UNIT 6 THE SIMPLEX METHOD

Structure

6.1 INTRODUCTION

In Unit 2, you were introduced to the notion of a convex set and its geometrical meaning. In Units 3 and 4, the graphical method of solving a LPP was discussed and it was shown that this method fails when you are dealing with a LPP involving three or more variables. Again in Unit 5, the meanings of various types of solutions of a LPP.have been explained' namely feasible solutions, basic solutions, basic feasible solutions and optimal solutions. It was shown proved that, the set of all feasible solutions for a LPP forms a convex set and that the optimal solution, if it exists, occurs at one of the extreme points of the convex set. Further, that every extreme point of a convex set of feasible solutions of a LPP, corresponds to a basic feasible solutions of the problem. **How to find that extreme point which corresponds to the optimal solution?** In other words, **how to identify an optimal solution out of the basic feasible solutions of a LPP?** We discuss this question now in this unit.

To answer this question, we use an algebraic method popularly called **Simplex Method.** We shall start with a basic feasible solution which is nor optimal. Then we shall show that how it is possible to active, in a finite number of steps (called iterations), an optimal basic feasible solution. Computational procedure in the Simplex method requires us first to understand how to put a given linear programming problem in a tableau format to follow a well organised procedure. This useful tabular format displaying all relevant quantities was developed by Orden, Dantzig and Hoffmann and is called the **Simplex Tableau.** You will see as you learn the computational form of the Simplex Method, that **a** new table is constructed. at each iteration of the Simplex Algorithm.

Objectives

After you have completed this unit, you should, be able to

- put a given linear programming problem in a tableau format and use it in solving a linear programming problem
- describe the steps of Simplex algorithm and use it to solve any linear programming problem
- to obtain initial basic feasible solution of a LPP by introducing artificial variables and use it by the two-phase method

6.2 SIMPLEX ALGORITHM

The Simplex Method is a computational process suitable for numerical solution of a linear programming problem. It was given by a famous American mathematician G.B. Dantzig in 1947. The word "Simplex" has nothing to do with the method as such. Its origin can be traced back to a special problem that was studied in the early development of its algorithm. During World War II, a group worked on allocation problems for the U.S. Air Force. A few models were developed by this group to allocate resources in such a way so as to maximize or minimize some linear objective function.

However, it was Dantzig a member of this group, who ultimately formulated the general linear programming problem and devised the simplex method for its solution. Problems of linear programming type were formulated and discussed even before the method was developed by Dantzig. However, the simplex method is the most efficient and reliable procedure that is generally used to solve a LPP. The method is extensively applied with the help of modern computers when the LPP involves a large number of constraints and variables.

But before we discuss the method, let us first understand the meaning of an Algorithm.

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Meaning of **An** Algorithm

You have come across the term **'algorithm'** several times in your previous studies of Mathematics. For example, you are familiar with the process of long division which is an algorithm, Similarly, the procedure for calculating the square roots is an algorithm. In short, an **algorithm is an** iterative solution procedure. It is simply a process in which the steps are repeated (iterated) over **and over** again until the desired result is achieved. Can you recall the iterative steps involved in the long division algorithm or the square root algorithm? Just try it. Thus, an algorithm is a procedure starting with the first step known as the **intial** step, developing a criteria to know when and where to stop and reach the last step where the desired result is obtained. This can be summarized in the following way :

(i) Structure of a General **Algorithm**

First Step	:	Ready to start the iterations
Subsequent steps	:	Performing the iterations
Concluding step	:	Has the desired result been achieved?
If Yes	:	Stop
If No	:	Repeat the interations

For the algorithm of the Simplex Method, we way have the following similar structure :

(ii) Structure of the **Simplex** Algorithm

The Simplex Algorithm is an algebraic procedure in which each iteration involves a system of equations to obtain a new trial solution for the optimality test. According to the procedure, we have to take the following three steps :

I. Initial Step : Start with a basic feasible solution of a given LPP

II. Initerative Step : Move to a better basic feasible solution

III. Optimality Test Step : The current basic feasible solution is optimal

IV. If yes : Stop
If No : Repeat the interative step

We give a complete description of the general algorithm through the following example of a LPP :

EXAMPLE 1 :

Maximize

$$Z = 3x_1 + 5x_2$$

Subject to

$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$
$$x_1 \ge 0,\ x_2 \ge 0.$$

SOLUTION : By introducing the slack variables (Unit 5), the problem becomes :

Maximize

$$Z = 3x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5$$

Subject to

$$x_1 + x_3 = 4 \quad (1)$$
$$2x_2 + x_4 = 12 \quad (2)$$
$$3x_1 + 2x_2 + x_5 = 18 \quad (3)$$
$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ x_4 \ge 0,\ x_5 \ge 0 \quad (4)$$

where x_3, x_4, x_5 are the slack variables,

Note that this problem is identical to the original and its form is much more convenient for algebraic manipulation and for identification of the feasible solutions. While dealing with this problem, it is easier to manipulate the equation of the objective function along with the constraints equations simultaneously. Therefore, before we use the steps of the Algorithm, we rewrite the problem once again in an equivalent way in the equations form as follows :

Maximize

$$Z$$

Subject to

$$z - 3x_1 - 5x_2 + 0x_3 + 0x_4 + 0x_5 = 0$$
$$0z + x_1 + 0x_2 + x_3 + 0x_4 + 0x_5 = 4$$
$$0z + 0x_1 + 2x_2 + 0x_3 + x_4 + 0x_5 = 12$$
$$0z + 3x_1 + 2x_2 + 0x_3 + 0x_4 + x_5 = 18$$

where $x_1, x_2, x_3, x_4, x_5 \geq 0$.

Now to use the algorithm, we have to find answers to the following corresponding questions :

I. Initial Step : How is the basic feasible solution selected ?

II. Iterative Step : While seeking a better basic feasible solution, how is the direction of the movement chosen? Where do we stop ? How to identify the new solution ?

III. Optimality Test : How to determine whether the latest basic feasible solution is the optimal solution ?

For the Step I, we can start with any basic feasible solution which is convenient to us. When the problem is in the inequality form, then the obvious choice is the origin i.e. all variables are taken as equal to zero i.e. $x_1 = 0, x_2 = 0$ is the starting basic feasible solution. But after the introduction of the slack or surplus variables, we take **original variables** viz. x_1, x_2 as the **non-basic variables** and **the slack variables** viz. x_3, x_4, x_5 as **the basic variables** for the initial (starting) basic feasible solution.

This is clearly illustrated as below where the basic variables are shown in bold face namely x_3, x_4, x_5 :

$$x_1 + x_3 = 4$$
$$2x_2 + x_4 = 12$$
$$3x_1 + 2x_2 + x_5 = 18.$$

With the non-basic variables x_1, x_2 chosen as $x_1 = 0$, $x_2 = 0$, we get the basic variables $x_3 = 4$, $x_4 = 12$, $x_5 = 18$. Hence the initial basic feasible solution is (0, 0, 4, 12, 18) for which Z = 0.

For the step II (the iterative step), we move from the current basic feasible solution to the next better basic feasible solution. This is done by replacing one non-basic variable (called the **Entering Basic Variable)** by **a** basic variable (called the **Leaving Basic Variable)** and identifying the new basic feasible solution. We, therefore, come to the stage of tackling the question : **How is the direction of the movement chosen? In** other words, **what is the criteria of choosing the Entering Basic Variable?**

Since the objective is to increase the value of the objective function Z, therefore, the variable that has the largest coefficient in the equation for Z would be the one to increase Z and hence should be chosen as the **entering basic variable.** In this case, the choice of the entering variable is x_2. Can you guess why? Try it.

Now the question arises : How to identify the **Leaving Basic Variable, The** possibilities for the leaving basic variable is that it is one of the non-basic variables x_3, x_4, x_5. We have to look for one of these x_3, x_4, x_5 for which the entering variable x_2 achieves the maximum value and none of x_3, x_4, x_5 becomes negative. This is done as follows :

Basic Variable	Equation	Max-value of x_2
x_3 :	$x_1 + x_3 = 4 \Rightarrow x_3 = 4 - x_1$	No limit
x_4 :	$2x_2 + x_4 = 12 \Rightarrow x_4 = 12 - 2x_2$	: $x_2 \le \frac{12}{2} = 6$
x_5 :	$3x_1 + 2x_2 + x_5 = 18 \Rightarrow x_5 = 18 - 3x_1 - 2x_2$	: $x_2 \le \frac{18}{2} = 9$

Table I

Since x_4 (the slack variable for the constraint $2x_2 \ge 12$) gives $x_2 = 6$ which satisfies the two conditions viz. that it is the maximum value for which none of x_3, x_4, x_5 becomes negative, therefore, x_4 is the variable we are looking for. Hence x_4 is the **leaving basic variable.** Note that $x_2 = 9$ is greater than $x_2 = 6$ but then if we take $x_2 = 9$, then x_4 becomes negative which we do not want.

Let us now calculate the new values' of the variables. Since we put only one non-basic variable at a time into the basis, we keep x_1 at 0 level, i.e. $x_1 = 0$. From the first row in Table I, we get $x_3 = 4$. From the second row, we get $x_4 = 12 - 2x_2 = 12 - (2 \times 6) = 0$. From the third row, we get $x_5 = 18 - (3 \times 0) - (2 \times 6) = 6$. So, our new basic feasible solution is

$$x_1 = 0,\ x_2 = 6,\ x_3 = 4,\ x_4 = 0,\ x_5 = 6,\ Z = 30$$

and x_2, x_3 and x_5 are our new basic variables.

Next, we shall check whether our solution is optimal. For this, we need to write our objective function in terms of the current non-basic variables, x_1 and x_4. So, we need to write x_2 in terms of the current non-basic variable x_1 and x_4 and use this to eliminate x_2. 'Also, to find the leaving variable we need to write all the current basic variables in terms of non-basic variables.

