we are done. Otherwise, there is at least one vertex which is repeated. Suppose j is the smallest integer such,that the vertex u_j is repeated. Then there is an integer $t > j$ such that $u_j = u_t$. Now consider the walk W_1 obtained by removing the part $\{e_{j+1}, \cdots, e_t\}$, that is
$W_1 = \{u = u_0, e_1, \cdots, u_j = u_t, e_{t+1}, \cdots e_k, u_k = v)$. Clearly W_1 is a u-v walk contained in the walk W and its length $l(W_1) = k - t + j < k$, since $j < t$. Hence, by induction, we can get a path P joining u and v contained in W_1. Since P is contained in the walk W_1 and W_1 is contained in the walk, the path P is contained in the walk W. Thus, $p(k)$ is true for W.

Therefore by induction, $p(n)$ is true for all n. Hence the result

The Theorem above says that, **if** there **is a** walk joining; two vertices in a graph, then we can **always** find a path joining **them**.

Now in many of the practical situations it is very important to know which of the vertices in a graph can be joined by a walk, and hence by a path. For instance, in the graph G, obtained by taking union of K_6 and K_5 (see Fig.6).

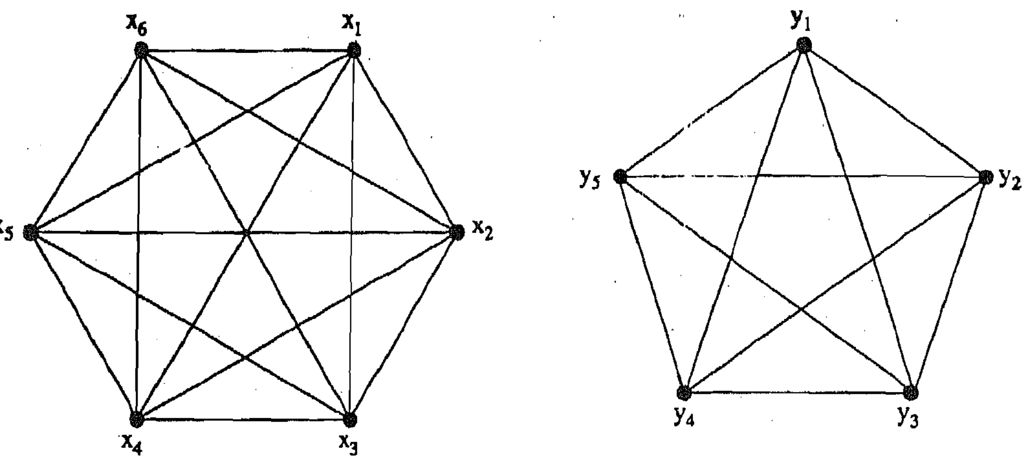

Fig.6

You can see that there is no (x_1-y_5) walk. Hence there is no way we can traverse from the vertex x_1 to the vertex y_5.

So, in the internal structure of a graph it sometimes matters whether two vertices are joined by a walk or not. This leads'us to the definition of a connected graph which we will introduce in the next subsection.

11.2.2 Components

As you have noticed, almost all graphs we have discussed so far have been 'in one piece'. The exceptions are null graphs and the union of graphs each of which is in one-piece. We can formalise this difference by introducing the concept of connectedness which we shall define in this sub-section. We are not only interested in the main graphs being connected, but we are also interested in knowing the subgraphs which are connected, which are known as components. Here we shall discuss them in detail.

A graph, whose edge-set is empty is called a null graph.

Definition: A graph $G = (V, E)$ is called connected if for any two vertices $u, v \in V$, there exists a u-v walk in G. If G is not connected, then it is called disconnected.

This means that in a connected graph any two distinct vertices are joined by a walk. From Fig. 6 you can see that both the graphs K_6 and K_5 are connected, but their union is not connected since there is no walk connecting the vertices of K_6 to the vertices of K_5

Here are some exercises for you.

E6) Can a graph with one vertex be connected? Give reasons far your answer.

E7) Which of the graphs given in Fig.7 are connected?

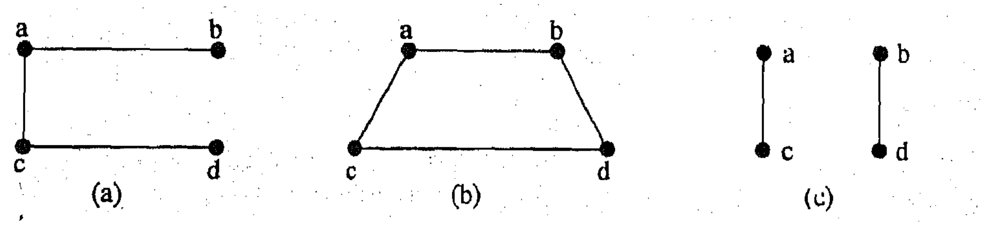

Fig.7

E8) If a graph G is connected , then all its subgraphs are connected. Prove or disprove this statement.

While solving E8, you would have realised that subgraphs of connected graphs need not be connected. But what about disconnected graphs? You can see that some subgraphs of such graphs are connected. Let us discuss them now.

Definition: Let $G = (V, E)$ be a graph A subgraph H of G is called a component.

i) if H is connected and it is not a subgraph of any other connected subgraph of G; and

ii) whenever K is a connected subgraph of G, and H is contained in K, then $H = K$.

Thus, a component is, in a sense, a 'maximal' connected subgraph of G. The number of components of G is denoted by $c(G)$.

Now, consider the graph G given by Fig.6. You can see that K_6 and K_5 are its components, and G is the union of these components.

Let us consider another example.

Example 3: Consider the graph G given by Fig.8. Find three components of this graph.

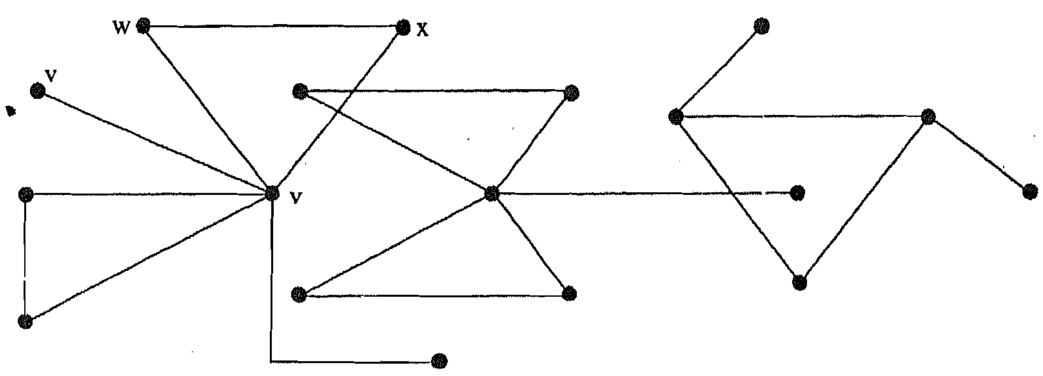

Fig.8

Solution: The three components of G are G_1, G_2 and G_3 (given in Fig.9(a), (b) and (c)).

