UNIT 4 REAL FUNCTIONS

STRUCTURE

4.1 INTRODUCTION

Leonard Euler

Real Analysis is often referred to as the Theory of Real **Functions.** The word 'function' was first introduced in 1694 by L.G. Leibniz **[1646-1716],** a famou . **German** mathematician, who is also credited along with Isacc Newton for the invention.of Calculus, **Leibniz** used the term functian to denote a quantity connected with a curve. A Swiss mathematician, L. **Euler [1707-1783]** treated function as an expression made up of a variablt and some constants. Euler's idea of a function was later generalized by an eminent French mathematician J. Fourier **[1768-1830].** Another German mathematician, L. Dirichlet (1805-1859) defined function as **a relationship** between a variable (called an independent variable) and another variable (called the dependent variable). This is the definition which, you know, is now used in Calculus.

The concept of a function has undergone many refinements. With the advent of Set Theory in 1895, this concept was modified **as** a correspondence between any two **non-**empty sets. Given any twa non-empty sets S and T, a function **f from** S into **T,** denoted as $f: S \to T$, defines a rule which assigns to each $x \in S$, a unique element $y \in T$. This is expressed **by** writing **as** $y = f(x)$. This definition, as you will **recall, was** given in Section 1.2. A function $f: S \to T$ is said to be a

(i) Complex-valued function of a complex variable if both S and **T** are sets of complex numbers;

(ii) Complex-valued **function** of a real variable if S is a set of real **numbers** and T is a set of **complex** numbers;

(iii) Real-valued function of a **complex** variable if S is a set of complex numbers and T is a set **of** real numbers;

(iv) **Real-valued** function **of** a real variable if both S and T are some sets of real numbers,

Since we are **dealing with** the course on Real **Analysis,** we shall confine our **discussion to** those functions whose **domains as** well as codomains are **some** subsets of the set of real numbers. We shall call such,functions **as Real** Functions.

In this unit, we shall deal with the algebraic and transcendental functions. **Among** the **transcendental** functions, we shall define the trigonometric functions, the **exponential** and logarithmic functions. Also, we shall talk about some special real functions including the bounded and monotonic functions. We shall frequently **use** these functions to illustrate various concepts in Blocks 3 and 4.

OBJECTIVES

After going through this unit, you should be able to

- identify various types of algebraic functions
- define the trigonometric and the inverse trigonometric functions
- describe the exponential and logarithmic functions
- discuss some special functions including the bounded and monotonic functions.

4.2 ALGEBRAIC FUNCTIONS

In Unit 1, we identified the set of natural numbers and built up various sets of numbers with the help of the algebraic operations of addition, subtraction, multiplication, division etc. In the same way, let us construct new functions from the real functions which we have chosen for our discussion. Before we do so, let us review the algebraic combinations of the functions under the operations of addition, subtraction, multiplication and division on the real-functions.

4.2.1 ALGEBRAIC COMBINATIONS OF FUNCTIONS

Let f and g be any two real functions with the same domain $S \subset R$ and their co-domain as the set R of real numbers. Then we have the following definitions:

DEFINITION 1: SUM AND DIFFERENCE OF TWO FUNCTIONS

(i) The Sum of f and g, denoted as $f + g$, is a function defined from S into R such that

$(f + g)(x) = f(x) + g(x)$, If $x \in S$.

(ii) The Difference of f and g, denoted as $f - g$, is a function defined from S to R such that

$(f - g)(x) = f(x) - g(x), \forall x \in S$.

Note that both f(x) and g (x) are elements of R. Hence each of their sum and difference is again a unique member of R.

DEFINITION 2: PRODUCT OF TWO FUNCTIONS

Let $f: S \to R$ and $g: S \to R$ be any two functions. The product of f and g, denoted as f.g, is defined as a function $f.g: S \to R$ by

$(f . g)(x) = f(x) . g(x), \forall x \in S$.

DEFINITION 3: SCALAR MULTIPLE OF A FUNCTION

Let $f: S \to R$ be a function and
k be same fixed real number. Then the scalar multiple of 'f' is a function
$kf: S \to R$ defined by

$(kf)(x) = k. f(x), \forall x \in S$.

This is also called the scalar multiplication.

DEFINITION 4: QUOTIENT OF TWO FUNCTIONS

Let $f: S \to R$ and $g: S \to R$ be any two functions such that $g(x) \neq 0$ for each x in S. Then a function $\frac{f}{g}: S \to R$ defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \forall x \in S$$

is called the quotient of the two functions.

EXERCISE 1)
Let f, g, h be any three functions, defined on S and taking values in R, as $f(x) = ax^2$, $g(x) = bx$ for every x in S, where a, b, are fixed real numbers. Find $f + g$, $f - g$, $f . g$, f/g and kf, when k is a constant.

4.2.2 NOTION OF AN ALGEBRAIC FUNCTION

You are quite familiar with the equations $ax + b = 0$ and $ax^2 + bx + c = 0$, where $a, b, c \in R$, $a \neq 0$. These equations, as you know are, called linear (or first degree) and quadratic (or second degree) equations, respectively. The expressions $ax + b$ and $ax^2 + bx + c$ are, respectively, called the first and second degree polynomials in x. In the same way an expression of the form $ax^3 + bx^2 + cx + d$ ($a \neq 0$, a, b, c, d $\in$ R) is called a third degree polynomial (cubic polynomial) in x. In general, an expression of the form $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n$ where $a_0 \neq 0$, $a_i \in R$, i = 0, 1, 2, ..., n, is called an nth degree polynomial in x.

A function which is expressed in the form of such a polynomial is called a polynomial function. Thus, we have the following definition:

DEFINITION 5: **POLYNOMIAL** FUNCTION

Let a_i (i = 0, 1, ..., n) be fixed real numbers where n is some fixed non-negative **integer.** Let S be a subset of R. A function f: S → R **defined** by

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n, \forall x \in S, a_0 \neq 0$$

is called a polynomial function of degree n.

Let us consider some particular cases of a polynomial function on R:

Suppose f: S → R is such that

(i) $f(x) = k, \forall x \in S$ (k is a fixed real number). This is a polynomial function. This is generally called a constant function on S.

For example,

$f(x) = 2$, $f(x) = -3$, $f(x) = \pi$, $\forall x \in R$, are all constant functions.

(ii) One special case of a constant function is, obtained by taking

k = 0 i.e. when

$f(x) = 0, \forall x \in S$.

This is called the zero function on S

EXERCISE 2)

Draw **the graph** of a constant function. Draw the graph of the zero function.

Let f: S → R be such that

(iii) $f(x) = a_0 x + a_1, \forall x \in S, a_0 \neq 0$.

This is a polynomial function and is called a linear function on S. For example,
$f(x) = 2x + 3$, $f(x) = -2x+3$,
$f(x) = 2x - 3$, $f(x) = -2x - 3$, $f(x) = 2x$ for every
$x \in S$ are all linear functions

(iv) The function f: S → R defined by

$f(x) = x, \forall x \in S$

is called the identity function on S.

(v) f: S → R given as

$f(x) = a_0 x^2 + a_1 x + a_2, \forall x \in R, a_0 \neq 0$.

is a polynomial function of degree two and is called a quadratic function on S.

For example, $f(x) = 2x^2 + 3x - 4$, $f(x) = x^2 + 3$, $f(x) = x^2 + 2x$,
$f(x) = -3x^2$,
for every $x \in S$ are all quadratic functions.

DEFINITION 6: RATIONAL **FUNCTION**

A function which can be expressed as the quotient of two polynomial functions is called a rational function.

Thus a function f: S → R defined by

$$f(x) = \frac{a_0 x^n + a_1 x^{n-1} + ... + a_n}{b_0 x^m + b_1 x^{m-1} + ... + b_m}, \forall x \in S.$$

is called a rational **function.**

Here $a_0 \neq 0$ $b_0 \neq 0$, a_i, $b_j \in R$ where i, j are some fixed seal numbers and the polynomial function in the denominator is never zero.

EXAMPLE 1: The following are all rational functions on R.

$$\frac{2x+3}{x^2+1}, \quad \frac{4x^2+3x+1}{3x-4} \quad (x \neq \frac{4}{3}) \quad \text{and} \quad \frac{3x+5}{x-4} \quad (x \neq 4).$$

The functions which are not rational are known as irrational functions. A typical example of an irrational function is the square root function which we define as follows:

DEFINITION 7: SQUARE ROOT FUNCTION

Let S be the set of non-negative real numbers. A function f: $S \to R$ defined by

$$f(x) = \sqrt{x}, \forall x \in S$$

is called the square root function.