From (1), we can write

$$x_3 = 4 - x_1 \tag{1-a}$$

Dividing (2) by 2, we get

$$x_2 + \frac{1}{2} x_4 = 6 \text{ or } x_2 = \frac{1}{2}(12 - x_4) = 6 - \frac{x_4}{2} \tag{2-a}$$

Next, we multiply (2) by -1 and add it to (3) to get

$$3x_1 + 2x_2 + x_5 - (2x_2 + x_4) = 18 - 12$$

(ie) $3x_1 - x_4 + x_5 = 6$ or $x_5 = 6 - 3x_1 + x_4$ (3–a)

Now, we use equation (2–a) to eliminate x_2 from the objective function to get

$$Z = 3x_1 + 5x_2 = 3x_1 + 5\left(6 - \frac{x_4}{2}\right) = 3x_1 - \frac{5x_4}{2} + 30$$

In general the co-efficients of the non-basic variables x_1 and x_4 are called **relative cast co-efficients** because they are defined relative to a basis.

In the above equation, the coefficient x_1 is positive, so we can improve Z by increasing x_1. Since the co-efficient of x_4 is negative, Z will decrease if we increase x_4. So, we keep x_4 at 0 level. We now have the following situation:

Basic variable	Equation	Maximum value of x_1
x_2	$x_2 = 6 - \frac{x_4}{2} = 6$	No limit
x_3	$x_3 = 4 - x_1$	$x_1 \le 4$
x_5	$x_5 = 6 - 3x_1 + x_4 = 6 - 3x_1$	$x_1 \le \frac{6}{3} = 2$

Table-II

Since x_1 gives $x_1 \le 2$ and this satisfies both $x_1 \le 4$ and $x_1 \le 2$, we let $x_1 = 2$, x_5 now becomes zero and leaves the basis. From the first row of Table-11, we get $x_2 = 6$. From the second row, we get $x_3 = 4 - x_1 = 4 - 2 = 2$. So, the new basic feasible solution is

$$x_1 = 2,\ x_2 = 6,\ x_3 = 2,\ x_4 = 0,\ x_5 = 0,\ Z = 36.$$

Let us check whether our solution is optimal. Once again, we write the current basic variables in terms of current non-basic variables. For this let us rewrite equation (1–a) to (3–a) in a convenient form.

$$\left.\begin{aligned} x_1 + x_2 &= 4 \\ x_2 + \frac{x_4}{2} &= 6 \\ 3x_1 + x_4 + x_5 &= 6 \end{aligned}\right\} \qquad (*)$$

We divide the last equation in (*) by 3 to get

$$x_1 + \frac{x_4}{3} + \frac{x_5}{3} = 2 \text{ or } x_1 = 2 - \frac{x_4}{3} - \frac{x_5}{3} \qquad (1\text{–b})$$

Multiplying this by -1 and adding it to the first equation in (*) we get,

$$(x_1 + x_3) - \left(x_1 + \frac{x_4}{3} + \frac{x_5}{3}\right) = 4 - 2 \text{ or } x_3 = 2 + \frac{x_4}{3} + \frac{x_5}{3} \qquad (2\text{–b})$$

The second equation in (*) can be rewritten as

$$x_2 = 6 - \frac{x_4}{2} \tag{3–b}$$

Now, we use (1 – b) to eliminate x_1 from the objective function.

$$Z = 3x_1 - \frac{5x_4}{2} + 30$$

$$= 3(2 - \frac{x_4}{3} - \frac{x_5}{3}) - \frac{5x_4}{2} + 30$$

$$= 36 - \frac{7x_4}{2} - x_5$$

We see that all the co-efficients of all thenon-basic variables (the relative cost co-efficients relative to the current basic variables) are all negative. So, the solution cannot be improved further and we have obtained the optimal solution.

Let us summarise the various steps that we have carried out.

In the first iteration, we chose the slack variables x,, x_4 and x,, as the basic variables and the variables x,, x_2 and x_3 were taken as non-basic variables. The objective function is already in terms of the non-basic variables, We chose x,, whose relative cost co-efficient is the largest, as the entering variable. Then, we found the limit on the value that can be assigned to x,. The variable x_4, which gives the lowest limit on x_2 was chosen to be the departing variables.

Optimability Check : To check the optimality of our new basic feasible solution, we calculated the relative cost co-efficients with respect to the new basis. We found that the cost coefficient of x, is positive. So, the solution is not optimal and it can be improved.

Second Iteration : We entered x, into the basis and removed x_5 by using the same process that we used in the first iteration. We wrote the basic variables in terms of non-basic variables to eliminate the basic variables from the objective function.

Optirnability Check : We calculated the relative cost coefficients and found that all of them are negative. So, we have arrived at the optimal solution.

Now, we discuss the general case and give the algorithm'. For this we concentrate on a general linear programming problem in its standard form that we introduced in Unit 5:

Maximize

$$Z = CX.$$

subject to

$$\begin{cases} AX = B \\ X \geq 0 \end{cases} \qquad (5)$$

Where A is an m x n matrix, C is 1 x n, X is n x I vector and B is a m x 1 vector. For convenience, let us discuss a maximization problem only.

Let $X = (X_S, 0)$ be a basic feasible solution of (5) with a basis matrix $S = (b_1, b_2 \ldots b_m)$. Then (Refer to Unit 5)

$$SX_s = B \quad \text{i.e.}$$

$$b_1 x_{S_1} + b_2 x_{S_2} + \ldots + b_m x_{S_m} = B \qquad (6)$$

$$X_s = S^{-1} B \qquad (7)$$

The value of the objective function is, therefore given by

$$C_s X_s = Z_s$$

$$\text{or} \quad C_{S_1} x_{S_1} + C_{S_2} x_{S_2} + C_{S_3} x_{S_3} + \ldots + C_{S_m} x_{S_m} = Z_S \qquad (8)$$

where $C_S = (C_{S_1}, C_{S_2}, \ldots, C_{S_m})$ i.e. the m-component row vector C_s (corresponding to the basis matrix S) consists of prices of basic variables.

We wish to examine the possibility of having another basic feasible solution with inîproved value of the objective function.

Let A_j be any column of the matrix A not included in the basis matrix B. Then A_j can be written as a linear combination of m linearly independent column vectors $b_1, b_2, \ldots, b_m$ of S (in fact the columns of matrix A which constitute a basis S) i.e. there exist scalars $y_{1j}, y_{2j}, \ldots, y_{mj}$ not all zero such that

$$A_j = y_{1j} b_1 + y_{2j} b_2 + \ldots + y_{mj} b_m \qquad (9)$$

$$= (b_1, b_2 \ldots b_m) \begin{pmatrix} y_{1j} \\ y_{2j} \\ \vdots \\ y_{mj} \end{pmatrix}$$

$$= SY_j \qquad \text{where } Y_j = \begin{pmatrix} y_{1j} \\ y_{2j} \\ \vdots \\ y_{mj} \end{pmatrix}, \quad S = [b_1, b_2, \ldots, b_m]$$

or

$$Y_j = S^{-1} A_j,$$

as S is non-singular. Let in (9), $y_{rj} \neq 0$ for some r. Dividing (9) throughout by y_{rj}, we get

$$\frac{1}{y_{rj}} A_j = \frac{y_{1j}}{y_{rj}} b_1 + \frac{y_{2j}}{y_{rj}} b_2 + \ldots + \frac{y_{r-1\,j}}{y_{rj}} b_{r-1} + b_r + \frac{y_{r+1\,j}}{y_{rj}} b_{r+1} + \ldots + \frac{y_{mj}}{y_{rj}} b_m$$

or

$$b_r = -\frac{y_{1j}}{y_{rj}} b_1 - \frac{y_{2j}}{y_{rj}} b_2 \ldots - \frac{y_{r-1\,j}}{y_{rj}} b_{r-1} + \frac{1}{y_{rj}} A_j - \frac{y_{r+1\,j}}{y_{rj}} b_{r+1} \ldots - \frac{y_{mj}}{y_{rj}} b_m. \quad (10)$$

Let A_j be inserted into the basis replacing the vector b_r (which goes out), we then have a new basis

$$\hat{S} = (b_1, b_2, \ldots, b_{r-1}, A_j, b_{r+1}, \ldots, b_m)$$

Put the value of b_r from (10) in (6), we get

$$b_1 x_{s_1} + b_2 x_{s_2} + \ldots + b_{r-1} x_{s_{r-1}} +$$

$$\left[-\frac{y_{1j}}{y_{rj}} b_1 - \frac{y_{2j}}{y_{rj}} b_2 \ldots - \frac{y_{r-1\,j}}{y_{rj}} b_{r-1} + \frac{1}{y_{rj}} A_j - \frac{y_{(r+1)j}}{y_{rj}} b_{r+1} \ldots - \frac{y_{mj}}{y_{rj}} b_m\right] x_{s_r}$$

$$+ b_{r+1} x_{s_{r+1}} + \ldots + b_m x_{s_m} = B$$

$$\left(x_{s_1} - \frac{y_{1j}}{y_{rj}} x_{s_r}\right) b_1 + \left(x_{s_2} - \frac{y_{2j}}{y_{rj}} x_{s_r}\right) b_2 + \ldots + \left(x_{s_{r-1}} - \frac{y_{r-1j}}{y_{rj}} x_{sr}\right) b_{r-1}$$

$$+ \frac{x_{s_r}}{y_{rj}} A_j + \left(x_{s_{r+1}} - \frac{y_{r+1\,j}}{y_{rj}} x_{s_r}\right) b_{r+1} + \ldots + \left(x_{s_m} - \frac{y_{mj}}{y_{rj}} x_{s_r}\right) b_m = B \qquad (11)$$

which can be written as

$$\hat{x}_{s_1} b_1 + \hat{x}_{s_2} b_2 + \ldots + \hat{x}_{s_{r-1}} b_{r-1} + \hat{x}_{s_r} A_j + \hat{x}_{s_{r+1}} b_{r+1} + \ldots + \hat{x}_{s_m} b_m = B. \qquad (12)$$

$$\text{where } \hat{x}_{s_i} = x_{s_i} - \frac{y_{ij}}{y_{rj}} x_{s_r}, \; i = 1 \ldots m; \; i \neq r$$

$$\hat{x}_{s_r} = \frac{x_{s_r}}{y_{rj}}$$

For $(\hat{x}_s, 0)$ to be a feasible solution to (5), we require

$$\hat{x}_{s_i} \geq 0, \; i = 1 \ldots m$$

i.e. $x_{s_i} - \frac{y_{ij}}{y_{rj}} x_{s_r} \geq 0, \; i = 1, ..., m, \; i \neq r$

$$\frac{x_{s_r}}{y_{rj}} \geq 0 \qquad (13)$$

Now, replace b_r with $y_{rj} \neq 0$ from the basis. This choice is not arbitrary because condition (13) is supposed to be satisfied. Also from (13), you may see that $x_{s_r} \neq 0$. Therefore we must have $y_{rj} > 0$. If $y_{rj} > 0$ and $y_{ij} \leq 0$, $i = 1, ..., m; \; i \neq r$, then (13) is automatically satisfied.