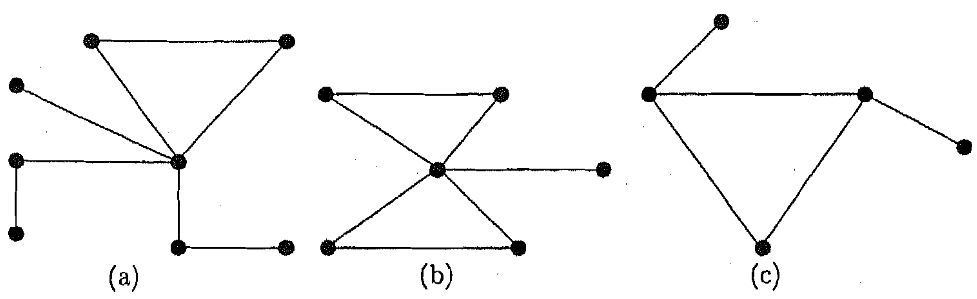

Fig.9

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Here also G is the disjoint union of components G_1, G_2 and G_3.

You can now try this exercise.

E9) Consider the graph G given by Fig.10. Then find

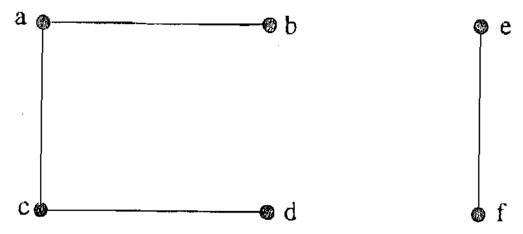

Fig.10

i) all the connected subgraphs of G.

ii) all the components of G. Are they disjoint? Give reasons for your answer.

E10) Consider the graph given in Fig.11. Show that tlie graph can be written as the disjoint union of its components.

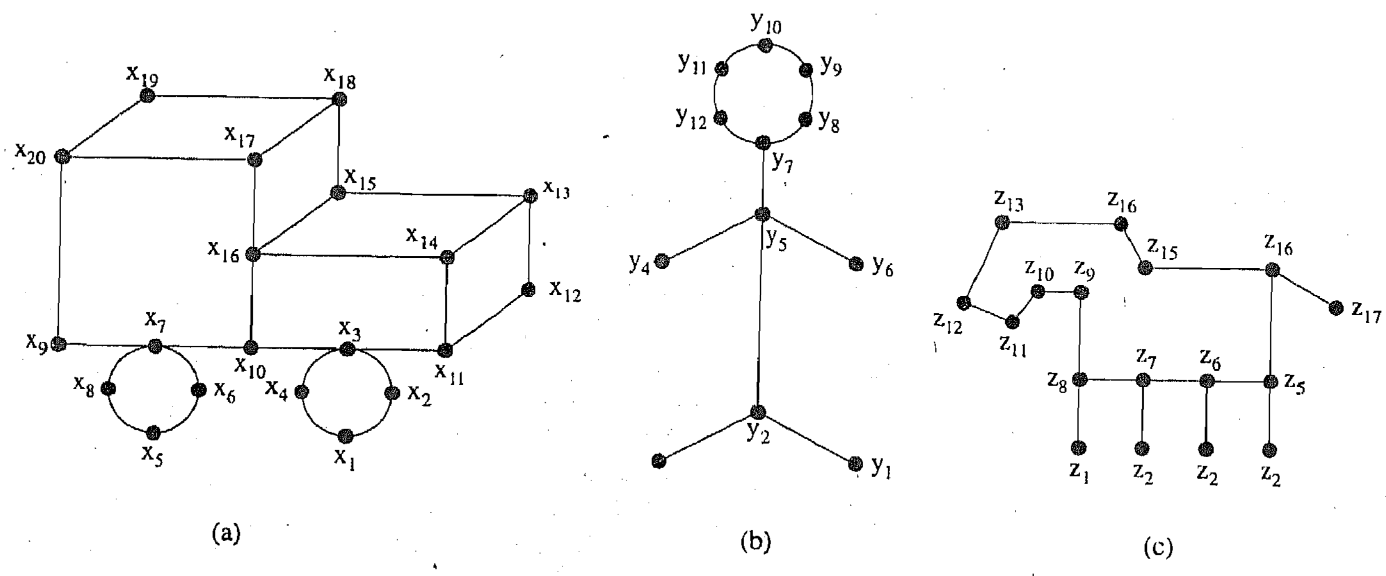

Fig.11

In tlie case of Example 3 and E10, we saw that the graphs given in each case can be written as a disjoint union of the corresponding components. Tliis phenomenon generalises to any graph as you will see in the following theorem. We shall only state tlie theorem; the proof of which, though it is not very difficult, is omitted.

Theorem 2: Every graph can be partitioned into components.

Now that we know each graph can be partitioned into components, we shall find something more about components in a graph. One direction of interest is to investigate bounds for tlie number of edges of a graph on n vertices with a given number of components. We shall now state a general result which gives the required bound as a special case.

Theorem 3: If G is a graph with n vertices and has k components,

$$n - k \le n \le \frac{1}{2}(n - k)(n - k + 1)$$

Note: If G is connected, then k = I and we get the bounds as

$$n - 1 \le n \le \frac{1}{2}n(n - 1)$$

Another approach used in the study of connected graphs is to ask the question 'how connected' is a connected graph'?:. One possible interpretation of this question is to ask how many edges or vertices must be removed from the graph in order to disconnect it. We shall discuss this in the next subsection.

11.2.3 Connectivity

Let us now consider the graph showing an electric circuit (see Fig.12). This graph is connected. Suppose we break the wire connecting d and e in the electric circuit. This means that in the graph showing the circuit, we are actually removing an edge corresponding to the wire. Now when we break this wire, the circuit becomes disconnected. This means that the removal of that edge in the graph makes the graph disconnected.

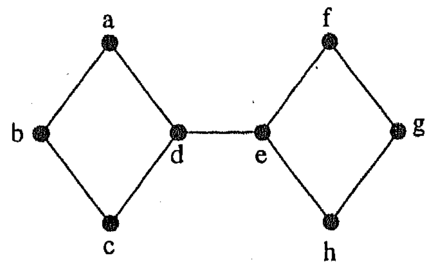

Fig.12

Note: Whenever we talk about removing an edge say xy, we mean, removing only the connection between x and y, that is the edge not the incident vertices x and y.