You may recall that $\sqrt{x}$ is the non-negative real number whose square is x. Also it is defined for all $x \geq 0$.

EXERCISE 3)

Draw the graph of the function $f(x) = \sqrt{x}$ for $x \geq 0$.

Polynomial functions, rational functions and the square root function are some of the examples of what are known as algebraic functions. An algebraic function, in general, is defined as follows

DEFINITION 8: ALGEBRAIC FUNCTION

An algebraic function f: $S \to R$ is a function defined by $y = f(x)$ if it satisfies identically an equation of the form

$$p_0(x)\, y^n + p_1(x)\, y^{n-1} + \ldots + p_{n-1}(x)\, y + p_n(x) = 0$$

where $p_0(x)$, $p_1(x)$, ... $p_{n-1}(x)$, $p_n(x)$ are Polynomials in x for all x in S and n is a positive integer.

EXAMPLE 2: Show that f: $R \to R$ defined by

$$f(x) = \sqrt{\frac{x^2-3x+2}{4x-1}}$$

is an algebraic function.

Solution

Let $y = f(x) = \dfrac{\sqrt{x^2-3x+2}}{\sqrt{4x-1}}$

Then $(4x-1)y^2 - (x^2-3x+2) = 0$

Hence f(x) is an algebraic function.

In fact, any function constructed by a finite number of algebraic operations (addition, subtraction, multiplication, division and root extraction) on the identity function and the constant function, is an algebraic function.

EXAMPLE 3: The functions f : $R \to R$ defined by

(i) $f(x) = \dfrac{(x^2+2)\sqrt{x-1}}{x^2+4}$

or $f(x) = \dfrac{x^3-2x}{\sqrt{x.(3x^2+5)}}$

are algebraic functions.

EXAMPLE 4: Prove that every rational function is an algebraic function.

SOLUTION : Let f: $R \to R$ be given as

$$f(x) = \frac{p(x)}{q(x)}, \quad \forall x \in R,$$

where p(x) and q(x) are some polynomial functions such that $q(x) \neq 0$ for any $x \in R$. Then we have

$$y = f(x) = \frac{p(x)}{q(x)}$$

$$q(x)\, y - p(x) = 0$$

which shows that $y = f(x)$ can be obtained by solving the equation

$$q(x)\, Y - p(x) = 0.$$

Hence f(x) is an algebraic function.

EXERCISE 4)

Verify that a function $f: R \to R$ defined by

$f(x) = \sqrt{x + \sqrt{x}}$

is an algebraic function.

A function which is not algebraic is called a Transcendental Function. Examples of elementary transcendental functions are the trigonometric functions, the exponential functions and the logarithmic functions, which we discuss in the next section.

4.3 TRANSCENDENTAL FUNCTIONS

In Unit 1, we gave a brief introduction to the algebraic and transcendental numbers. Recall that a number is said to be an algebraic if it is a root of an equation of the form

$$a_0 x^n + a_1 x^{n-1} + \dots x + a_{n-1} x + a_n = 0$$

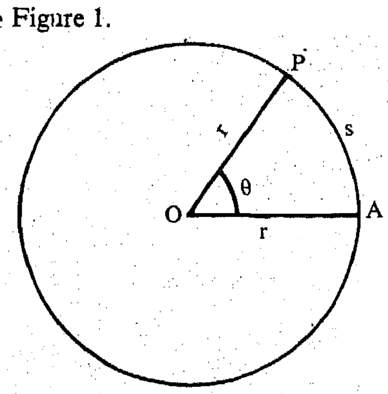
with integral coeffioients and $a_0 \neq 0$, where n is a positive integer. A number which is not algebraic is called a transcendental number. For example the numbers e and π are transcendental numbers. In fact, the set of transcendental numbers is uncountable.

'Based on the same analogy, we have the transcendental functions. In Section 4.2, we have discussed algebraic functions. The functions that are non-algebraic are called transcendental functions. In this section, we discuss some of these functions.

4.3.1 TRIGONOMETRIC FUNCTIONS

You are quite familiar with the trigonometric functions from the study of Geometry and Trigonometry. The study of Trigonometry is concerned with the measurement of the angles and the ratio of the measures of the sides of a triangle. In Calculus, the trigonometric functions have an importance much greater than simply their use in relating sides and angles of a triangle. Let us review the definitions of the trigonometric functions sin x, cos x and some of their properties. These functions form an important class of real functions.

Consider a circle $x^2 + y^2 = r^2$ with radius r and centre at O. Let P be a point on the circumference of this circle. If θ is the radian measure of a central angle at the centre of the circle as shown in the Figure 1.



then you know that the lengths of the arc AP is given by

$$s = \theta r.$$

You already know how the trigonometric ratios sin θ, cos θ, etc., are defined for an angle θ measured in degrees or radians. We now define sin x, cos x, etc., for $x \in R$.

If we put r = 1 in above relation, then we get $\theta = s$. Also the equation of circle becomes $x^2 + y^2 = 1$. This, as you know,, is the Unit Circle. Let C represents this circle with centre O and radius 1. Suppose the circle meets the x-axis at a point A as shown in the Figure 2.

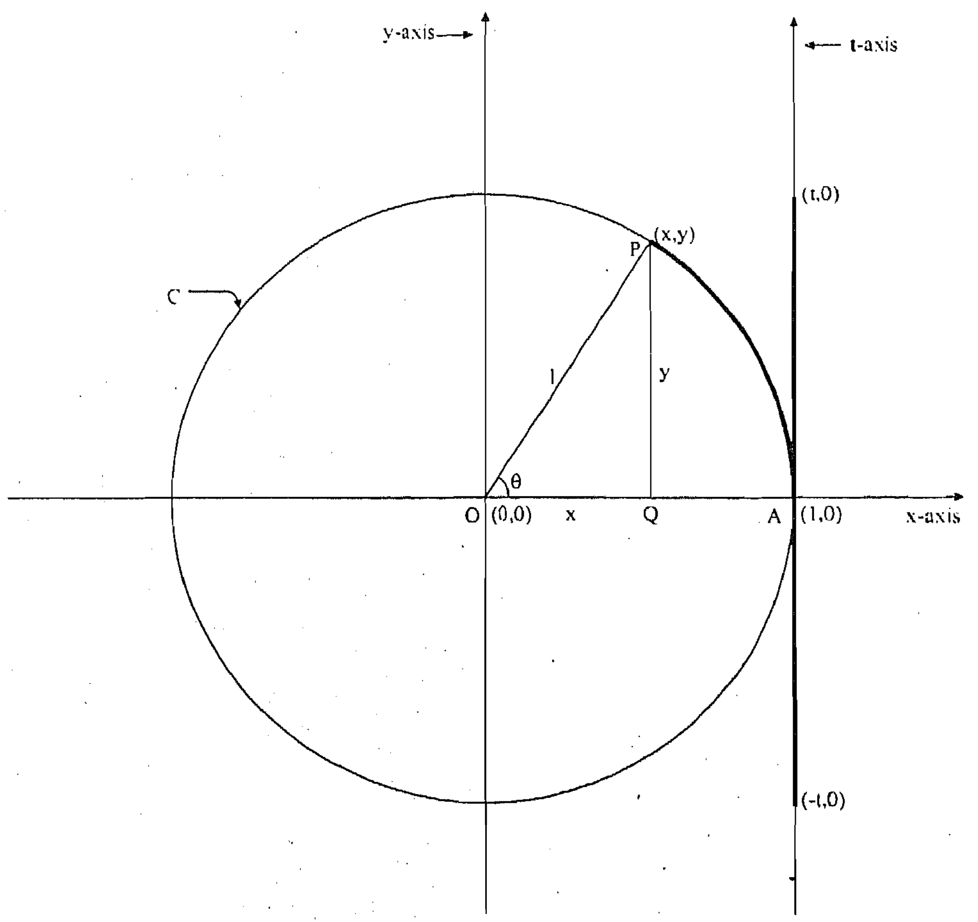


Fig. 2

Through the point A = (1,0); we draw a verticle line labeled as t-axis with origin at A and positive direction upwards. Now, let t be any real number and we will think of this as a point on this verticle number line i.e., t-axis.