But if $y_{ij} > 0$, we must select from the various vectors b_i's of S with $y_{ij} > 0$, (the vector b_r to be removed from the basis) in such a way that (13) always holds.

If we consider those values of i for which $y_{ij} > 0$, then condition (13) requires

$$\frac{x_{s_i}}{y_{ij}} \geq \frac{x_{s_r}}{y_{rj}} \quad i = i, ..., m; \; i \neq r$$

Therefore, let

$$\frac{x_{s_r}}{y_{r_j}} = \min_{i} \left\{ \frac{x_{s_i}}{y_{ij}} \;\middle|\; y_{ij} > 0 \right\} \qquad (14)$$

If we select the vector b_1 to be thrown out of the basis S according to (14), then conditions in (13) are always satisfied and the new value of the objective function is given by

$$\hat{Z}_S = C_{s_1}\hat{x}_{s_1} + ... + C_{s_i}\hat{x}_{s_r} + ... + C_{s_m}\hat{x}_{s_m}$$

$$= \sum_{\substack{i=1 \\ i \neq r}}^{m} C_{s_i}\hat{x}_{s_i} + C_{s_r}\hat{x}_{s_r}$$

$$= \sum^{m} C_{s_i} \left[x_{s_i} - \frac{y_{ij}}{y_{rj}} x_{s_r} \right] + C_{s_r} \frac{x_{s_r}}{y_{rj}}$$

$$= \sum_{i=1}^{m} C_{s_i} x_{s_i} + \frac{x_{s_r}}{y_{rj}} \left[C_j - \sum_{i=1}^{m} C_{s_i} y_{ij} \right]$$

$$= \sum_{i=1}^{m} C_{s_i} x_{s_i} + \frac{x_{s_r}}{y_{rj}} \left(C_j - \sum_{i=1}^{m} C_{s_i} y_{ij} \right)$$

$$= Z_S + \frac{x_{s_r}}{y_{rj}} \left(C_j - Z_j \right) \text{ (using (9)}$$

or $\quad \hat{Z}_S = Z_s + \theta \; (C_j - Z_j)$

where $Z_j = \sum_{i=1}^{m} C_{s_i} y_{ij}$

$$= C_{s_1} Y_{1j} + C_{s_2} Y_{2j} + \,..\, + C_{s_m} Y_{mj} = C_s Y_j$$

and $\quad \theta = \dfrac{x_{s_r}}{y_{rj}}$

Now if $C_j - Z_j > 0$, then obviously $\hat{Z}_s > Z_s$ $\qquad (\because \theta > 0)$

You have seen, therefore, the new value of the objective function is the original value plus the quantity $\theta(C_j - Z_j)$. Here $Z_s = C_s Y_j$ refers to original basic solution and C_j is the cost associated with newly introduced variable x_j. Therefore if we choose any vector A_j to enter the basis for which $C_j - Z_j > 0$ (or $Z_j - C_j < 0$) and at least one $y_{ij} > 0$, we can obtain a new basic feasible solution with $\hat{Z}_s > Z_s$.

Let us denote $Z_j - C_j$ by A, i.e. 4 $= Z_j - C_j$.

In actual computations, you will notice there are large number of A_j's with A, < 0. Thus the question arises. Which vector A_j is to be entered into the basis? To bring maximum increase in the value of the objective function at any iteration, we should choose that vector A, to enter into the basis corresponding to which – A, is largest (A, has the smallest value) : The vector which is inserted into the basis is called the entering vector, whereas, the vector which is deleted (or is removed) from the basis is called the departing vector.

Thus the method is used iteratively to get a basic feasible solution from another basic feasible solution with an improved value of the objective function. The process is continued until there are no vectors for which A, < 0 i.e. the algorithm terminates when $\Delta_j \geq 0$ for all columns A_j of A. And we say, we have reached an optimal solution because no further improvement in the value of the objective function is possible.

To summarize now the steps of simplex algorithm, let us now consider the following general linear programming problem in its feasible **canonical form.**

Maximize

$$Z = C_1 x_1 + C_2 x_2 + \dots + C_n x_n$$

Subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\leq b_m \\ x_1 \geq 0,\ x_2 \geq 0,\ \dots\ x_n &\geq 0 \end{aligned} \qquad (15)$$

I. Let us assume that $b_i \geq 0$, (i = 1, 2,... m). Next, slack variables are added to convert each less than or equal to constraint into an equality. Each slack variable is assigned a value zero.

The problem can then be reformulated as

Maximize

$$Z = C_1x_1 + C_2x_2 + \dots + C_nx_n + 0x_{n+1} + 0x_{n+2} + \dots + 0x_{n+m}$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + x_{n+1} = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + x_{n+2} = b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + x_{n+m} = b_m$$
$$x_1, \dots, x_{n+m} \geq 0 \qquad (16)$$

Now, you may note that the slack variable x_{n+i} is added to the i^{th} inequality, i = 1, 2, ... m. The column of the coefficient matrix corresponding to the slack variable x_{n+i} is e_i where e_i is a unit vector of dimension m with 1 at the i^{th} position and zero everywhere else.

II In matrix notation, you can write the problem as

Maximize

$$Z = CX$$

Subject to $AX = B, X \geq 0$

Now examine the matrix **A.** You may notice that it contains an $m \times m$ identity matrix I_m corresponding to columns of slack variables. To begin the simplex method, we always use this identity matrix I_m as the initial basis matrix S. At this point you may refer back to Unit 5. It is clear. therefore that $x_1, x_2, \dots, x_n$ are non-basic variables and $x_{n+1}, \dots x_{n+m}$ are m basic variables. An initial basic feasible solution to problem (16) can be obtained by putting non-basic variables equal to zero. Thus we have

$$x_1 = 0, x_2 = 0, \dots, x_n = 0, x_{n+1} = b_1, x_{n+2} = b_2, \dots, x_{n+m} = b_m$$

as the intial basic feasible solution and Z = 0

In other words, since $S = I_m \Rightarrow S^{-1} = I_m$, therefore the initial basic feasible solution is given by $X_s = B$ and $Z_s = 0$ is the corresponding value of the objective function.

III. Set up the initial Simplex tableau as follows :

	$c_j \rightarrow$	c_1	c_2	......	c_n	0	...	0	
C_s	Variables	A_1	A_2	......	A_n	A_{n+1}	...	A_{n+m}	Solution
0	x_{n+1}	a_{11}	a_{12}			1		0	b_1
0	x_{n+2}	a_{21}	a_{22}		a_2	0			b_2
0	x_{n+m}	a_{m1}			a_{mn}	0		1	b_m
	Δ_j	Δ_1	Δ_2		Δ_n	Δ_{n+1}		A_{n+m}	$Z_s = 0$

This is a useful tableau form displaying all the quantities of interest. Such a format, as you know, is called **Tableau Format.** As pointed out earlier, a new tableau is constructed at each iteration i.e. each time a new vector is introduced into the basis.

The first row of the Simplex tableau indicates the values of C_j where C_j is the coefficient of x_j in the objective function Z. It is pointed out here that the values of C_j would remain same in the succeeding tables. The second row of the table provides the column headings for the table and these column headings also remain unchanged in succeeding tables of Simplex Algorithm.

The first column of the tableau gives C_s, the coefficients of basic variables in the objective function (i.e. prices corresponding to variables in the basis) the second column tells which variables one in the basis.

The last column of the tableau, under the heading "Solution" gives the current values of basic variables, together with the value of the objective function for the basic feasible solution described by she given tableau. The remaining columns list all the vectors A_j of the coefficient matrix **A.** Actually these columns list the values of Y_j, but you notice that Y_j happens to be the same as A_j since $Y_j = S^{-1} A_j$ and $S^{-1} = I_m$.

The last entry in each of these columns gives $\Delta_j = Z_j - C_j$,

where $Z_j = C_s Y_j = C_s S^{-1} A_j = C_s A_j$, $j = 1.2, \ldots, n, n+1, \ldots, n+m$.

Thus, it is clear that Z_j is obtained by multiplying entries in the first column C of the tableau by corresponding entries in the column designated as A, and summing up all products thus obtained.

IV. Optimality Test

If all $\Delta_j \geq 0$, the given basic feasible solution is optimal.

When one or more $\Delta_j < 0$, the current solution can be improved further by removing one basic variable (vector) from the basis and replacing it by some non-basic variable (vector).

V. Criteria for a vector to be inserted into the basis

Compute

$$\Delta_k = \min_j \Delta_j \, , \, \Delta_j < 0$$

The vector A; enters into the basis (Corresponding variable x_k, becomes basic). If there is a tie for the minimum value of **A,,** any one of the tied vectors can be chosen to enter the basis.