When we remove an edge uv from the graph, we denote tlie resulting graph by G – uv.

We just saw a situation in which tlie removal one edge disconnects the graph. But this is not always the case. For instance if we remove the edge ab in Fig. 12, the resulting graph is not disconnected. You can also see this situation in the graph given in Fig. 13, which represents the roads connecting the

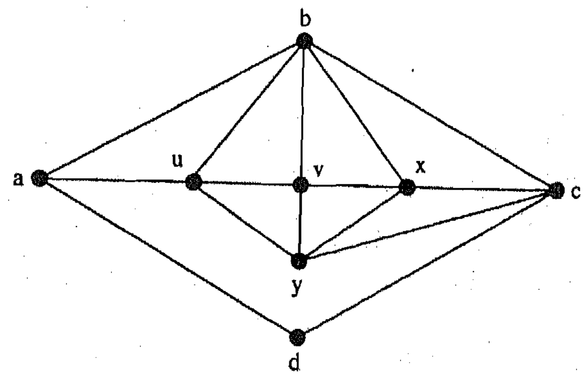

Fig.13

main towns in a state, In this case the removal of any single edge will not disconnect the graph since there always exist alternate connections.

These types of edges lead us to the following definition

Definition An edge e of a graph G is called a bridge in G if the removal of e disconnects G.

For example, the edge uv in the graph given by Fig.12, is a bridge.

Here are some related exercises for you.

E11) Find the bridges in each of the graphs in Fig.14.

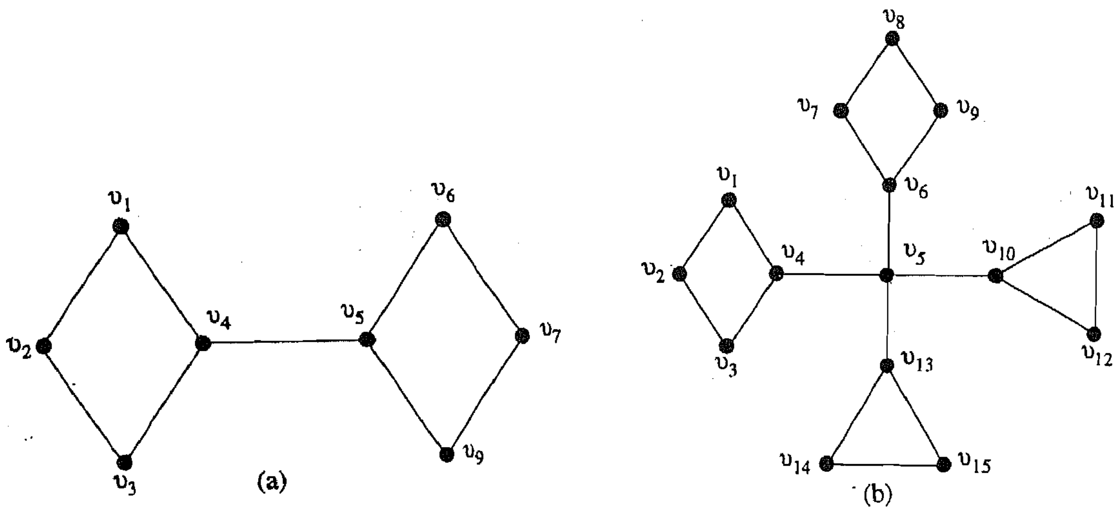

Fig.14

E12) Give an example of a graph without a bridge.

Let us consider another graph given by Fig.15.

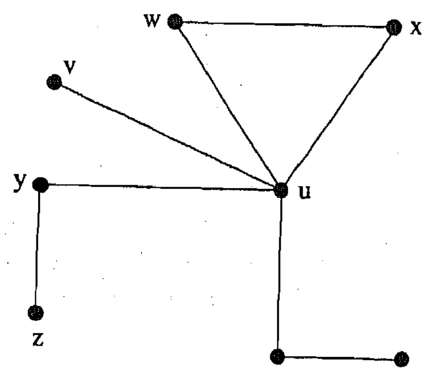

Fig.15

This graph is connected. Here, if we remove the edge uv, then the resulting graph gets disconnected, the components being {v) and G/ {v). The number of components of the resulting graph (G − uv) is 2. On the other hand, if we remove the edge uw, then the graph does not get disconnected. Note that the edge uw belongs to the cycle {u, w, x, u), but the edge uv does not belong to any such cycle. The cycle seems to provide an alternate connection between the vertices u and w.

In fact it follows from the definition of a bridge that **an** edge e **of a graph G is a bridge if and only if** e does **not belong to any cycle of** *G*.

While doing the exercise E11, you must have obtained a graph which does not have a bridge. We cannot disconnect such a graph by removing just one edge; we need to remove more than one edge to disconnect it. Therefore, given a graph, it is natural to **ask** 'what is the minimum number of edges whose removal disconnects G?'. This number is given a special name according to the following definition.

Definition: The edge-connectivity $\lambda(G)$ of a connected graph G is the smallest number of edges whose removal disconnects G.

For example, the edge-connectivity of the graph given in Fig.14 is 1. In fact the edge-connectivity of any graph with a bridge is 1.

Let us consider an example.

Example 4: Find the edge-connectivity of the graph G given in Fig.16.

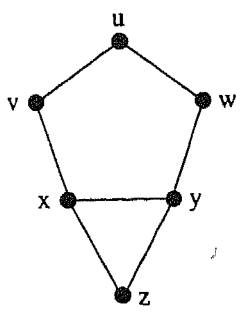

Fig.16

Solution: First note that this graph does not have any bridges. Therefore its edge-connectivity is more than 1. Now, if we remove the edges xz, zy, then the graph gets disconnected. Similarly there are other sets of two edges, namely {xv, vu}, and {uw, wy}, whose removal disconnects G. Therefore we get that the edge connectivity is 2.

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Why don't you try this exercise now?

E13) Find the edge connectivity of

i) the graph given in Fig.15;

ii) the following graphs.

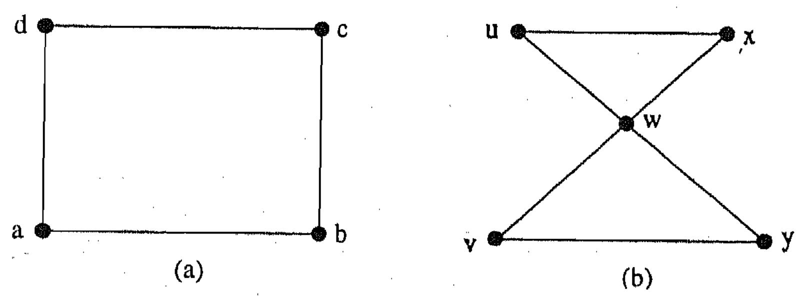

Fig.17

Let us now look at a set of edges of a connected graph.