Imagine this t-axis as a line of thread that can be wrapped around the circle C, Let p(t) = (x, y) be the point where 't' ends up when this wrapping takes place. In other words, the line segment from A to point (t, 0) becomes the arc from A to P, positive or negative i.e., counterclockwise or clockwise, depending on whether t > 0 or t < 0. Of course, when t = 0, P = A. Then, the trigonometric functions 'sine' and 'cosine', for arbitrary $t \in R$, are defined by

$$\sin t = \sin \theta = y, \text{ and } \cos t = \cos \theta = x,$$

where 'θ' is the radian measure of the angle subtended by the arc AP at the centre of the circle C. More generally, if t is any real number, we may take $(0 \le \theta \le 2\pi)$ to be the angle (rotation) whose radian measure is t. It is then clear that

$$\sin(t + 2\pi) = \sin t \text{ and } \cos(t + 2\pi) = \cos t.$$

You can easily see that as θ increases from '0' to $\pi/2$, PQ increases from 0 to 1 and OQ decreases from 1 to 0. Further, as θ

increases from $\frac{\pi}{2}$ to π, PQ decreases from 1 to 0 and OQ decreases from 0 to -1. Again as θ increases from π to $\frac{3\pi}{2}$ PQ decreases from 0 to -1 and OQ increases from -1 to 0. As θ increases from $\frac{3\pi}{2}$ to 2π. OQ increases from 0 to 1 and PQ increases from -1 to 0. The graphs of these functions take the shapes as shown in Figure **3.**

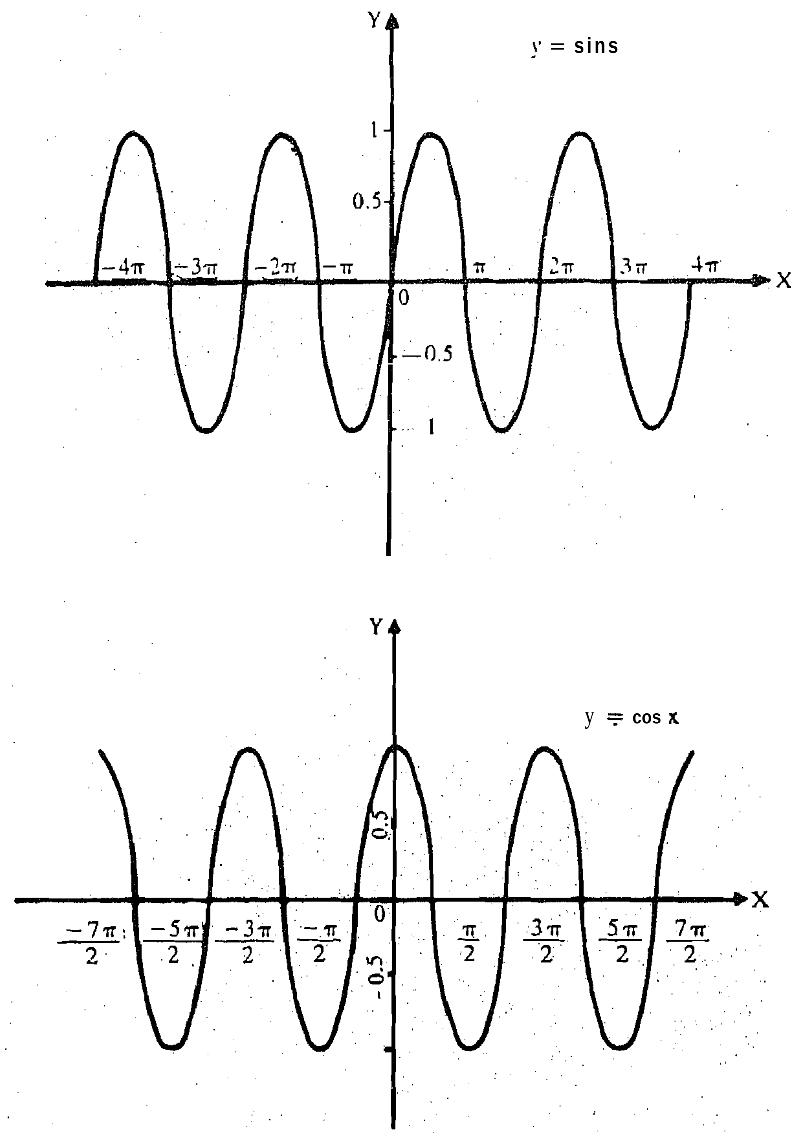


Fig. 3

Thus, we define sin x and cos x as follows:

DEFINITION 9: SINE FUNCTION

A function f: $\mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \sin x, \ \forall \ x \in R$$

is called the sine of x. We often write $y = \sin x$.

DEFINITION 10: COSINE FUNCTION

A function f: $R \to \mathbf{R}$ defined by

$$f(x) = \cos x, \ \forall \ x \in R$$

is called the cosine of x and we write $y = \cos x$.

Note that the range of each of the sine and cosine, is $[-1, 1]$. In terms of the real functions sine and cosine, the other four trigonometric functions can be defined as follows:

(i) A function $f: S \longrightarrow R$ defined by

$f(x) = \tan x = \frac{\sin x}{\cos x}$, $\cos x \neq 0, \forall x \in S = R - \{(2n+1)\frac{\pi}{2}\}$ is called the

Tangent Function. The range of the tangent function is $]-\infty, +\infty[= R$ and the domain is $S = R - \{(2n+1)\frac{\pi}{2}\}$, where n is a non-negative integer.

(ii) A function $f: S \longrightarrow R$ defined by

$f(x) = \cot x = \frac{\cos x}{\sin x}$, $\sin x \neq 0, \forall x \in S = R - \{n\pi\}$,

is said to be the **Cotangent Function.** Its range is also same as its co-domain i.e. range $=]-\infty, \infty[= R$ and the domain is $S = R - \{n\pi\}$, where n is a non-negative integer.

(iii) A function $f: S \longrightarrow R$ defined by

$f(x) = \sec x = \frac{1}{\cos x}$, $\cos x \neq 0, \forall x \in S = S - \{2n+1)\frac{\pi}{2}\}$.

is called the **Secant Function.** Its range is the set

$S =]-\infty, -1] \cup [1, \infty[$ and domain is $S = R - \{2n+1)\frac{\pi}{2}\}$,

(iv) A function $f: S \longrightarrow R$ defined by

$f(x) = \operatorname{cosec} x = \frac{1}{\sin x}$, $\sin x \neq 0, x \in S = R - \{n\pi\}$,

is called the Cosecant function. Its range is also the set $S =]-\infty, -1] \cup [1, \infty[$ and domain is $S = R - \{n\pi\}$.

The graphs of these functions are shown in the Figure 4 on pages 76-77.

EXAMPLE 5: Let $S = [-\frac{\pi}{2}, \frac{\pi}{2}]$. Show fhat the function $f: S \longrightarrow R$ defined by

$$f(x) = \sin x, \forall x \in S$$

is one-one. When is f only onto? Under what conditions f is both one-one and onto?

SOLUTION: Recall from Unit 1 that a function f is one-one if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

for every x_1, x_2 in the domain of f.

Therefore, here we have for any $x_1, x_2 \in S$,

Recall the trigonometric identities which you have learnt in your previous study of Trigonometry.

$$f(x_1) = f(x_2) \Rightarrow \sin x_1 = \sin x_2$$

$$\Rightarrow \sin x_1 - \sin x_2 = 0$$

$$\Rightarrow 2\sin\frac{x_1 - x_2}{2}\cos\frac{x_1 + x_2}{2} = 0$$

$$\Rightarrow \text{Either } \sin\frac{x_1 - x_2}{2} = 0, \text{ or } \cos\frac{x_1 + x_2}{2} = 0.$$

If $\sin\frac{x_1 - x_2}{2} = 0$, then $\frac{x_1 - x_2}{2} = 0, \pm\pi, + 2\pi, \ldots.$

If $\cos\frac{x_1 + x_2}{2} = 0$, then $\frac{x_1 + x_2}{2} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots.$

Since $x_1, x_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, therefore we can only have

$$-\frac{\pi}{2} \le \frac{x_1 - x_2}{2} \le \frac{\pi}{2}$$

and

$$-\frac{\pi}{2} \le \frac{x_1 + x_2}{2} \le \frac{\pi}{2}$$

Thus $\frac{x_1 - x_2}{2} = 0$ i.e. $x_1 = x_2$. Also, if $\frac{x_1 + x_2}{2} = \pm \frac{\pi}{2}$

i.e. then $x_1 + x_2 = \pm \pi$.

Since $x_1, x_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,

therefore, $x_1 = x_2 = \frac{\pi}{2}$ or $x_1 = x_2 = -\frac{\pi}{2}$

Hence $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, which proves that f is one-one. Then function f(x) = sin x defined as such, is not onto because you know that the range of sin x is $[-1, 1] \neq R$.