Once you have selected the vector **A,** to enter into the basis, you may find yourself under anyone of the following two possibilities.

(i) Corresponding to the vector **A,** that qualifies for entry into the basis (i.e. Δ_k is most negative), $y_{ik} \leq 0 \;\forall\; i = 1, 2, ..., m$. This means there is an unbounded solution involving vectors already in the basis and the vector A_k.

(ii) On the other hand, if in the column corresponding to A,, $y_{ik} > 0$ for at least one i, a new basic feasible solution can be found having new value of the objective function greater than or equal to the present value.

The column corresponding to the vector **A,** that enters into the basis is called the distinguished column or the pivot column.

VI. Criteria for a vector to be removed from the basis

Choose **a** vector r to be removed from the basis, using

$$\frac{x_{s_r}}{y_{rk}} = \min_i \left\{ \frac{x_{s_i}}{y_{ik}} \,\middle|\, y_{ik} \geq 0 \right\}$$

If there is a tie in selecting the miximum ratio, any one of the tied columns can be removed and replaced by A_k. The row r corresponding to the minimum of the ratios above is called the distinguished row or the pivot row, variable x_{n+r} becomes non-basic now.

Element y_{rk} (actually a,) lying at the intersection of distinguished row and distinguished column is called the **distinguished element** or the **pivot element.**

VII. Computing the new tableau : It is now necessary to compute a new tableau, vector A, has entered into the basis, therefore in the new tableau, we would like the vector **A,** to be a unit vector i.e. we would like to have unity in the distinguished or pivot position and zero everywhere else in the column designated as A_k. This can be achieved as follows.

(i) Divide the pivot row 'r' by the pivot element $y_{rk} > 0$.

(ii) Subtract suitable multiples of new r^{th} row from other rows to get zero elsewhere in the pivot column, i.e. you may compute

$$\hat{y}_{ij} = y_{ij} - \frac{y_{ik}}{y_{rk}} y_{rj} \quad i = 1, ..., m;\ i \neq r,\ j = 1, 2, ..., n+m$$

$$\hat{y}_{rj} = \frac{y_{rj}}{y_{rk}}$$

and we have

$$\hat{x}_{S_i} = x_{S_i} - \frac{y_{ij}}{y_{rj}} x_{S_r} \quad i = 1, ..., m;\ i \neq r,\ j = 1, 2, ..., n+m$$

$$\hat{x}_{rj} = \frac{x_{S_r}}{y_{rj}}$$

for the new basic feasible solution. Return to srrp 4, you will reach the Optimal Basic Feasible solution (if it exists) in a finite number of steps.

To understand the Simplex algorithm, let us consider the product mix problem which was discussed by graphical method in Unit 3. Its feasible canonical form is as follows:

EXAMPLE 3 :

Maximize

$$Z = 4x_1 + 5x_2$$

Subject to

$$\begin{cases} 2x_1 + 3x_2 \leq 12 \\ 3x_1 + x_2 \leq 8 \\ x_1, x_2 \geq 0 \end{cases} \qquad (17)$$

Step I : Convert all constraints into equations

Adding slack variables x_3 and x_4 to the two constraints, we obtain

$$\left.\begin{array}{l} 2x_1 + 3x_2 + x_3 = 12 \\ 3x_1 + x_2 + x_4 = 8 \\ x_1, x_2, x_3, x_4 \geq 0 \end{array}\right\} \qquad (18)$$

Step II : Obtain initial basic feasible solution

You can see the constraint system (2) is a system of two equations in four unknowns x_1, x_2, x_3 & x_4 and is of the form $AX = B$, $X \geq 0$.

You can see the constraint system (18) is a system of two equations in four unknowns x_1, x_2, x_3 & x_4 and is of theform $AX = B$, $X \geq 0$.

Coefficient matrix A, vectors X and B are given by

$$A = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } B = \begin{pmatrix} 12 \\ 8 \end{pmatrix}$$

You notice, there are four columns in **A.** Let us denote these columns by

$$A_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } A_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Since rank of **A** = 2, therefore two of these four columns A,, A„ A„ **A,** are linearly independent. Hence as in unit 5, you can see that a basic solution to this system of equations will have at most two components different from zero. Let us therefore keep the vectors A, and A, in the basis i.e. let

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

be the initial basis matrix. The corresponding basic variables are x_3 and x_4 and other two variables x_1 and x_2 are non-basic variables. An initial basic feasible solution to the system of constraints (18) can be obtained by putting non-basic feasible variables x, = 0, x, = 0 i.e. the intial basic feasible solution is given by

$$x_1 = 0, \; x_2 = 0, \; x_3 = 12, \; x_4 = 8 \text{ and } Z = 0$$

The last row of the simplex table contains the –ve of relative cost co-cfficients.

Step III : Put the problem in a tableau form

	C_j	4	5	0	0	
C_S	Variables in the basis	A_1	A_2	A_3	A_4	Solution
0	x_3	2	[3]	1	0	12 → D.V
0	x_4	3	1	0	1	8
	Δ_j	– 4	–5 ↑ E.V.	0	0	0

The first column of the tableau gives C_s the coefficients of basic variables x_3 and x_4 in the objective function. The second column tells us which variables are in the basis. Thus, you may see variables x_3 and x_4 corresponding to slack vectors A, and A, are the basic variables.

The last column of the tableau, under the heading 'Solution' gives the current values of the basic variables, together with the value of the objective function for the basic feasible solution described by the given tableau. Recall the equation $Y_j = S^{-1}A_j$ from the theory of the Simplex Method. The remaining columns have the values $y_j = S^{-1} A_j$ for dl vectors A_j in **A.** Note hers that $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $|S| \neq 0$, therefore S^{-1} exists and is given by

$S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence $Y_j = A_j$ for all columns A_j of **A**.

The last entry in each of these columns gives $\Delta_j = Z_j - C_j$,

and $Z_j = C_S Y_j = C_S S^{-1} A_j = C_S A_j$ since $S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Now, let us learn how to compute $\Delta_j = Z_j - C_j$ for the various columns A_j of A.

First let us compute $\Delta_j = Z_1 - C_1$. It is known that $C_1 = 4$. To find Z_1, we have

$$Z_1 = C_S A_1$$

which is obtained by multiplying entries in the first column C_S of the tableau by corresponding entries in the column designated as **A,** and summing up all products thus obtained. In this example we have

$$C_S = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Thus, $Z_1 = 0 \times 2 + 0 \times 3 = 0$ i.e. $\Delta_1 = Z_1 - C_1 = 0 - 4 = -4$

Similarly, you can compute and verify that

$$\begin{array}{|l|} \hline \Delta_2 = Z_2 - C_2 = -5 \\ \Delta_3 = Z_3 - C_3 = 0 \\ \Delta_4 = Z_4 - C_4 = 0 \\ \hline \end{array}$$

Step IV : Apply Optimality Criteria

Let us now see whether or not the solution obtained is optimal. For this we proceed as follows:

Examine the last row of the tableau. If all $\Delta_j \geq 0$, then no further improvement in the value of the objective function would be possible. Accordingly the basic feasible solution obtained in step III will be **the** optimal solution. But this is not the case since two entries **A,** and **A,** are negative. Therefore, take the next step.

Step V : Choosing a vector to enter the basis

Choose the column having the smallest Δ_j i.e. Choose

$$\Delta_k = Z_k - C_k = \min_j (\Delta_j) , \Delta_j < 0. \quad (19)$$

In this example, it is $\Delta_2 = -5$. Therefore the vector A_2 enters into the basis and corresponding variable x_2 becomes an **Entering Variable** (E.V.) and hence it is a new basic variable.
Note here, that is, if there is a tie for the minimum value in (19), then any one of the tied vectors can be chosen to enter the basis.

Also, if all $y_{i2} \le 0$ for i = 3, 4, there would have been an unbounded solution. But you notice in this case $y_{i2} > 0$, i = 3, 4. The entering column A_2 into the basis is called the **distinguished column** or the **pivot column,** as mentioned earlier.

Step VI : Choosing a vector to be removed from the basis

Choose a vector to be removed from the basis using

$$\frac{x_{Sr}}{y_{r2}} = \min_{i}\left\{\frac{x_{Si}}{y_{i2}} \;/\; y_{i2} > 0\right\} \quad , \; i = 3, 4$$

$$= \min\left\{\frac{12}{3}, \frac{8}{1}\right\} = \frac{12}{3} = 4 \quad \text{i.e. } r = 3 \qquad (20)$$

Accordingly remove vector A_3 from the basis. This is called **Departing Variable (D.V.)**

The first row in the tableau is called the **pivot row** or the **distinguished row. Note again that if there is a tie in (20), then any one of the tied vectors can be removed from the basis.** As you know that element lying at the intersection of distinguished row and distinguished column is called the **distinguished element** or the **pivot element.** In this example $\boxed{3}$ is the pivot element in the present tableau.

Step VII : Computing the new tableau

It is now necessary to compute a new tableau since vector A_2 has entered with the basis, In the new tableau, we would like the vector A_2 to be a unit vector i.e. we would like to have unity in the distinguished or pivot position and zero everywhere else in that column. This can be achieved as follows.

(i) Divide the pivot row by the pivot element i.e. in this case divide the first row throughout by $y_{32} = 3$

(ii) Subtract suitable multiple of the new first row from other rows (i.e. the second row) to get zero everywhere else in the pivot column i.e. in this case we can compute

$$\hat{y}_{4j} = y_{4j} - \frac{y_{42}}{y_{12}} Y_{1j} \quad , \; j = 1, 2, 3, 4.$$

Thus, we obtain entries in the new second row and the new table is given by

C_S	C_j Variables in the basis	4 A,	5 A_2	0 A_3	0 A_4	Solution
5	x_2	$\frac{\ }{3}$	1	$\frac{1}{3}$	0	**4**
0	x_4	$\boxed{\frac{7}{3}}$	0	$-\frac{1}{3}$	1	**4** →
	Δ_j	$-\frac{\ }{3}$ ↑	0	$\frac{5}{3}$	0	**20**

We move to step 4, and compute Δ_j and you may notice **A,** $= -\frac{2}{3}$ is negative, therefore the column A, enters with the basis. We put the symbol vertical arrow ↑ below this column in the tableau indicating that A_1 is the entering vector or the pivot vector.