Definition: A cut set S of a connected graph G is a set S of edges with the following properties:

i) the removal of all the edges in S disconnects G;

ii) the removal of any proper subset of S will not disconnect G.

For example consider the following graph given in Fig.18

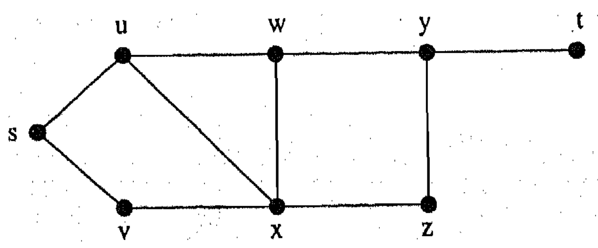

Fig.18

The set {uw, ux, vx) and {uw, wx, xz} are cut sets for this graph; whereas the set {uw, wx, xz, yz} is not a cut set since this set has a subset {uw, wx, xz) whose removal disconnects G.

Note that two cut sets of a graph need not have the same number of edges. For example, in the above graph in Fig.18, the sets {uw, ux, vx) and {wy, xz) are both cutsets.

Also note **that** the edge connectivity $\lambda(G)$ of a graph G is the **size** of the smallest **cutset** of G.

Try this exercises now.

E14) Which of the following sets of edges are cutsets of the graph given in **Fig.**19. and what is its edge-connectivity.?

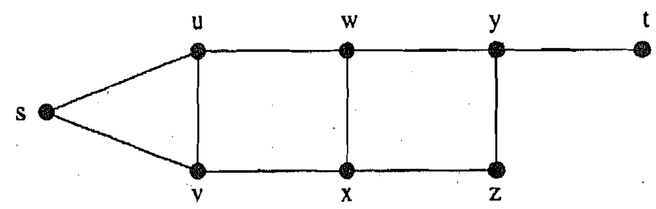

Fig.19

a) {su, sv}

b) {uv, wx, yz}

c) {ux, vx, wx, yz}

d) {yt}

e) {wx, xz, yz}

f) {uw, wx, wy}

We can also think of connectivity in terms of the minimum number of vertices which need to be removed in order to disconnect a graph. Note that when we remove a vertex, then if there is any edge incident with that vertex, that also gets removed. Let us see some examples. Let us consider the graphs given in Fig.17. Graph 17(b) can be disconnected by removing just one vertex w. But Graph 17(a) cannot be disconnected by removing one single vertex, but the removal of two non-adjacent vertices (such as a and c) disconnects it.

Now we can define vertex-connectivity and vertex-cut-set on similar lines as we have done for edges. Why don't you try it for yourself (see E 15).

E15) How would you define vertex-connectivity and cut vertex-set ?

E16) Find the vertex-connectivity and a cut vertex-set for the graph given in **Fig.17** (b).

In the next section we shall introduce you to another type of graphs known as bipartite graphs.

11.3 BIPARTITE GRAPHS

In this section we shall define bipartite graphs and explain their importance through various problems.

Let us first start with the following problem.

Four workmen x_1, x_2, x_3 and x_4 are available to fill five jobs y_1, y_2, y_3, y_4 and y_5. x_1 is qualified for the jobs y_1 and y_2; x_2 is qualified for the jobs y_1 and y_3. x_3 is qualified for the job y_4; and x_4 is qualified for the jobs y_2, y_3 and y_5. The assignment problem is concerned with the following questions:

i) Can each person be assigned to a single job for which he is qualified?

ii) If so, how should the assignment be made?

iii) If not, at most, how many of them can be assigned?

The problem of the kind stated above is known as assignment problem. To solve this problem it is convenient to consider the following graph theoretic model of the situation.

The graph G has vertices $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$ and y_5 and edges defined in the following way: there is an edge joining x_i and y_j if x_i is qualified for the job y_j. The graph is shown in Fig.20.

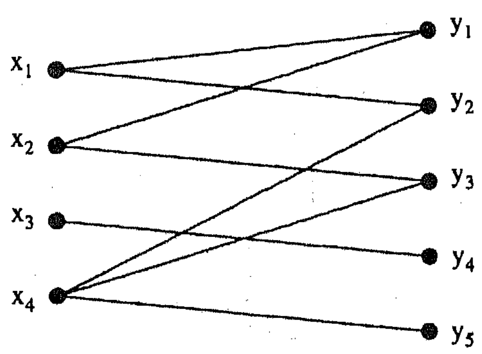

Fig.20

Then the problem of assigning people to jobs for which they are qualified is equivalent to the problem of selecting a subset of the set of edges such that each x will be connected to exactly one y by one of these edges.

Now, if you look at the graph given in Fig.20, you will see that the set of its vertices can be divided into two disjoint subsets such that no two vertices in a subset are adjacent. Let us formally define such graphs.

Definition : A graph G is said to be bipartite if $V(G) = X \cup Y$, where X and Y are non-empty subsets such that $X \cap Y = \phi$ and every edge in E(G) has ono end vertex in the set X and the other end vertex in the set Y. The sets X, Y form a partition of the set V(G) and we often say that $X \cup Y$ is a bipartition of the graph G.

An alternative way of thinking of a bipartite graph is in terms of colouring its vertices with two colours, say red and blue - a graph is bipartite if we can colour each vertex red or blue in such a way that every edge has a red end and a blue end.

Bipartite graphs are useful in studying various real-life problems as for example, for modelling neural networks. Many different kinds of models have been formed for studying the neural networks. One such model that emulates the essential working of the network using graph theory is given in Fig.21. As you can see that this is a bipartite graph and the properties of bipartite graphs are used in studying this model,

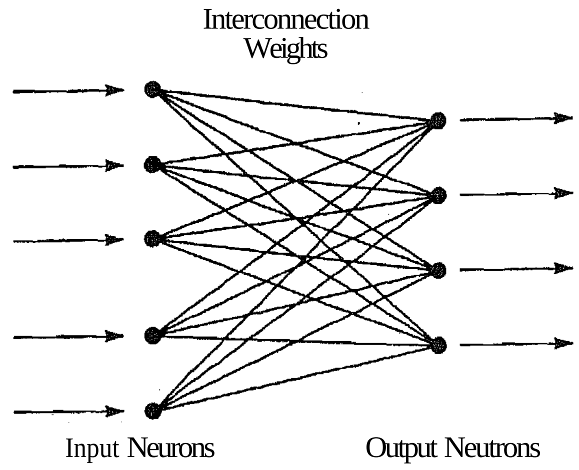

Fig.21

Given a bipartite graph, you may wonder if the bipartition is unique. The following example will give you an answer to this question.