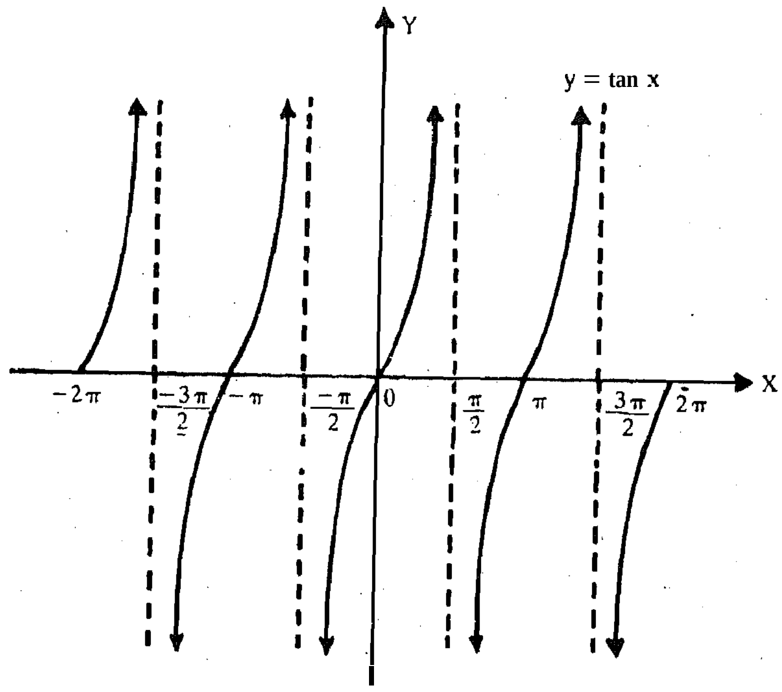


Fig.4 (i)

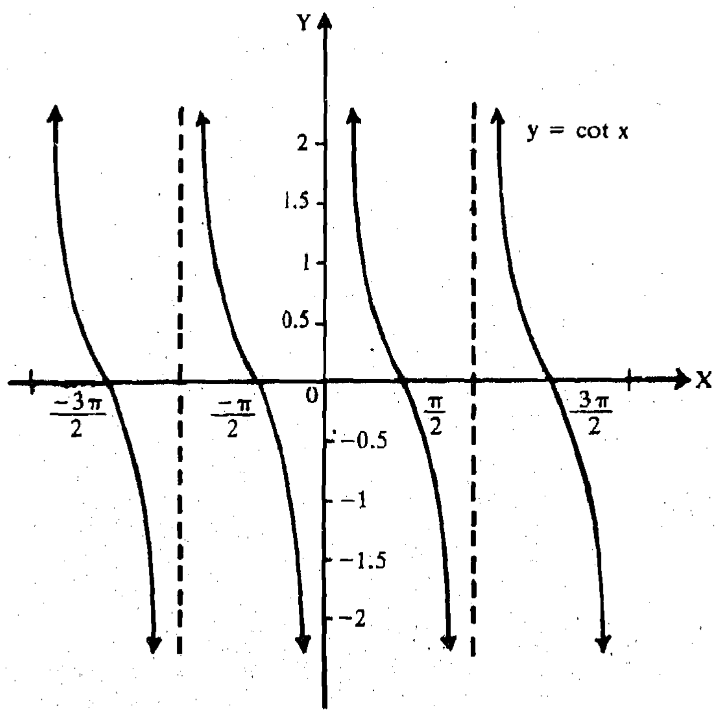


Fig. 4-(ii)

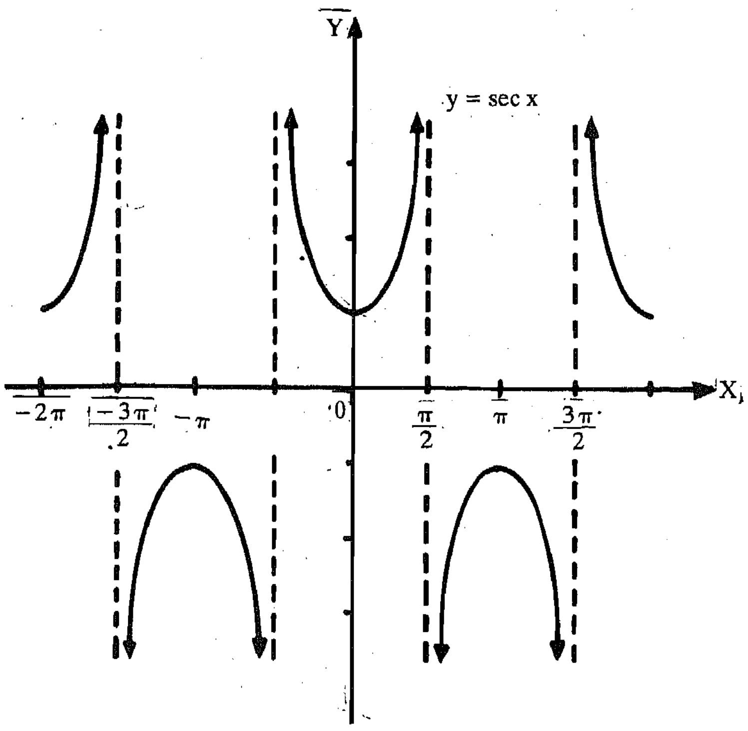


Fig. 4 (iii)

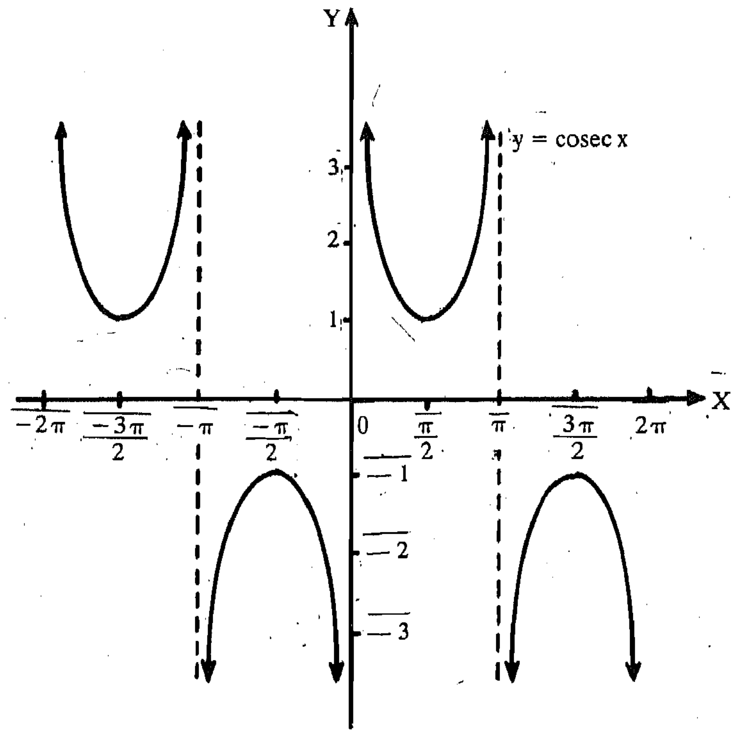


Fig. 4 (iv)

If you define f: R $\longrightarrow$ [−1, 1] as

$f(x) = \sin x, \ \forall x \in R.$

Then f is certainly onto. But then it is not one-one. However the function.

f: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$ defined by

$f(x) = \sin x, \ \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

is both one-one and onto.

EXERCISE 5)

Two functions g and h are defined as follows:

(i) **g:** $S \to R$ **defined** by
g(x) = cos x, $x \in S = [0, \pi]$

(ii) **h:** $S \to R$ defined **by**
$h(x) = \tan x,\ x \in S =]-\frac{\pi}{2}, \frac{\pi}{2}[$

Show that **the functions** are one-one. Under what conditions the function **are** one-one and onto?

4.3.2 INVERSE TRIGONOMETRIC FUNCTIONS

In Section 1.2 we discussed inverse functions. You know that if a function is one-one and onto, then it will have an inverse. If a function is not one-one and onto, then sometimes it is possible lo restrict its domain in some suitable manner such that the restricted function is one-one and onto. Let us use these ideas to define the inverse trigonometric functions. We begin with the inverse of the sine function.