Again, to decide the departing vector, we compute

$$\frac{x_{Sr}}{y_{r1}} = \min_i \left\{ \frac{x_{Si}}{y_{i1}} \,\middle|\, y_{i1} > 0 \right\}$$

$$= \min\left\{\frac{4}{2/3}, \frac{4}{7/3}\right\} = \min\left\{6, \frac{12}{7}\right\} = \frac{12}{7}. \text{ Hencer} = 4 \text{ and } y_{41} = \frac{7}{3}$$

is the pivot element. We indicate the departing the departing variable by the horizontal arrow → as shown in the tableau II. Proceeding in the same manner, we obtain the following new tableau:

C_S	C_j Variables in the basis	4 A_1	5 A_2	0 A_3	0 A_4	Solution
5	x_2	0	1	$\frac{3}{7}$	$-\frac{2}{7}$	$\frac{20}{7}$
4	x_1	1	0	$-\frac{1}{7}$	$\frac{3}{7}$	$\frac{12}{7}$
	Δ_j	0	0	$\frac{11}{7}$	$\frac{2}{7}$	$\frac{148}{7}$

Since all $\Delta_j \geq 0$, no further improvement in the value of the objective function is possible and current solution is optimal. It is given by

$$x_1 = \frac{12}{7},\ x_2 = \frac{20}{7},\ x_3 = 0,\ x_4 = 0 \text{ and } \max Z = \frac{148}{7}$$

same as you obtained using graphical method in Unit 3.

EXERCISE **1:** Use Simplex Algorithm to solve the following programming problems by explaining the relevant steps.

(a) Maximize $Z = 2x_1 + 3x_2$
Subject to $x_1 + x_2 \leq 1$
$3x_1 + x_2 \leq 4$
$x_1, x_2 \geq 0$

(b) Maximize $Z = 2x_1 + 3x_2$
Subject to $5x_1 + x_2 \leq 7$
$3x_1 + 2x_2 \leq 11$
$x_1, x_2 \geq 0$

EXERCISE **2:** Solve the following problems by Simplex Method

(a) Maximize $Z = 6x_1 - 2x_2$
Subject $4x_1 + x_2 \leq 6$
$2x_1 + 4x_2 \leq 11$
$x_1, x_2 \geq 0$

(b) Maximize $Z = 5x_1 + 3x_2$
Subject to $3x_1 + 5x_2 \leq 15$
$5x_1 + 2x_2 \leq 10$
$x_1, x_2 \geq 0$

EXERCISE 3 : Solve the following Linear programming problems by Simplex Method showing only the iterative steps in the table.

(a) Maximize $Z = 4x_1 + 7x_2$
Subject to $2x_1 + x_2 \leq 10$
$2x_1 + 4x_2 \leq 20$
$x_1, x_2 \geq 0$

(b) Maximize $Z = 11x_1 + 13x_2$
Subject to $6x_1 + 4x_2 \leq 3$
$7x_1 + x_2 \leq 5$
$x_1, x_2 \geq 0$

EXERCISE 4 : Solve the following problems :

(a) Maximize $Z = x_1 + 3x_2 - 2x_3$
Subject to $3x_1 + x_2 + 2x_3 \leq 7$
$-2x_1 + 4x_2 \leq 12$
$-4x_1 + 3x_2 + 8x_3 \leq 10$
$x_1, x_2, x_3 \geq 0$

(b) Maximize $Z = 2x_1 + 4x_2 + x_3 + x_4$
Subject to $x_1 + 3x_2 + x_4 \leq 4$
$2x_1 + x_2 \leq 3$
$x_2 + 4x_3 + x_4 \leq 3$
$x_1, x_2, x_3, x_4 \geq 0$

EXERCISE 5 : Use Simplex Method to solve the following

(a) Maximize $Z = 3x_1 + x_2 + 3x_3$
Subject to $2x_1 + x_2 + x_3 \leq 2$
$x_1 + 2x_2 + 3x_3 \leq 5$
$2x_1 + 2x_2 + x_3 \leq 6$
$x_1, x_2, x_3 \geq 0$

(b) Maximize $Z = 3x_1 + 4x_2 + x_3 + 5x_4$
Subject to $8x_1 + 3x_2 + 2x_3 + 2x_4 \leq 10$
$2x_1 + 5x_2 + x_3 + 4x_4 \leq 5$
$x_1 + 2x_2 + 5x_3 + x_4 \leq 6$
$x_1, x_2, x_3, x_4 \geq 0$

EXERCISE 6 : (a) The first tableau format for any linear programming problem is given below in an incomplete form

C_B	C_j Variables in the basis	... A_1	... A_2	... A_3	... A_4	Solution
0	x_3	3	1	1	0	3
0	x_4	2	2	0	1	4
	Δ_j	–4	–1	0	0	

(i) Complete the first row C_j, j = 1, 2, 3, 4

(ii) Write the corresponding linear programming problem

(b) An intermediate Simplex tableau for some linear programming problem is given below :

	C_j	0	–1	3	0	–2	0	
C_B	Variables	A_1	A_2	A_3	A_4	A_5	A_6	Solution
0	x_1	1	$\frac{5}{2}$	0	$\frac{1}{4}$	2	0	10
3	x_3	0	$-\frac{1}{2}$	1	$\frac{1}{4}$	0	0	3
0	x_6	0	$-\frac{5}{2}$	0	$-\frac{3}{4}$	8	1	1
	Δ_j							

(i) Calculate Δ_j ; j = 1, 2, - - , 6

(ii) Does this table correspond to an optima1 basic feasible solution, if not

(iii) Determine entering and departing vectors

(iv) Indicate the pivot element

(v) Write the next tableau

6.3 ARTIFICIAL VARIABLE METHODS

So far, we have discussed how simplex method could be applied to solve any linear programming problem in which the objective function was of maximization form and constraints were all less than or equal to '≤' type. What will happen if you come across problems where the constraints may be '2' type or '=' type, objective function may also be in the form of a minimization function i.e. suppose you come across a linear programming problem of the form

Minimize

$$Z = C_1x_1 + C_2x_2 + \dots + C_nx_n$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \geq b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \geq b_m$$
$$x_1 \geq 0, \dots, x_n \geq 0.$$

In Unit 5, you have learnt how to convert a minimization problem to a maximization problem and also there you studied in detail, how to convert constraints with '≥' type into equations by means of introduction of surplus variables. Thus problem can be rewritten as

Maximize

$$-Z = -C_1x_1 - C_2x_2 + \dots + C_nx_n$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n - x_{n+1} = b_1$$
$$a_{21}x_1 + a_{21}x_2 + \dots + a_{2n}x_n - x_{n+2} = b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n - x_{m+m} = b_m$$
$$x_1 \geq 0, \dots x_n \geq 0 \;,\; x_{n+1} \geq 0 \;, \dots ,\; x_{n+m} \geq 0$$

Where $x_{n+1}, \dots, x_{n+m}$ are surplus variables. You may obtain an initial basic feasible solution to this problem by setting $x_1 = 0, x_2 = 0, \dots, x_n = 0$. Thus, we obtain $x_{n+1} = -b_1, x_{n+2} = -b_2 \dots x_{n+m} = -b_m$.

Now, the following relevant question arises: **This basic solution is not feasible, it violates non-negativity restrictions, how can you apply simplex method to solve such a problem?** For such **a** problem a slight modification is required which we discuss in this section.

As mentioned in Unit 5, that for the problems with (≥) or (=) constraints, the slack variables cannot provide a starting feasible solution. To find a starting feasible solution in such cases, we use the methods of **"artificial variables".** The methods have acquired the natne **"artificial variables"** because in these methods, we take the help of some variables which are fictitious and have no physical meaning. These variables are eliminated from the Simplex Tableau as soon as they become non-basic. Two methods using artificial variables are generally used.

These are known as **"M-technique"** or the **"Method of Penality"** and the **"Two-Phased Method".** We discuss only the **"Two-Phased Method"** in this section, because it is more convenient than the M-Method.

To discuss the **"Two-Phased Method",** we consider the general linear programming problem in its standard form that we introduced to you in Unit 5 namely the following:

Maximize

$$Z = C_1x_1 + C_2x_2 + \dots + C_nx_n$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots \quad a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \qquad (21)$$
$$x_1 \geq 0, \; x_2 \geq 0 \;, \dots ,\; x_n \geq 0.$$

Here, you may presume that, you have already added slack or surplus variables. If no identity matrix appears in the coefficient matrix of this problem, you find yourself in a difficult position. **Then how to find an initial basic feasible solution to (21)?**

You have noticed that it is desirable to have an identity matrix to constitute an initial basis matrix, therefore instead of original set of constraints for problem (21), we consider a new set of constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + x_1^a = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + x_2^a = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + x_m^a = b_m \\ x_1 \ge 0,\ \ x_2 \ge 0, \dots, x_n \ge 0,\ x_1^a \ge 0, \dots,\ x_m^a \ge 0 \end{array}\right\} \quad (22)$$

i.e. we add m-artificial variables, x_1^a, x_2^a, ..., x_m^a (superscript 'a' stands for artificial), one to each of the m-constraints with column vectors corresponding to these variables as b_i; i = 1, 2, ..., m.