Example 5: Consider the graph given in Fig.22. Find two different partitions that make G bipartite.

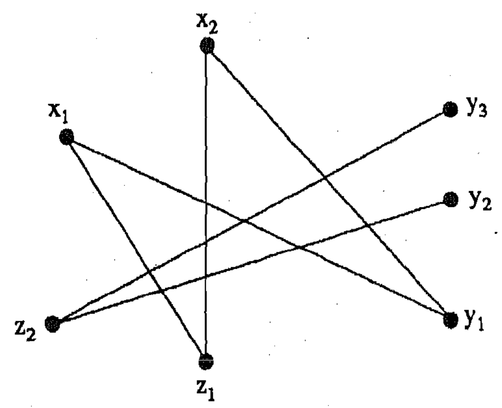

Fig.22

Solution: The vertex set is $\{x_1, x_2, y_1, y_2, y_3, z_1, z_2\}$. One way of partitioning this can be by taking $X = \{x_1, x_2, z_2\}, Y = \{z_1, y_1, y_2, y_3\}$. Another way can be $X_1 = \{x_1, x_2, y_3\}, Y_1 = \{z_2, z_1, y_1, y_2\}$. Both these partitions make G bipartite.

***

We shall now state a theorem which gives *a* characterisation for bipartite graphs. Before giving the statement let us just note the following.

Note: When a graph G is a cycle on n vertices, we often say that G is an n-cycle. A cycle C_n is said to be an even cycle if n is a positive even integer and it is called an odd cycle if n is a positive odd integer. The positive integer n is called the length of the cycle.

Now we state the theorem without giving its proof.

Theorem 4: **A** graph G is bipartite if and only if G does not contain any odd cycles as subgraphs.

You can, try some exercises now.

E17) Check whether the fallowing graphs are bipartite or not.

i) complete graph K_3 (see Sec. 10.1, Unit 1)

ii) liypercubes Q_2 and Q_3 (see Sec. 10.2, Unit 2)

Q_n is bipartite

E18) Show that the subgraph of a bipartite graph is bipartite.

E19) Show that if $G_1, \ldots, G_n$ are bipartite, tlien $\bigcup_{i=1}^{n} G_i$ is bipartite.

Let us now go back to the assignment problem. In that problem we are interested iii finding special subgraphs of bipartite which gives a solution to the problem. We have defined such graphs below.

Definition: In a bipartite graph G, let X and Y denote the two disjoint subsets of vertices. A matching in G is a set of edges such that no two edges in the set are incident with the same vertex (in X or in Y). In other words a matching defines a one-to-one correspondence between the vertices in a subset of X arid the vertices in a subset of Y.

For example, the following figure sliows a bipartite graph and one of its matchings. Fig. 23(b) gives the matching.

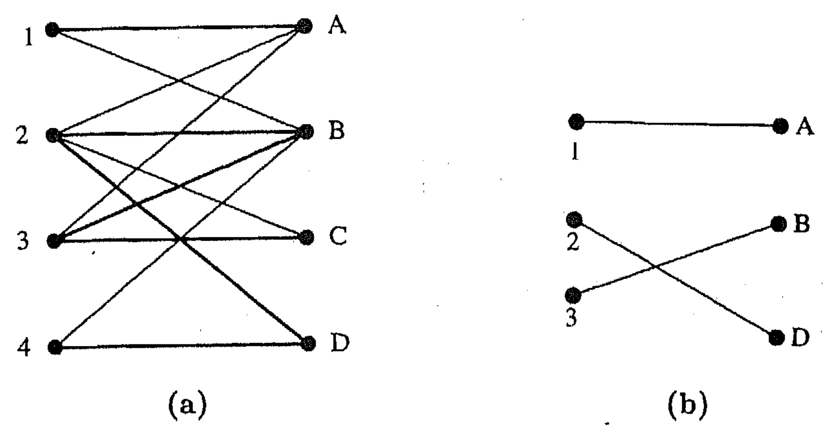

Fig.23

Can you find any other matching? We leave this as an exercise for you to check (see E 20 (i)).

Related to the concept of matching, we have another concept.

Definition: **A** matching of **X** into Y is called a complete matching of X **and** Y if, there is an edge incident with every vertex in X. In other words, a matching is complete if a one-to-one correspondence is defined between **all** the vertices in X and the vertices in a subset of Y.

Is the matching given in Fig.23(b) a complete matching? No, because in this matching, the vertex 4 is left out.

In graph-theoretic terminology, the assignment problem can be stated in the following way: if $G = G(X, Y)$ is a bipartite graph, when does there exist a complete matching from X to Y in G? So, for a given bipartite graph, we want to know whether there is a complete matching of the set of vertices in X into the set of vertices in Y. The following theorem gives a necessary and sufficient condition for the existence of a complete matching in a bipartite graph. As before we shall only state the theorem, omitting the proof.

Theorem **5:** Let $G = G(X, Y)$ be a bipartite graph. **A** complete matching of X into Y exists in G if **and only if** $|A| \leq |R(A)|$ for every subset **A** of X where $|A|$ denotes the number of elements in **A** (also called cardinality of **A**) and R(A) denotes the set of vertices in Y that are adjacent to the vertices in A.

Next we shall apply the above theorem to the assignment problem in the following example.

Example 6: Verify the conditions of Theorem 5 for the assignment problem given at the beginning of this section(See Fig.20).

Solution: To check the theorem we have to consider all subsets of the vertex set X = $\{x_1, x_2, x_3, x_4\}$, their cardinality, corresponding sets R(A) and their cardinality. The following table gives a list of all the possibilities.