Refer to the graph of f(x) = sin x in Fig. 3. The x-axis cuts the curve y = sin x at the points $x = 0$, $x = \pi$, $x = 2\pi$,....... This shows that function f(x) = sin x is not one—one. However, we have already shown in Example 5 that if we restrict the domain of

f(x) = sinx to the interval $[-\pi/2, \pi/2]$, then the function

f: $[-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ defined by

$$f(x) = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

is one-to-one as well as onto. Hence it will have the inverse. The inverse function is called the inverse sine of x and is denoted as sin x. In other words,

$$y = \sin^{-1} x \Leftrightarrow x = \sin y,$$

where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $-1 \le x \le 1$.

Be careful about the notation used. The superscript – 1 that appears in y = sin-' is not an exponent, but is the symbol f^{-1} used to denote the inverse of a function f. Two avoid this, notation y = arcsin, x instead of y = $\sin^{-1}$ x is used sometimes.

Thus, we have the following definition:

DEFINITION 11: INVERSE SINE FUNCTION

A function $g : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ defined **by**

$$g(x) = \sin^{-1} x,\ \forall\ x \in [-1, 1]$$

is called the inverse sine function.

Again refer back to the graph of f(x) = cos x in Figure 3. You can easily see that cosine function is also not one-one. However, if you restrict the domain of f(x) = cos x to the interval $[0, \pi]$, then the function f: $[0, \pi] \longrightarrow [-1, 1]$ defined by

$$f(x) = \cos x,\ 0 \le x \le \pi,$$

is one-one and onto. Hence it will have the inverse. The inverse function is called the inverse cosine of **x** and is denoted by $\cos^{-1}$ x (or by arc cosx). In other words,

$$y = \cos^{-1} x \Leftrightarrow x = \cos y,$$

where $0 \le y \le \pi$ and $-1 \le x \le 1$.

Thus, we have the following definition:

DEFINITION 12: **A** function g: $[-1, 1] \longrightarrow [0, \pi]$ defined **by**

$$g(x) = \cos^{-1} x,\ \forall\ x \in [-1, 1],$$

is called the inverse cosine function.

You can easily see from Figure 4 that the tangent function, in general, is not one-one. However, again if we restrict the domain of f(x) = tan x to the interval $]-\pi/2, \pi/2[$, then the function

$f:]-\frac{\pi}{2}, \frac{\pi}{2}[\to R$ defined by

$$f(x) = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$$

is one-one and onto. Hence it has an inverse. The inverse function is called the inverse tangent of x and is denoted by $\tan^{-1} x$ (or by arctan x). In other words,

$$y = \tan^{-1} x \Leftrightarrow x = \tan y,$$

where $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $-\infty < x < +\infty$.

Thus, we have the following definition:

DEFINITION 13: Inverse Tangent Function

A function $g: R \to]-\frac{\pi}{2}, \frac{\pi}{2}[$ defined by

$$g(x) = \tan^{-1} x, \forall x \in R$$

is called the inverse tangent function.

Now you can try the following exercise to define the remaining three inverse trigonometric functions:

EXERCISE 6)
Define the inverse cotangent, inverse secant and inverse cosecant function. Specify their domain and range.

Now, before we proceed to define the logarithmic and exponential functions, we need the concept of the monotonic functions. We discuss these functions as follows:

4.3.3 MONOTONIC FUNCTIONS

Consider the following functions:

(i) $f(x) = x, \forall x \in R$.

(ii) $f(x) = \sin x, \forall x \in [-\pi/2, \pi/2]$.

(iii) $f(x) = -x^2, \forall x \in [0, \infty[$.

(iv) $f(x) = \cos x, \forall x \in [0, \pi]$.

Out of these functions, (i) and (ii) are such that for any x_1, x_2 in their domains,

$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2),$$

whereas (iii) and (iv) are such that for any x_1, x_2 in their domains,

$$x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2).$$

The functions given in (i) and (ii) are called monotonically increasing while those of (iii) and (iv) are called monotonically decreasing. We define these functions as follows:

Let $f: S \to R$ $(S \subset R)$ be a function

(i) It is said to be a monotonically increasing function on S if

$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2) \text{ for any } x_1, x_2 \in S$$

(ii) It is said to be a monotonically decreasing function on S if

$$x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2) \text{ for any } x_1, x_2 \in S.$$

(iii) The function f is said to be a monotonic function on S if it is either monotonically increasing or monotonically decreasing.

(iv) The function f is said to be strictly increasing on S if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2), \text{ for } x_1, x_2 \in S.$$

(v) It is said to be strictly decreasing on S if

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2), \text{ for } x_1, x_2 \in S.$$

You can notice immediately that if f is monotonically increasing then $-f$ i.e. $-f: R \to R$ defined by $(-f)(x) = -f(x), \forall x \in R$ is monotonically decreasing and vice-versa.

EXAMPLE 6: Test the monotonic character of the function $f: R \to R$ defined as

$$f(x) = \begin{cases} x^2, x \le 0 \\ -x^2, x > 0 \end{cases}$$

SOLUTION: For any $x_1, x_2 \in R, x_1 \le 0, x_2 \le 0$

$$x_1 < x_2 \Longrightarrow x_1^2 > x_2^2 \Longrightarrow f(x_1) > f(x_2)$$

which shows that f is strictly decreasing.

Again if $x_1 > 0, x_2 > 0$, then

$x_1 < x_2 \Longrightarrow x_1^2 < x_2^2 \Longrightarrow -x_1^2 > -x_2^2 \Longrightarrow f(x_1) > f(x_2)$

which shows that f is strictly decreasing for $x > 0$. Thus f is strictly decreasing for every $x \in R$.

Now, we discuss an interesting property of a strictly increasing function in the form of the following theorem:

THEOREM 1: Prove that a strictly increasing function is always one-one.

PROOF: Let $f: S \to T$ be a strictly increasing function. Since f is strictly increasing, therefore,

$x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$ for any $x_1, x_2 \in S$.

Now to show that $f: S \to T$ is one-one, it is enough to show that

$$f(x_1) = f(x_2) \Longrightarrow x_1 = x_2.$$

Equivalently, it is enough to show that distinct elements in S have distinct images in T i.e. $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$, for $x_1, x_2 \in S$.

Indeed,

$$\begin{aligned} x_1 \neq x_2 &\Longrightarrow x_1 < x_2 \text{ or } x_1 > x_2 \\ &\Rightarrow f(x_1) < f(x_2) \text{ or } f(x_1) > f(x_2) \\ &\Longrightarrow f(x_1) \neq f(x_2) \end{aligned}$$

which proves the theorem.

EXAMPLE 7: Let $f: S \to T$ be a strictly increasing function such that $f(S) = T$. Then prove that f is invertible and $f^{-1}: T \to S$ is also strictly increasing.

SOLUTION: Since $f: S \to T$ is strictly increasing, therefore, f is one-one. Further, since $f(S) = T$, therefore f is onto. Thus f is one-one and onto. Hence f is invertible. In other words, $f^{-1}: T \to S$ exists.

Now, for any $y_1, y_2 \in T$, we have $y_1 = f(x_1), y_2 = f(x_2)$. If $y_1 < y_2$, then we claim $x_1 < x_2$.

Indeed if $x_1 \ge x_2$, then $f(x_1) \ge f(x_2)$ (why?).
This implies that $y_1 \ge y_2$ which contradicts that $y_1 < y_2$.
Hence $y_1 < y_2 \Longrightarrow x_1 < x_2 \Rightarrow f^{-1}(y_1) < f^{-1}(y_2)$
which shows that f^{-1} is strictly increasing.

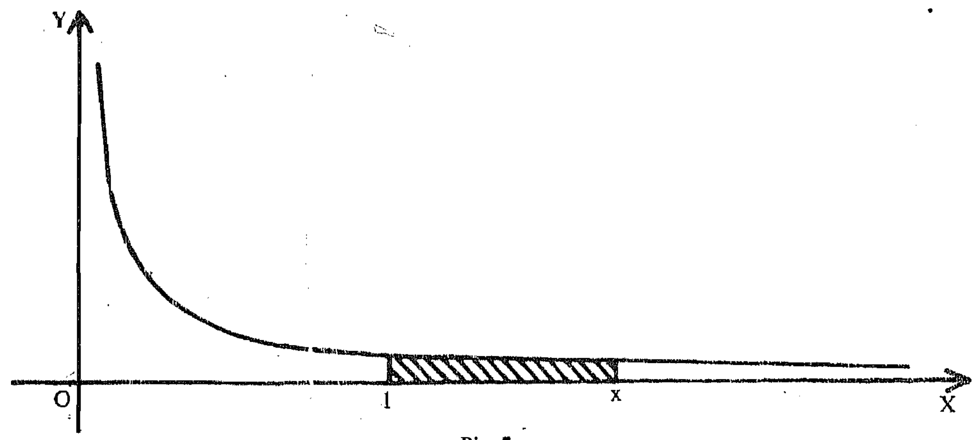
You can similarly solve the following exercise for a strictly decreasing function:

EXERCISE 7)
Let $f: S \to T$ be a strictly decreasing function such that $f(S) = T$. Show that f is invertible and $f^{-1}: T \to S$ is also strictly decreasing.