Here, you may ask a question as to what have we gained by using this trick? Answer is, we have gained what we were looking for i.e. an intial basis iden'tity matrix now appears in (22) and immediately you may get an initial basic feasible solution to the system (22) as

$$x_1 = 0,\ \ x_2 = 0, \dots, x_n = 0$$

and

$$x_1^a = b_1\ ,\ \ x_2^a = b_2, \dots, x_m^a = b_m.$$

This solution is feasible since $b_i \ge 0$, i = 1, ..., m

Our ultimate aim is to get that basic feasible solution to the system (21) in which all artificial variables x_i^a, i = 1, 2 ,..., m are zero i.e. the solution should be in terms of original variables x_j, j = 1, ... , n. For this purpose again, we make use of Simplex method, this will enable us to insert original column A_j into the identity basis matrix in the usual step-by-step procedure, thus removing artificial vectors from the basis. You will find in this way you end up with a basic feasible solution to (21) in terms of original variables,

While adding artificial variables to the equality constraints in the system of constraints (21), we have to keep in our mind, that we need add artificial variables to only those constraints corresponding to which we do not have unit vector e_i in the original system.

To handle artificial variables Dantzig along with others have suggested a slight modification in Simplex method and called it Two-Phase-method. The general idea of Two-Phase Method is **that, we try to solve a given linear programming problem in two parts :**

Firstly, artificial variables are driven to zero, this is called **phase** I. In **phase**

I, we assign each artificial variable, the price –1 and to each of the original variables we assign the price 0. In this phase, instead of considering the actual objective function, we maximize the function

$$Z^0 = -x_1^a - x_2^a \dots - x_m^a ,$$

where x_i^a is the i^{th} artificial variable.

Let us now illustrate two Phase **Method** of **handling artificial variables** by means of the following examples :

EXAMPLE 4

Minimize

$$Z = 3x_1 + 5x_2$$

Subject *to*

$$6x_1 + 7x_2 \geq 11$$
$$4x_1 + 2x_2 \geq 13$$
$$x_1, x_2 \geq 0$$

SOLUTION.

We convert the objective function to a maximization function and introduce surplus variables x_3 and x_4 to convert constraints into equations and problem takes the form

– Maximize

$$-Z = -3x_1 - 5x_2$$

Subject to

$$6x_1 + 7x_2 - x_3 = 11$$
$$4x_1 + 2x_2 - x_4 = 13$$
$$x_1, x_2, x_3, x_4 \geq 0$$

If you examine the coefficient matrix A of the constraint equations for this problem $\mathbf{A} = \begin{bmatrix} 6 & 7 & -1 & 0 \\ 4 & 2 & 0 & -1 \end{bmatrix}$, this coefficient matrix has no identity matrix present, therefore initial basic feasible solution is not available to you. Therefore, we add artificial variables to the two constraints and we have

$$6x_1 + 7x_2 - x_3 + x_1^a = 11$$
$$4x_1 + 2x_2 - x_4 + x_2^a = 13$$
$$x_1, \dots, x_4 \geq 0, \; x_1^a , \; x_2^a \geq 0$$

In phase I of the two phase method we solve the following linear programming problem using **Simplex method.**

Maximize

$$Z^0 = -x_1^a - x_2^a$$

Subject to

$$6x_1 + 7x_2 - x_3 + x_1^a = 11$$
$$4x_1 + 2x_2 - x_4 + x_2^a = 13$$
$$x_1, x_2, x_3, x_4, x_1^a, x_2^a \geq 0$$

Obvious initial basic feasible solution for this problem is

$$x_1 = x_2 = x_3 = x_4 = 0, \ x_1^a = 11, \ x_2^a = 13 \ \text{and} \ Z^0 = -24.$$

Put the problem in the table form

C_S^0	C_j^0 Variables in the basis	0 A_1	0 A_2	0 A_3	0 A_4	−1 A_1^a	−1 A_2^a	Solution
−1	x_1^a	□	7	−1	0	1	0	11
−1	x_2^a	4	2	0	−1	0	1	13
	Δ_j^0	−10	−9	1	1	0	0	−24

where C_j^0 is the coefficient of x_j in Z^0 and C_S^0 is the column denoting coefficients of basic variable in Z^0. Most negative Δ_j^0 is − 10. Therefore column A_1 enters the basis. To decide the departing vector we choose minimum $\left\{\frac{11}{6}, \frac{13}{4}\right\} = \frac{11}{6}$ as both entries in column A_1 are positive.

Clearly vector A_1^a corresponding to artificial variable x_1^a is removed from the basis and in the usual manner we obtain the following new table, in which [6] is the pivot clement.

C_S^0	C_j^0 Variables in the basis	0 A_1	0 A_2	0 A_3	0 A_4	−1 A_1^a	−1 A_2^a	Solution
0	x_1	1	$+\frac{7}{6}$	$-\frac{1}{6}$	0	$\frac{1}{6}$	0	$\frac{11}{6}$
−1	x_2^a	0	$-\frac{16}{6}$	[$\frac{4}{6}$]	−1	$-\frac{4}{6}$	1	$\frac{34}{6}$ →
	Δ_j^0	0	$\frac{16}{6}$	$-\frac{4}{6}$ ↑	1	$\frac{10}{6}$	0	$-\frac{34}{6}$

Now most negative Δ_j^0 is $-\frac{4}{6}$. Column A_3 enters the basis. You may notice only one entry in the column A_3 is positive. Therefore, corresponding vector A_2^a is to be removed from the basis and artificial variable x_2^a becomes non-basic and $\boxed{\frac{4}{6}}$ is the pivot element consequently, we obtain the **following** new table

	C_j^0	0	0	0	0	–1	–1	
C_S^0	Variables in the basis	A	A_2	A_3	A_4	A_1^a	A_2^a	Solution
0	x_1	1	$\frac{1}{2}$	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{13}{4}$
0	x_3	0	-4	1	$-\frac{3}{2}$	–1	$\frac{3}{2}$	$\frac{17}{2}$
	Δ_j^0	0	0	0	0	1	1	

All $\Delta_j^0 \geq 0$, therefore we have the optimal for phase I. Now we neglect the last two columns i.e. columns corresponding to artificial vectors as we now have a basic solution in terms of original variables. **Phase II** now begins and we have to consider now the objective function

Maximize

$$-Z = 3x_1 - 5x_2$$

and initial table for phase II is the final table for phase I without the columns of artificial vectors. Therefore we now have the following table:

	C_j	–3	–5	0	0	
C_S	Variables	A_1	A_2	A_3	A_4	Solution
–3	x_1	1	$\frac{1}{2}$	0	$-\frac{1}{4}$	$\frac{13}{4}$
0	x_3	0	-4	1	$-\frac{3}{2}$	$\frac{17}{2}$
	Δ_j	0	$\frac{7}{2}$	0	$\frac{3}{4}$	$-\frac{39}{4}$

Since $\Delta_j \geq 0$, therefore, we have optimal for phase II also. The optimal solution is given by

$x_1 = \frac{13}{4}, \; x_2 = 0, x_3 = \frac{17}{2}, x_4 = 0$ –Maximum $(-Z) = \frac{-39}{4}$.

Therefore, Minimum $Z = \frac{39}{4}$.

EXAMPLE 5 (Mixed Constraints)

Maximize

$$Z = 3x_1 - x_2$$

Subject to

$$2x_1 + x_2 \geq 2$$
$$x_1 + 3x_2 \leq 3$$
$$x_2 \leq 4$$
$$x_1, x_2 \geq 0.$$

SOLUTION

Firstly we convert all constraints into equations, therefore we introduce surplus variable x_3 to the first constraint and slack variables x_4 and x_5 to the other two constraints and problem reduces to

Maximize

$$Z = 3x_1 - x_2$$

Subject to

$$2x_1 + x_2 - x_3 = 2$$
$$x_1 + 3x_2 + x_4 = 3$$
$$x_2 + x_5 = 4$$
$$x_1 \ldots x_5 \geq 0$$

Coefficient matrix A for this problem is given by $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

No **3** X 3 unit matrix is present that can be taken as initial basis matrix. You may notice, two unit vectors are present in A, therefore we need add artificial variable x_1^a only to the first constraint and we have to solve the following problem in phase I.

Phase I

Maximize

$$Z^0 = -x_1^a$$

Subject to

$$2x_1 + x_2 - x_3 + x_1^a = 2$$
$$x_1 + 3x_2 + x_4 = 3$$
$$x_2 + x_5 = 4$$
$$x_1 \ldots x_5, x_1^a \geq 0$$

Obviously, initial solution is

$$x_1 = x_2 = x_3 = 0,\ x_1^a = 2,\ x_4 = 3,\ x_5 = 4, \text{ and } Z^0 = -2$$

We now put the problem in the table form

	C_j^0	0	0	0	0	0	−1	
C_S^0	Variables in the basis	A_1	A_2	A_3	A_4	A_5	A_1^a	Solution
−1	x_1^a	[2]	1	−1	0	0	1	2 →
0	x_4	1	3	0	1	0	0	3
0	x_5	0	1	0	0	1	0	4
	Δ_j^0	−2 ↑	−1	1	0	0	0	

Most negative Δ_j^0 is −2, therefore column A_1 enters the basis. Now only two entries in column A_1 are +ve. Therefore to decide which vector is to be removed from the basis, we consider $\min\left\{\frac{2}{2}, \frac{3}{1}\right\} = \frac{2}{2}$. This implies that A_1^a is to be removed from the basis and [2] is the pivot element. We have the following next table

	C_j^0	0	0	0	0	0	−1	
C	Variables in the basis	A_1	A_2	A_3	A_4	A_5	A_1^a	Solution
0	x_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	1
0	x_4	0	$\frac{5}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	2
0	x_5	0	1	0	0	1	0	4
	Δ_j^0	0	0	0	0	0	1	0

All $\Delta_j^0 \geq 0$. Therefore we have optimal for phase I i.e. phase I terminates, and consequently we get initial basic feasible solution for the original problem. Note that we neglect the column A_1^a corresponding to artificial variable x_1^a and proceed to phase II.