Table 1

A	|A|	R(A)	|R(A)|
ϕ	0	ϕ	0
$\{x_1\}$	1	$\{y_1, y_2\}$	2
$\{x_2\}$	1	$\{y_1, y_3\}$	2
$\{x_3\}$	1	$\{y_4\}$	1
$\{x_4\}$	1	$\{y_2, y_3, y_5\}$	3
$\{x_1, x_2\}$	2	$\{y_1, y_2, y_3\}$	2
$\{x_2, x_3\}$	2	$\{y_1, y_3, y_4\}$	2
$\{x_3, x_4\}$	2	$\{y_2, y_3, y_4, y_5\}$	4
$\{x_1, x_4\}$	2	$\{y_1, y_2, y_3, y_5\}$	4
$\{x_2, x_4\}$	2	$\{y_1, y_2, y_3, y_5\}$	4
$\{x_1, x_3\}$	2	$\{y_1, y_2, y_4\}$	3
$\{x_1, x_2, x_3\}$	3	$\{y_1, y_2, y_3, y_4\}$	4
$\{x_2, x_3, x_4\}$	3	$\{y_1, y_2, y_3, y_4, y_5\}$	5
$\{x_1, x_3, x_4\}$	3	$\{y_1, y_2, y_3, y_4\}$	4
$\{x_1, x_2, x_4\}$	3	$\{y_1, y_2, y_3, y_5\}$	4
$\{x_1, x_2, x_3, x_4\}$	4	$\{y_1, y_2, y_3, y_4 y_5\}$	5

The table shows that the condition $|A| \leq |R(A)|$ is satisfied for all subsets **A** of X. Hence the conditions of Theorem 1, is satisfied.

* * *

The example above shows that there exists a complete matching from X into Y for the assignment problem. Therefore the assignment problem is solved. You can now try this exercise now.

E20) For the bipartite graph given in Fig.23, find a matching, apart from the one given in Fig.23. Does the graph given in Fig. 23(a) have a complete matching? Give reasons for your answer.

Let us now see another type of graph which has come into prominence because of its applications to electrical networks.

11.4 TREES

We are all familiar with the idea of a family tree. The concept of a tree in graph theory'first arose in connection with pork of a mathematician G. Kirchoff on electric networks in the 1840s, and with the work of another mathematician Cayley on the enumeration of chemical molecules in the 1870s. More recently, trees are used many areas, ranging from linguistics to computing.

For mathematicians, the interest and importance of trees arises from the fact that, in many ways, a tree is a special type of graph, which has several interesting properties some of which we sliall bring out in this section.

Let us first see what a tree means.

Consider the following graphs. Can you find any difference in their structures?

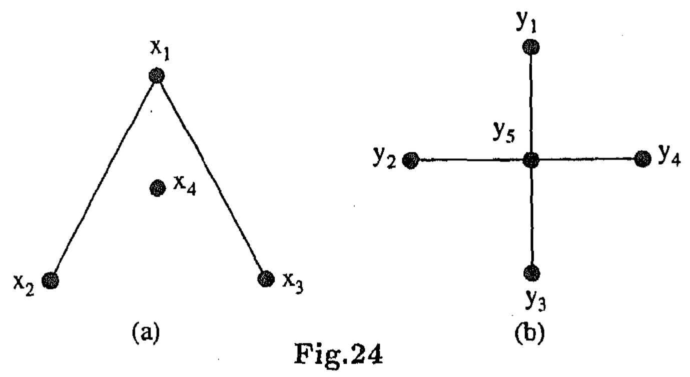

Fig.24

You miglit have noticed tliat (a) is disconnected. Also, (a) has no cycles. On the otlier hand, (b) is connected arid has no cycles. From the following definition you will see that (b) is an exaniple of a tree.

Definition: A graph with no cycles is called **acyclic.** A tree is a connected acyclic graph. A forest is a graph, each of whose components is a tree.

The following figure shows a forest with four components, (a), (b), (c) and (d) each of which is a tree,

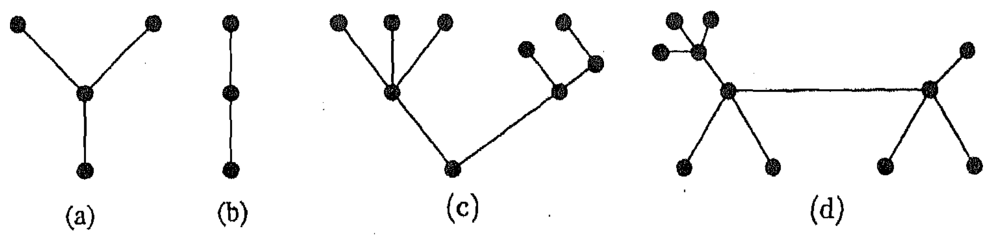

Fig.25

A tree has several interesting properties which we shall list in the following theorem.

Theorem **6:** Let G be a graph with n vertices. Then the following statements are equivalent.

i) G is a tree.

ii) G is acyclic and has $(n - 1)$ edges.

iii) G is connected, 'and lias $(n - 1)$ edges,

iv) G is connected, and every edge is a bridge.

v) any two vertices of G are connected by exactly one path.

Proof: If $n = 1$, all the five results are trivial. We shall, therefore. assume that $n \geq 2$. Now, from Unit **2** you know that if we prove (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (v) and (v) $\Rightarrow$ (i) then all the statements are equivalent so let, us do this. We sliall prove the equivalent statements one by one.

(i) => (ii): By the definition, G does not have any cycles. Therefore it is acyclic. Now we will sliow that G has $(n - 1)$ vertices. We will prove this by induction.

If $n = 1$ then the number of edges is 0. Therefore, the result is true for $n = 1$.

So, now let us assume that every tree on p vertices has $(p-1)$ edges for any positive integer p such that $1 < p < n$. Then we have to show that every tree on n vertices has $(n-1)$ edges. Now suppose we remove any edge. Since G is acyclic; the removal of any edge disconnects G into two graphs G_1 and G_2, such that G_1 and G_2 are connected and acyclic. Therefore G_1 and G_2 are trees and each has vertices less than n. Let n_1 and n_2 be the vertices in G_1 and G_2. Then $n_1 + n_2 = n$. Since n_1 and n_2 are less than n, by our induction assumption, the number of edges in G_1 and G_2 are $n_1 - 1$ and $n_2 - 1$ respectively. Therefore the total number of edges in both the graphs is $n_1 + n_2 - 2 = n - 2$. This together with the edge which is removed will give the total number of edges in the original graph. Therefore the total number is $n - 1$. Thus we got that every tree on n vertices has $n - 1$ edges. This is true for all n. Hence the result

(ii)=> (iii): Suppose that G is disconnected. Let $c(G) = t > 1$. Let $G_1, G_2, \ldots, G_t$ be components of G such that the number of vertices in each G_i is p_i for $i = 1, 2, \ldots t$, and the number of edges in each G_i is q_i, for $i = 1, 2, \ldots, t$. Then

$$p = p_1 + p_2 + \ldots + p_t, q = q_1 + \ldots + q_t$$

Now since every G_i is connected and acyclic, each G_i is a tree for $i = 1, 2, \ldots, t$. Therefore, by what we have shown while proving (i) $\Rightarrow$ (ii), $q_i = p_i - 1, 1 \leq i \leq t$. Then

$$p - 1 = q = q_1 + \ldots + q_t = p - t$$

This is possible only if $t = 1$. This contradicts our assumption that $t > 1$. Therefore, G is connected.