4.3.4 LOGARITHMIC FUNCTION

You know that a definite integral of a function represents the area enclosed between the curve of the function, X-Axis and the two Ordinates. You will now see that this can be used to define logarithmic function and then the exponential function.

We consider the function $f(x) = \frac{1}{x}$ for $x > 0$. We find that the graph of the function is as shown in the Figure 5,



Pig. 5

DEFINITION 14: **LOGARITHMIC** FUNCTION

For $x \geq 1$, we define **the natural logarithmic** function log x as

$$\log x = \int_1^x \frac{1}{t}\, dt$$

In the Figure 5, log x represents the area between the curve $f(t) = \frac{1}{t}$, X – Axis and the two **ordinates** at 1 and at x. For $0 < x < 1$, **we** define

$$\log x = -\int_x^1 \frac{1}{t}\, dt$$

This means that for $0 < x < 1$, log x is the negative of the area under the graph of $f(t) = \frac{1}{t}$, X - Axis and the two **ordinates** at x and **at** 1

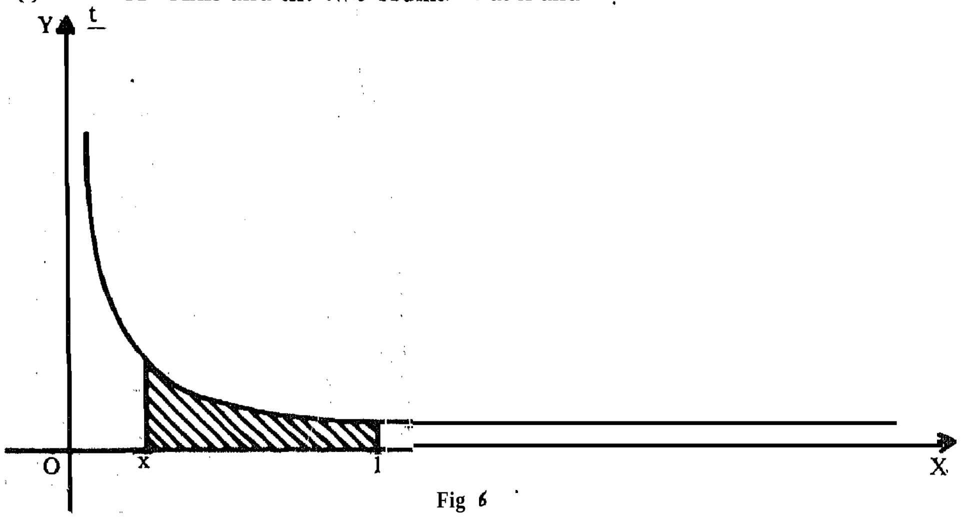


Fig 6

We also see **by** this definition that

$\log x < 0$ if $0 < x < 1$

$\log 1 = 0$

and

$\log x > 0$ if $x > 1$.

It also follows by **definition** that if $x_1 > x_2 > 0$, then $\log x_1 > \log x_2$. This shows that log x is strictly increasing. The reason for this is **quite** clear if we realise **by** $\log x_1$ as the area under the graph as shown in the Figure 7.

The **logarithmic** function defined here is called the Natural logarithmic function. For **any** $x > 0$, and for any positive real number $a \neq 1$, we can define

$$\log_a x = \frac{\log x}{\log a}$$

This function is called the **logarithmic** function with respect to the base a.

If a = 10, then this function is called the common logarithmic **function.**

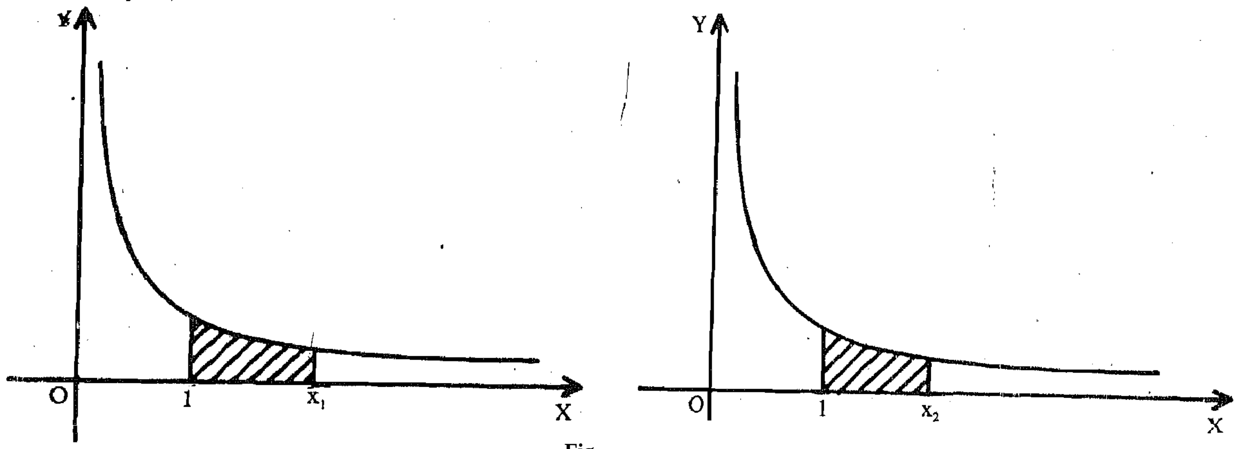


Fig.

Logarithmic function to the base a has the following properties

(i) $\log_a (x_1 x_2) = \log_a x_1 + \log_a x_2$

(ii) $\log_a \left[\frac{x_1}{x_2}\right] = \log_a x_1 - \log_a x_2$

(iii) $\log_a x^m = m \log_a x$ for every integer m.

(iv) $\log_a a = 1$.

(v) $\log_a 1 = 0$

By the definition of log x, we see that log 1 = 0 and as x becomes larger and larger, the area covered by the curve $f(t) = \frac{1}{t}$, X — axis and the ordinates at 1 and x, becomes larger and larger. Its graph is as shown in the Figure **8**,

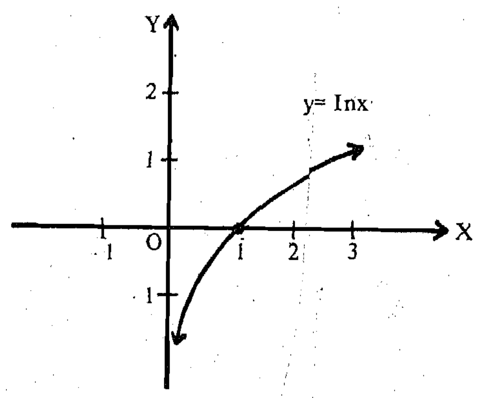


Fig. 8

You already know what is meant by inverse of a function. You had also seen in Unit 1 that if f is 1 — 1 and onto, then **f** is invertible. Let us apply that study to logarithmic function.