Phase II

To, Maximize $Z = 3x_1 - x_2$
the initial table is

C_S	C_j → Variables in the basis	3 A_1	− A_2	1 A_3	0 A_4	0 A_5	0 Solution	
3	x_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	1	
0	x_4	0	$\frac{5}{2}$	$\boxed{\frac{1}{2}}$	1	0	2	→
0	x_5	0	1	0	0	1	4	
	Δ_j	0	$\frac{5}{2}$	$-\frac{3}{2}$ ↑	0	0		

Most negative Δ_j is $-\frac{3}{2}$. Therefore column A, enters the basis. Now the only positive entry in this Column is $\frac{1}{2}$. Therefore, corresponding variable x_4 is to be removed from the basis and pivot element is $\boxed{\frac{1}{2}}$. Thus as usual the next table is

C_S	C_j → Variables in the basis	3 A_1	−1 A_2	0 A_3	0 A_4	0 A_5	Solution
3	x_1	1	3	0	1	0	3
0	x_3	0	5	1	2	0	4
0	x_5	0	1	0	0	1	4
	Δ_j	0	10	0	3	0	9

All $\Delta_j \geq 0$, therefore current solution is optimal and is given by

$$x_1 = 3, x_2 = 0, x_3 = 4, x_1 = 0, x_5 = 4 \text{ and Maximum } Z = 9$$

Note that at the end of phase I, when the optirnality condition is satisfied or $Z^0 = 0$, three possibilities are there

- Maximum $Z^\circ < 0$; one or more artificial vector appears in the basis at a positive level, the original problem has no feasible solution.

- Maximum $Z^\circ = 0$; no artificial vector appears in the basis. We have found a basic feasible solution to the original problem.

- Maximum $Z^\circ > 0$; one or more artificial vector appears in the basis. In this case, we retain the columns corresponding to the artificial variables which are in the basis and remove the columns corresponding to non-basic artificial variables. We proceed with our iterations as usual, **making** sure that none of the artificial variables become positive. We do not go into the details **of** how this is done. If you are interested, you can refer to Mathematical Programming Techniques by N.S. **Kambo, (1984),** Affiliated East-West Press.

Let us illustrate this by the following example :

EXAMPLE 6

Maximize

$$Z = 4x_1 + 3x_2$$

Subject to

$$x_1 + x_2 \le 5$$
$$x_1 + 2x_2 \ge 8$$
$$3x_1 + 2x_2 \ge 14$$
$$x_1, x_2 \ge 0.$$

SOLUTION : Let us first convert inequalities into equations i.e. we introduce slack variable x_3 in the first constraint and surplus variables x_4 and x_5 into the other two constraints. The constraint set reduces to

$$x_1 + x_2 + x_3 = 5$$
$$x_1 + 2x_2 - x_4 = 8$$
$$3x_1 + 2x_2 - x_5 = 14$$
$$x_1 \dots x_5 \ge 0.$$

If we put $x_1 = 0$, $x_2 = 0$, we get $x_3 = 5$, $x_4 = -8$, $x_5 = -14$, therefore initial basic feasible solution is not known.

We add artificial variables to the last two constraints to get the following form.

Phase I

Maximize

$$Z^0 = -x_1^a - x_2^a$$

Subject to

$$x_1 + x_2 + x_3 = 5$$
$$x_1 + 2x_2 - x_4 + x_1^a = 8$$
$$3x_1 + 2x_2 - x_5 + x_2^a = 14$$
$$x_1 \dots x_5, x_1^a, x_2^a \ge 0.$$

Clearly now the initial solution is

$$x_1 = x_2 = x_4 = x_5 = 0\,;\, x_3 = 5,\, x_1^a = 8,\, x_2^a = 14,\; Z^0 = -22$$

We put the problem in the table'form.

	C_j^0	0	0	0	0	0	−1	−1	
C_S^0	Variables in the basis	A_1	A_2	A_3	A_4	A_5	A_1^a	A_2^a	Solution
0	x_3	1	1	1	0	0	0	0	5
−1	x_1^a	1	[2]	0	−1	0	1	0	8 →
−1	x_2^a	3	2	0	0	−1	0	1	14
	Δ_j^0	−4	−4 ↑	0	1	1	0	0	−22

Most −ve $\Delta_j^0 = -4$ and you may find that there is a tie. Therefore, select any one of the two tied vectors **A,** and A, to enter the basis. Let us choose **A,** to enter the basis. There for the vector which is to be removed from the basis we choose

$$\text{minimum}\left\{\frac{5}{1}, \frac{8}{2}, \frac{14}{2}\right\} = \frac{8}{2},$$

Hence artificial vector A_1^a is to be removed from the basis and □ is the., pivot element. Next table is given by

	C_j^0	0	0	0	0	0	−1	−1	
C_S^0	Variables in the basis	A_1	A_2	A_3	A_4	A_5	A_1^a	A_2^a	Solution
0	x_3	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	1
0	x_2	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	4
−1	x_2^a	2	0	0	1	−1	−1	1	6
	Δ_j^0	−2 ↑	0	0	−1	1	2	0	−6

Most −ve $\Delta_j^0 = -2$, therefore column A_1 enters the basis and we choose

Minimum $\left\{ \frac{\ldots}{1/2}, \frac{\ldots}{1/2}, \frac{\ldots}{2} \right\} = \min\{2, 8, 3\} = 2$

Thus column **A,** is to be **removed** from the basis and $\boxed{\frac{1}{2}}$ is the **pivot** element. **Next** table is given by

C_j^0		0	0	0	0	0	–1	–1	
C_S^0	Variables in the basis	A_1	A_2	A_3	A_4	A_5	A_1^a	A_2^a	Solution
0	x_1	1	0	2	1	0	–1	0	2
0	x_2	0	1	–1	–1	0	1	0	3
–1	x_2^a	0	0	4	–1	–1	1	1	2
	Δ_j^0	0	0	4	1	1	0	0	–2

All $\Delta_j^0 \geq 0$, optimality criteria is satisfied, but $Z^0 < 0$ and an artificial variable x_2^a remains in the basis at a positive level. Hence original problem has no feasible solution.

EXAMPLE 7 : (A case of unbounded solution)

Maximize

$$Z = 3x_1 + 2x_2 + x_3$$

Subject to

$$-3x_1 + 2x_2 + 2x_3 = 8$$
$$-3x_1 + 4x_2 + x_3 = 7$$
$$x_1, x_2, x_3 \geq 0$$

SOLUTION

Adding artificial variables x_1^a and x_2^a to the two equality constraints **and** in phase I, we proceed to solve the following linear programming problem.

Maximize

$$Z^0 = -x_1^a - x_2^a$$

Subject to

$$-3x_1 + 2x_2 + 2x_3 + x_1^a = 8$$
$$-3x_1 + 4x_2 + x_3 + x_2^a = 7$$
$$x_1, x_2, x_3, x_1^a, x_2^a \geq 0.$$

Initial solution is $x_1 = x_2 = x_3 = 0$, $x_1^a = 8$, $x_2^a = 7$, $Z^0 = -15$. We therefore put the problem in the table form.

	C_j^0	0	0	0	–1	–1	
C_S^0	Variables in the basis	A_1	A_2	A_3	A_1^a	A_2^a	Solution
–1	x_1^a	–3	2	2	1	0	8
–1	x_2^a	–3	$\boxed{4}$	1	0	1	7 →
	Δ_j^0	6	– 6 ↑	–3	0	0	–15

Most –ve, $\Delta_j^0 = -6$, column A_2 enters the basis and we choose

$$\text{minimum}\left\{\frac{8}{2}, \frac{7}{4}\right\} = \frac{7}{4}$$

i.e. column A_2^a corresponding to artificial variable x_2^a is to be removed from the basis and pivot element is $\boxed{4}$. Next table is given by

	C_j^0	0	0	0	–1	–1	
C_S^0	Variables in the basis	A_1	A_2	A_3	A_1^a	A_2^a	Solution
–1	x_1^a	$-\frac{3}{2}$	0	$\boxed{\frac{3}{2}}$	1	$-\frac{1}{2}$	$\frac{9}{2}$ →
0	x_2	$-\frac{3}{4}$	1	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{7}{4}$
	Δ_j^0	$\frac{3}{2}$	0	$-\frac{3}{2}$ ↑	0	$\frac{3}{2}$	$-\frac{9}{2}$

Most negative $\Delta_j^0 = -\frac{3}{2}$. Therefore Column A_3 enters the basis. To choose the departing vector, we calculate minimum $\left\{\frac{9/2}{3/2}, \frac{7/4}{1/4}\right\}$ = minimum $\{3, 7\} = 3$ which impiles that column corresponding to artificial variable x_1^a is to be removed from the basis, and $\boxed{\frac{3}{2}}$ is the pivot element. Therefore the next table is given by

	C_j^0	0	0	0	−1	−1	
C_S^0	Variables in the basis	A_1	A_2	A_3	A_1^a	A_2^a	Solution
0	x_3	−1	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	3
0	x_2	$-\frac{1}{2}$	1	0	$-\frac{1}{6}$	$\frac{1}{3}$	1
	Δ_j^0	0	0	0	1	1	0

It is clear that phase I terminates and we now have initial solution for phase II and in phase II wc consider the original objective function.

Phase II

Maximize

$$Z = 3x_1 + 2x_2 + x_3 ,$$

The initial table for phase II is the final table for phase I (without the columns of artificial vectors). Hence, we have

		3	2	1	
C_S	Variables in the basis	A_1	A_2	A_3	Solution
1	x_3	−1	0	1	3
2	x_2	$-\frac{1}{2}$	1	0	1
	Δ_j	−5	0	0	9

Now most negative Δ_j is −5, but no entry in the first column is strictly positive, we cannot select the departing vector, in this case also problem has an unbounded solution.