(iii) => (iv): Suppose there is an edge which is not a bridge. Then the removal that edge will result in a graph with n vertices and $(n - 2)$ edges. This not possible when G is connected by Theorem 3 in Sec. 11.2. Therefore every edge is 'a bridge.

(iv) => (v): Since T is connected each pair of vertices is connected by at least one path. If a given pair of vertices is connected by two paths, then they form a cycle, which contradicts the fact that every edge is a bridge. Therefore there is a unique path joining any two vertices.

(vi) => (i): We are assuming that any two vertices are connected by a 'unique path. So, the graph G is connected. It is also acyclic because if it contains a cycle $C = \{x_0, x_1, \ldots, x_n = x_0\}$, then we can find two distinct paths $P_1 = \{x_0, x_1\}$ and $P_2 = \{x_0, x_{n-1}, \ldots, x_2, x_1\}$ connecting the vertices' x_0 and x_1, which contradicts our assumption. Therefore, G is a tree.

The theorem above tells us that a tree has got several nice properties which a general graph does not have. In fact the importance of trees in graph theory is that every connected graph contains a tree which has all the vertices of the original graph, as you will now see.

Let us consider a connected graph G. Consider a cycle in it and remove one of its edges, such that the resulting graph is connected. We repeat this procedure with one of the remaining cycles, continuing until there are no cycles left. The graph which remains is a connected subgraph of G which does not have any cycle. Therefore, it is a tree. Note that this tree has all the vertices of G. Such a graph is called a spanning tree, as you will realise from the following definition.

Definition: A spanning tree for a graph G is a connected acyclic subgraph which contains all the vertices of G.

The following figure shows a connected graph and one of its spanning trees.

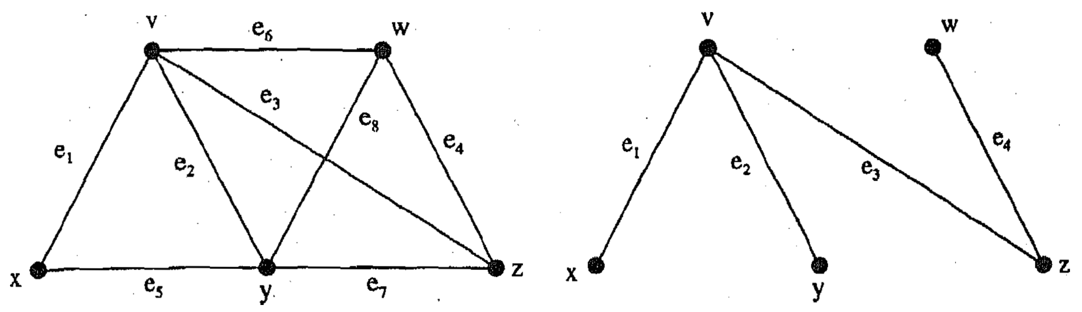

Fig.26

Does this graph have only one spanning tree? No, the graph in Fig. 27 gives another spanning tree for the graph.

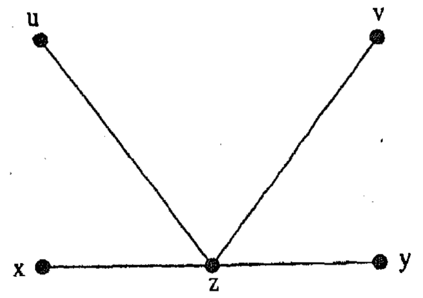

Fig.27

This shows that a connected graph can have several spanning trees. We shall now state tlie theorem, the proof of which is omitted.

Theorem 7: G is connected if and only if it has a spanning tree.

The theorem above tells tliat in a graph with k components, each component will have a spanning tree. Because of this result and because of the special structure of trees, in trying to prove a general result, in graph theory, it is sometimes convenient to try to prove the corresponding result for a tree.

You can try some exercises now.

E21) Draw three spanning trees of the following graph ,

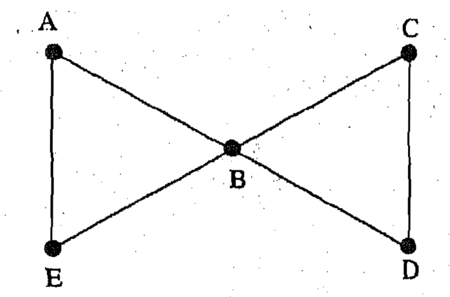

Fig.28

E22) Is every tree a bipartite graph? Give reasons for your answer.

So far we have seen three types of graphs: connected graphs, bipartite graphs and trees. You will see more types of graphs in the following units. Let us now summarise what we have covered in this unit.

11.5 SUMMARY

In this unit we have covered the following main points.

1) We have defined the terms walks, trails, paths, circuits and cycles in a graph.
2) We have defined connected graphs as well as components. We have also discussed, through various examples, how to find components of a graph. We have also shown the effect of removal of a vertex or an edge on the number $c(G)$ of the components of a graph G.
3) We have defined bipartite graphs and obtained a characterisation of these graphs in terms of cycles in them.
4) We have defined trees and discussed the importance of these graphs among the class of all connected graphs.

11.6 SOLUTIONS / ANSWERS

E1) {u, uv, vx, x, xy, y, yv, v, vb, b, bu, u, uv) is a walk of length 7. This is not the only one. Think of some others too.

E2) i) The walk corresponding to W_1 Fig. 4 is an example of a trail which is not a path. In W_1 the vertex x is repeated.

ii) False. The reason is the same as that given in (i).

E3) No, because the first and last vertices are the same.

E4) i) It is easy to find examples if you draw the walk. One example is {u, x, w, z, y, x, w, v) There are other examples.

ii) W = {u, v, y, z, w, x, z, u) is a circuit in which the vertex is repeated. Therefore, W is not a cycle.

iii) W_0 = {u, t, w, x, u) is a cycle such that all other cycles have length greater than $l(W_0)$.

E5) We use induction on k. If k = 1, then every vertex has at least one neighbour. Thus, there exists a path of length 1 starting at any vertex.

Now, by induction, assume that in every graph H with $\delta(H) \geq (k-1)$, there is a path of length $(k-1)$ starting at any given vertex.