4.3.5 EXPONENTIAL FUNCTION

We now come to define exponential function. We have seen that

$\log x:]0, \infty[\rightarrow \mathbf{R}$

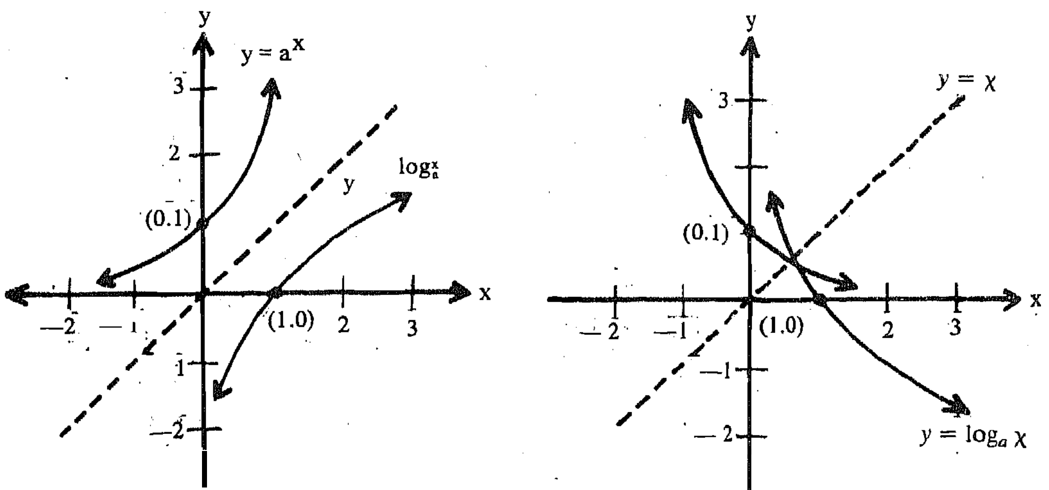
is strictly increasing function. The graph of the logarithmic function also shows that

$\log x:]0, \infty[\rightarrow \mathbf{R}$

is also onto. Therefore this function admits of inverse function. Its inverse function, called the Exponential function, Exp (x) has domain as the set R of **all real numbers** and range as $]0, \infty[$. If

$\log x = y$, then **Exp** (y) = x.

The graph of this function is the mirror image of logarithmic function as shown in the Figure 9.



(ii) a > 1   (i) 0 < a < 1

Fig. 9

The Exp (x) satisfies the following properties

(i) $\mathrm{Exp}\,(x + y) = \mathrm{Exp}\,x\ \mathrm{Exp}\,y$

(ii) $\mathrm{Exp}\,(x - y) = \mathrm{Exp}\,x / \mathrm{Exp}\,y$

(iii) $(\mathrm{Exp}\,x)^n = \mathrm{Exp}\,(nx)$

(iv) $\mathrm{Exp}\,(0) = 1$

We now come to define a^x for $a > 0$ and x any real number. We write

$$a^x = \mathrm{Exp}\,(x \log a)$$

If x is any rational number, then we know that $\log a^x = x \log a$. Hence $\mathrm{Exp}\,(x \log a) = \mathrm{Exp}\,(\log a^x) = a^x$. Thus our definition agrees with the already known definition of a in cace x is a rational number. The function a^x satisfies the following properties

(i) $a^x a^y = a^{x+y}$

(ii) $\dfrac{a^x}{a^y} = a^{x-y}$

(iii) $(a^x)^y = a^{xy}$

(iv) $a^x b^x = (ab)^x$, $a > 0$, $b > 0$.

Denote E (I) = e, so that log e = 1. The number e is an irrational number and its approximation say upto five places of decimals is 2.71828. Thus

$$e^x = \mathrm{Exp}\,(x \log e) = \mathrm{Exp}\,(x).$$

Thus Exp (x) is also denoted as e^x and we write for each $a > 0$, $a^x = e^{x \log a}$

EXAMPLE 8: Plot the graph of the function I: R → R defined by $f(x) = 2^x$.

SOLUTION:

x	−2	−1	0	1	2
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

The required graph takes the shape as shown in the Figure 10.

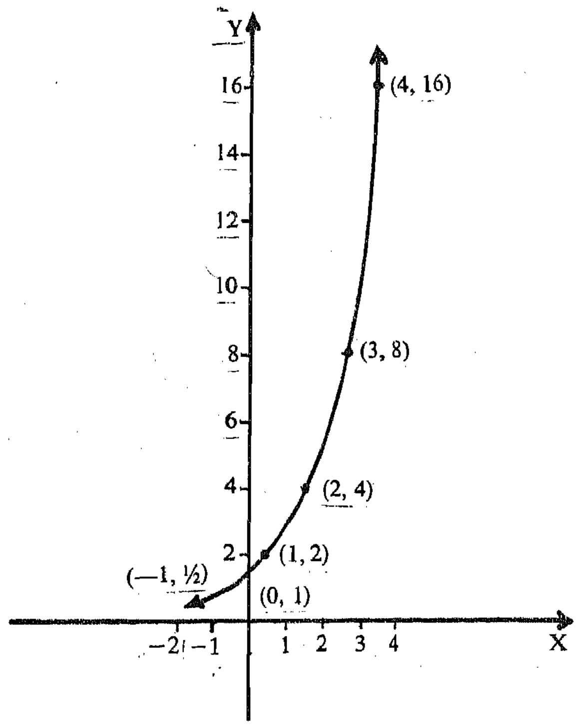


Fig. 10

EXERCISE 8)
Show the graph off: R → R defined by $F(x) = (\frac{1}{2})^x$

4.4 SOME SPECIAL FUNCTIONS

So far, we have discussed two main classes of real functions — Algebraic and Transcendental. Some functions have been designated as special functions because of their special nature and behaviour. Some of these special functions are of great interest to us. We shall frequently use these functions in our discussion in the subsequent units and blocks.

1. Identity Function

We have already discussed some of the special functions in Section **4.2.** For example, the **Identity function i : R → R**, defined as **i** (x) = x, ∀ x E R has already been discussed as an algebraic function. However, this function is sometimes, referred to as a special function because of its special characteristics, which are as follows:

(i) domain of **i** = **Range** of **i** = Codomain of i

(ii) The function i is one-one and onto. Hence it has an inverse i^{-1} which is also one-one and onto.

(iii) The function i is invertible and its inverse i^{-1} =

(iv) The graph of the identity function is a straight line through the origin which forms an angle of 45° along the positive direction of X-Axis as shown in the Figure 11.

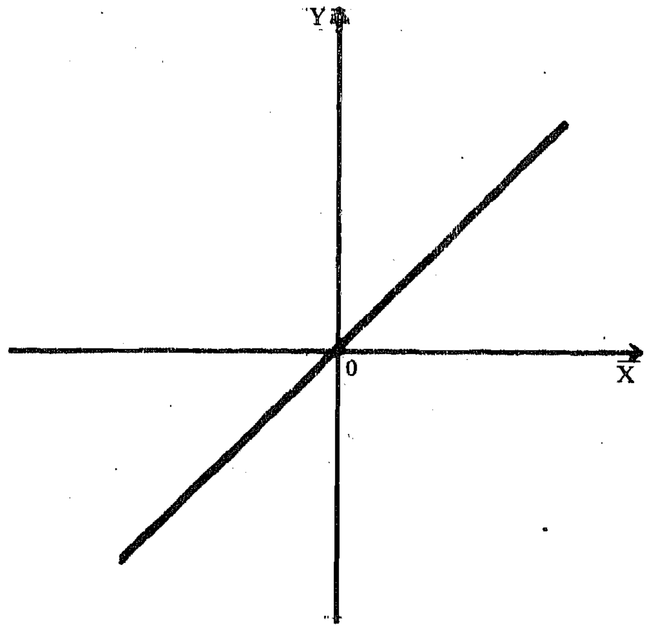


Fig. 11

2. Periodic Function

You know that
$\sin(2\pi + x) = \sin(4\pi + x) = \sin x$,
$\tan(\pi + x) = \tan(2\pi + x) = \tan x$.

This leads us to define a special class of functions, known as **Periodic** functions. All trigonometric functions belong to this class.

A function $f: S \rightarrow R$ is said to be periodic if there exists a positive real number k such that

$$f(x + k) = f(x), \forall x \in S$$

where $S \subset R$.

The smallest such positive number k is called the period of the function.

You can verify that the functions sine, cosine, secant and cosecant are periodic each with a period 2π while tangent and cotangent are periodic functions each with a period π.

EXERCISE 9)
Find the period of the function f where $f(x) = |\sin^3 x|$

---

3. Modulus Function

The modulus or the absolute (numerical) value of a real number has already been defined in Unit 1. Here we define the **modulus (absolute value) function as** follows:

Let S be a subset of R. A function $f: S \rightarrow R$ defined by

$$f(x) = |x|, \forall x \in S$$

is called the modulus function.

In short, it is written as Mod function.

You can easily see the following properties of this function:

(i) The domain of the Modulus function may be a subset of R or the set **R** itself.

(ii) The range of this function is a subset of the set of non-negative real numbers.

(iii) The **Modulus** function $f: R \rightarrow R$ is not an onto **function.** (Check why?).

(iv) The Modulus function $f: R \rightarrow R$ is not one-one. For instance, both 2 and -2 in the domain have the same image 2 in the range.

(v) The modulus function $f: R \rightarrow R$ does not have an inverse function (why)?

(vi) The graph of the Modulus function is $R \rightarrow R$ given in the Figure 12.