EXAMPLE 8 (A case of alternative optimal)

Maximize

$$Z = 2x_1 + 4x_2 + 6x_3$$

Subject to

$$x_1 + x, \le 5$$
$$x_1 \le 1$$
$$x_1 + 2x_2 + 3x_3 \le 10$$
$$x_1, x_2, x_3 \ge 0.$$

SOLUTION

Since all constraints are '≤' type, introducing slack variables x_4, x_5, x_6 the constraints can be put as

$$x_1 + x_2 + x_4 = 5$$
$$x_1 + x_5 = 1$$
$$x_1 + 2x_2 + 3x_3 + x_6 = 10$$
$$x_1 \ldots x_6 \geq 0$$

Obvious initial solution is

$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 5, x_5 = 1, x_6 = 10$ and $Z = 0$. Hence we put the problem into a table form.

	C_j	2	4	6	0	0	0	
C_S	Variables in the basis	A_1	A_2	A_3	A_4	A_5	A_6	Solution
0	x_4	1	1	0	1	0	0	5
0	x_5	1	0	0	0	1	0	1
0	x_6	1	2		0	0	1	10 →
	Δ_j	−2	−4	−6 ↑	0	0	0	0

Most negative A, , ≂ 6, column A, enters the basis, only one entry is +ve in the third column, therefore column corresponding x_6 is to be removed from the basis and [3] is the pivot element. Next table is given by

	C_j	2	4	6	0	0	0	
C_S	Variables in the basis	A,	A_2	A_3	A_4	A_5	A,	Solution
0	x_4	1	1	0	1	0	0	5
0	x_5	1	0	0	0	1	0	1
0	x_3	$\frac{1}{3}$	$\frac{2}{3}$	1	0	0	$\frac{1}{3}$	$\frac{10}{3}$
	Δ_j	0	0	0	0	0	2	20

All A, ≥ 0, the solution is optimal and is given by $x_1 = 0, x_2 = 0, x_3 = \frac{10}{3}$

Maximum Z = 20 .

You can see that, here $\Delta_1 = 0$, x_1 is non-basic variable and all entries in column 1 are positive, let us see what happens if we make x_1 a basic variable i.e. let us enter A_1 into the basis.

To decide the departing vector, we choose

$$\min\left\{\frac{5}{1}, \frac{1}{1}, \frac{10/3}{1/3}\right\} = \min\{5, 1, 10\} = 1,$$

this shows column corresponding to the basic variable x_5 is to be removed from the basis and new table is

	C_j	2	4	6	0	0	0	
C_S	Variables in the basis	A_1	A_2	A_3	A_4	A_5	A_6	Solution
0	x_4	0	1	0	1	–1	0	4
2	x_1	1	0	0	0	1	0	1
6	x_3	0	$\frac{2}{3}$	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	3
	Δ_j	0	0	0	0	0	6	20

Here you may notice that some Δ_j corresponding to non-basic variables are also zero and some entries y_{ij}'s in those columns are positive. This is an indication to the fact that, we have an alternative optimal solution

$$x_1 = 1, x_2 = 0, x_3 = 6 \text{ and } \max Z = 20 \text{ (again)}$$

Similarly for non-basic variable x_2, $\Delta_2 = 0$, and two entries in the column corresponding to x_2 are positive. Therefore we can make x_2 basic by entering the column A_2 into the basis and we choose

$$\min\left\{\frac{4}{1}, \frac{3}{2/3}\right\} = \min\left\{4, \frac{9}{2}\right\} = 4 .$$

This implies that, we can remove the vector A_4 from the basis and new table is given as

	C_j	2	4	6	0	0	0	
C_s	Variables in the basis	A,	A,	A,	A,	A,	A,	. Solution
4	x_2	0	1	0	1	–1	0	4
2	x_1	1	0	0	0	1	0	1
6	x_3	0	0	1	$-\frac{2}{3}$	$\frac{1}{3}$	1	$\frac{1}{3}$
	Δ_j	0	0	0	0	0	6	20

Again $\Delta_j \geq 0$ and optimal solution is now given by

$$x_1 = 1, x_2 = 4, x_3 = \frac{1}{3} \text{ and } Z = 20 \text{ (again)}$$

Also now A = 0 and the two entries in the column A, are both positive. Therefore, we can enter A, and remove x_3 from the basis. New table is 'given by

	C_j	2	4	6	0	0	0	
C_s	Variables in the basis	A,	A_2	A_3	A_4	A_5	A,	Solution
4	x_2	0	1	3	− 1	0	3	5
2	x_1	1	0	− 3	2		− 3	0
0	x_5	0	0	3	− 2	1	3	1
	Δ_j	0	0	0	0	0	6	20

Again all $\Delta_j \geq 0$, and the optimal solution is given by

$$x_1 = 0, x_2 = 5, x_3 = 0 \text{ and maximum } Z = 20.$$

Thus **we** are able to find four alternative optimal basic feasible solutions given by

$$\left(0,0,\frac{10}{3}\right), (1,0,6), \left(1,4,\frac{1}{3}\right), (0,5,0)$$

with optimal value of objective function to be the same $Z = 20$.

EXERCISE 7 : Solve the following LPP by the two-phased:

(a) Maximize

$$Z = 2x_1 + x_2 + 3x_3$$

Subject to

$$x_1 + x_2 + 2x_3 \leq 5$$
$$2x_1 + 3x_2 + 4x_3 = 12$$
$$x_1, x_2, x_3 \geq 0.$$

(b) Minimize

$$Z = x_1 + 2x_2$$

Subject to

$$2x_1 + 5x_2 \geq 6$$
$$x_1 + x_2 \geq 2$$
$$x_1, x_2 \geq 0.$$

6.4 SUMMARY

We conclude this unit by giving summary of what we have covered in this unit.

In this unit,' we have learnt simplex algorithm to solve any given linear programming problem.

In Section 6.2, we have developed Simplex algorithm through an example and then generalized it. We started with a basic feasible solution which was not optimal, then by changing single vector in the basis at a time, we reached the optimal basic feasible solution in a finite number of iterations.

— Then we learnt how to put any given linear programming problem into a tableau form

— Using tableau format, we then learnt how to solve a given linear programming problem. We :

(i) Introduced optimality criteria

(ii) Rule to insert a new vector into the basis

(iii) Rule to decide which vector is to be removed from the basis

(iv) Learnt how to change a simplex tableau to the next tableau

— Introduced artificial vectors to find an initial basic feasible solution, if it exists, to a linear programming problem.

— Developed two phase method to handle artificial variables.

In phase I, each artificial variable was assigned the price –1 whereas original variables were assigned price zero each. Instead of considering the original objective function, we maximized the function

$$Z^0 = -x_1^a - x_r^a - \dots - x_m^a ,$$

where x_i^a was the i^{th} artificial variable. Phase I, terminated when all artificial variables were driven to zero using Simplex Method. Final tableau for phase I was initial tableau for phase II and in phase II only we dealt with the original objective function of the problem to be solved.

— Finally we solved number of examples, involving '≥' type constraints, mixed constraints, infeasibility in original problem, case of unbounded solution and a case of an alternative optima.

6.5 ANSWERS / HINTS / SOLUTIONS

E1 **(a)** $x_1 = 0$, $x_2 = 1$; Max $Z = 3$

(b) $x_1 = 0$, $x_2 = \frac{11}{2}$; Max $Z = \frac{33}{2}$

E2 (a) $x_1 = \frac{3}{2}$, $x_2 = 0$; Max $Z = 9$

(b) $x_1 = \frac{20}{19}$, $x_2 = \frac{45}{19}$; Max $Z = \frac{235}{19}$

E3) (a) $x_1 = \frac{10}{3}$, $x_2 = \frac{10}{3}$; Max $Z = \frac{110}{3}$

(b) $x_1 = 0$, $x_2 = \frac{3}{4}$; Max $Z = \frac{39}{4}$

E4) (a) $x_1 = \frac{8}{7}$, $x_2 = \frac{25}{7}$, $x_3 = 0$; Max $Z = \frac{83}{7}$

(b) $x_1 = 1$, $x_2 = 1$, $x_3 = \frac{1}{2}$, $x_4 = 0$;

Max $Z = 6.5$

E5) **(a)** $x_1 = \frac{1}{5}$, $x_2 = 0$, $x_3 = \frac{8}{5}$; Max $Z = \frac{27}{5}$

E6 (a) (i) $C = (4, 1, 0, 0)$

(ii) Maximize

$$Z = 4x_1 + x_2$$

Subject to

$$3x_1 + x_2 \le 3$$
$$2x_1 + 2x_2 \le 4$$
$$x_1, x_2 \ge 0$$

(b) (i) $\Delta_1 = 0$, $\Delta_2 = -\frac{1}{2}$, $\Delta_3 = 0$, $\Delta_4 = \frac{3}{4}$, $\Delta_5 = 2$, $\Delta_6 = 0$

(ii) Table does not correspond to optimal solution as $\Delta_2 < 0$

(iii) At this particular iteration, entering vector is A_2 and departing vector is A_1

(iv) The pivot element is $\boxed{\frac{5}{2}}$

(v) Next tableau is

	C_j	0	−1	3	0	−2	0	
C_S	Variables in the basis	A_1	A_2	A_3	A_4	A_5	A_6	Solution
−1	x_2	$\frac{2}{5}$	1	0	$\frac{1}{10}$	$\frac{4}{5}$	0	4
3	x_3	$\frac{1}{5}$	0	1	$\frac{3}{10}$	$\frac{2}{5}$	0	5
0	x_6	1	0	0	$-\frac{1}{2}$	10	1	11
	Δ_j	$\frac{1}{5}$	0	0	$\frac{2}{5}$	$\frac{12}{5}$	0	11

E7 (a) $x_1 = 3$, $x_2 = 2$, $x_3 = 0$; Max. $Z = 8$

(b) $x_1 = \frac{4}{3}$, $x_2 = \frac{2}{3}$; Minz. = 8/3.