Let G be a graph with $\delta(G) \geq k > 1$. Let x_0 be any vertex in G. Choose any edge e_1 incident on x_0. Consider $G - e_1$. Removal of one edge reduces only the degrees of its end vertices by one. Thus $\delta(G - e_1) \geq (k - 1)$. Thus, by induction, there is a path {xi, $e_2, \ldots, e_k, x_k\}$ of length $(k-1)$ in G_1. Moreover, since the degree of x_{k-1} $d(x_{k-1})$, is at least k, we can choose x_k different from x_0, Xi,..., x_{k-2}. Also $\{x_0, el, x_1, e_2, \ldots, x_k\}$ is a path of length k in G. Therefore there exists a path of length k starting at any vertex in G. since this is true for all k, the result follows.

E6) It is connected. Because if it is disconnected then there exists two distinct vertices which are not joined by a path, which is not possible since the graph does not have two distinct vertices.

E7) (a) and (b) are connected, (c) is disconnected.

E8) The statement is false. For example, consider the graph K_3 given in Fig.29(a). Then the subgraph of this graph obtained by deleting the edges v_1v_3 and v_2v_3 given in Fig.29(b) is not connected.

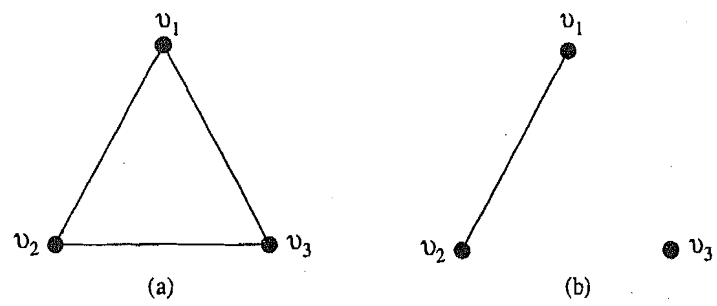

Fig.29

E9) a) Graphs having single vertices a, b, c, d, e, f and tlie graphs having the following vertices and edges

i) $V = \{a, b)$, $E = \{ab\}$

ii) $V = \{a, c)$, $E = \{ac)$

iii) $V = \{c, d)$, $E = \{cd)$

iv) $V = \{c, f\}$, $E = \{cf)$

v) $V = \{d, c, a\}$, $E = \{dc, ca\}$

vi) $V = \{b, a, c\}$, $E = \{ba, ac\}$

vii) $V = \{a, b, c, d)$, $E = \{ab, ac, cd)$

b) Two components are: the graph formed by the vertices a, b, c and d and the graph formed by the vertices e and f.

E10) Note that the graph consists of three pieces: one looks like a car, the other looks like a human being and the third looks like a dog. Then each piece is a connected graph which are not contained in any other connected subgraphs of G. Therefore, each piece is a component. Hence, we get that G is the disjoint union of its components.

E11) a) In (a) there is only one bridge given by v_4v_5. Whereas in (b) there are four bridges given by $v_4v_5, v_5v_{10}, v_5v_{13}, v_5v_6$.

E12) The graph given by the following Figure:

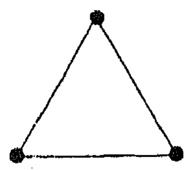

Fig.30

E13) i) Edge connectivity is 1 in both the cases.

ii) Edge connectivity is 2 in both the cases.

E14) The sets given in (a), (c), (d) and (f) are cut sets. The set given in (b) is not a cut set, since its removal docs not disconnect the graph; the set given in **(c)** also not a cut set, since we can disconnect the graph by removing just xz and yz.

E15) The vertex connectivity of a connected graph G is the smallest number of vertices whose removal disconnects G.

A cut vertex set H of a connected graph g is a set H of vertices with the following properties

i) the removal of all vertices in H disconnects G

ii) the removal of any proper subset of H will not disconnect G.

E16) Vertex connectivity is 1 and the set is {w}

E17) i) From the figure you can see K_3 has 3-cycles. So, by Theorem 4, it is not bipartite.

ii) Both Q_2 and Q_3 does not contain odd cycles, therefore, by Theorem 4 , they are bipartite.

E18) Let G be a bipartite graph with a bipartition $X \cup Y$. Let H be a subgraph of G. If V(H) is disjoint from either X or Y, $E(H) = \phi$. You can take any portion of V(H) into two subsets. It will serve as a bipartition,

If, on the other hand, V(H) intersects both the subsets X, Y of V(G), then $V(H) = X' \cup Y'$, where $X' = X \cap V(H), Y' = Y \cap V(H)$, serves as a bipartition of H.

E19) Let $G_i, 1 \leq i \leq n$ be bipartite graphs with the bipartitions $V(G_i) = X_i \cup Y_i$, respectively. Let $G = \cup_{i=1}^{n} G_i$. Then, E(G) is the disjoint union $\cup_{i=1}^{n} E(G_i)$. Clearly, $V(G) = A \cup B$, where $A = \cup_{i=1}^{n} X_i, B = \cup_{i=1}^{n} Y_i$, is a bipartition of V(G). This can be seen as follows: Let e be an edge in E(G). Since E(G) is disjoint union $E(G_1), \ldots, E(G_n)$ and the edge e belongs to only one of them. Without loss of generality, suppose $e \in E(G_r)$. Since G_r is bipartite with a bipartition $X_r \cup Y_r$, this means e has one end vertex in X, and the other in Y,, that is, e has one end vertex in A and the other in B. Thus, G is bipartite with a bipartition $A \cup B$.

E20) Fig.31 gives another matching.

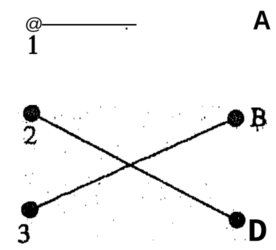

Fig.31

The following is a complete matching in the graph: {1A, 2B, 3C, 4D} (shown by thicker lines in Fig.23).

E21)

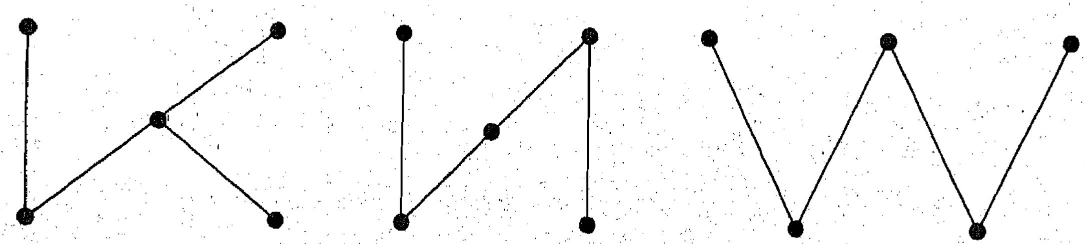

Fig.32

E22) Yes. Since a tree does not have cycles, by Theorem 4, it is a bipartite graph.