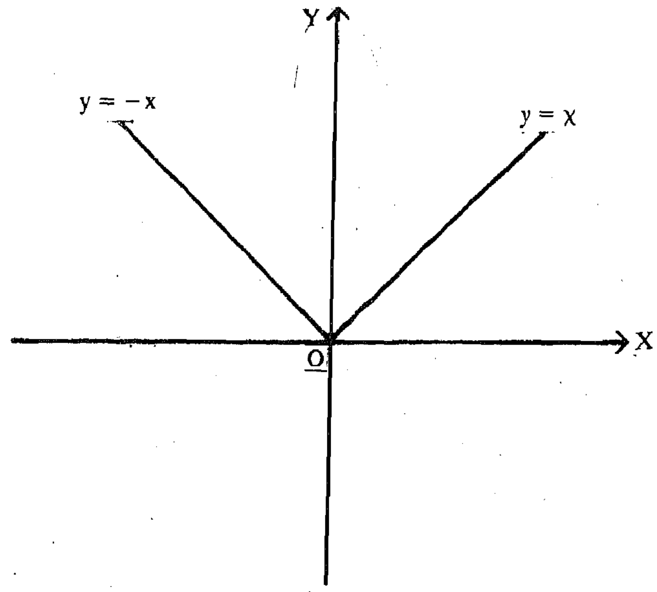


Fig. 12

It consists of two straight lines:

(i) $y = x \ (y \geq 0)$

and (ii) $y = -x \ (y \geq 0)$

through 0, the origin, making an angle of $\pi/4$ and $3\pi/4$ with the positive direction of X-axis:

4. Signum Function

A function f: R → R defined by

$$f(x) = \begin{cases} \frac{|x|}{x} & \text{when } x \neq 0 \\ x & \text{when } x = 0: \end{cases}$$

or equivalently by:

$$f(x) = \begin{cases} -1 \text{ if } x < 0 \\ 0 \text{ if } x = 0 \\ 1 \text{ if } x > 0. \end{cases}$$

is called the signum function. It is generally mitten as sgn (x).

Its range set is {−1, 0, 1}. Obviously sgn x is neither one-one nor onto. The graph of sgn x is shown in the Figure 13,

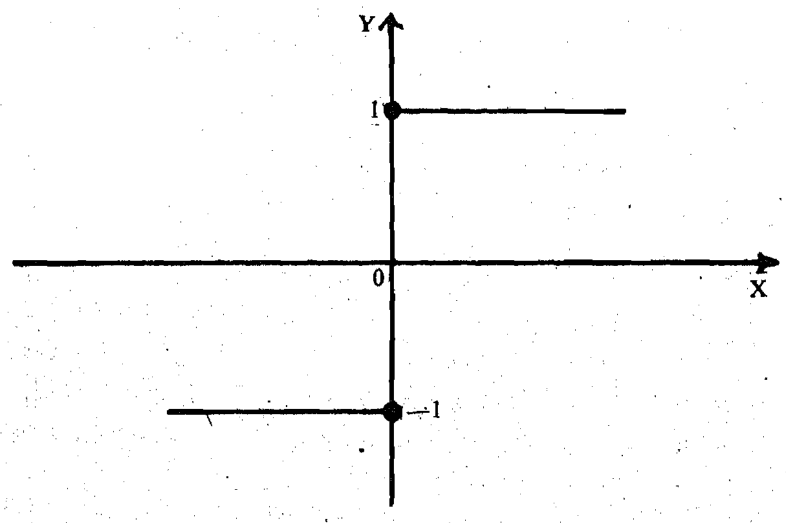


Fig. 13

5. Greatest Integer Function

Consider the number 4.01. Can you find the greatest integer which is less than or equal to this number? Obviously, the required integer is 4 and we write it as [4.01] = 4.

Similarly, if the symbol [x] denotes the greatest integer contained in x then we have

$[3/4] = 0$, $[5.01] - 5$,

$[-.005] = -1$ and $[-3.961 \approx -4$.

Based on these, the greatest integer function is defined as follows:
A function f: $R \to R$ defined by

$$f(x) = [x], \forall x \in R,$$

where [x] is the largest integer less than or equal to x is called the greatest integer function:

The following properties of this function are quite obvious:

(i) The domain is R and the range is the set Z of all integers.

(ii) The function is neither one-one nor onto.

(iii) If n is any integer and x is any real number such that x is greater than or equal to n but less than n + 1 i.e., if $n \le x < n + 1$ (for some integer n), then [x] = n i.e.,

The graph of the greatest integer function is shown in the Figure 14.

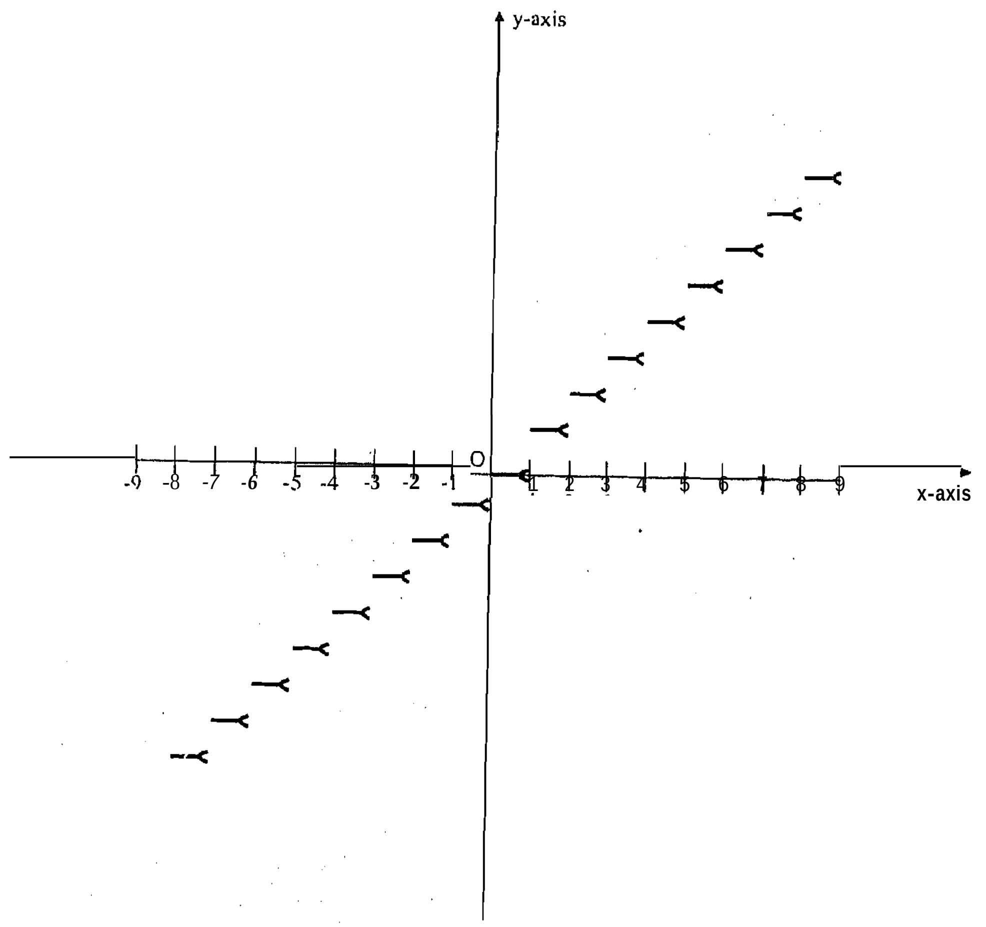


Fig. 14

EXAMPLE 9: Prove that

$$[x + m] = [x] + m, \forall x \in R \text{ and } m \in Z.$$

SOLUTION: You know that for every $x \in R$, there exists an integer n such that

$$n \le x < n + 1.$$

Therefore,

$$n + m \le x + m < n + 1 + m,$$

and hence

$$[x + m] = n + m = [x] + m,$$

which proves the result.

EXERCISE 10)

Test whether or not the function f: R → R defined by f(x) = x–[x] ∀ x ∈ R, is periodic. If it is so, find its period.

6. Even and Odd Functions

Consider a function f: R → R defined as

$f(x) = 2x, \forall x \in R.$

If you change x to −x, then you have

$f(-x) = 2(-x) = -2x = -f(x).$

Such a function is called an odd function.

Now, consider a function f:R → R defined as

$f(x) = x^2 \ \forall x \in R$

Then changing x to −x we get

$f(-x) = (-x)^2 = x^2 = f(x)$

Such a function is called an **even** function.

The definitions of even and odd functions are as follows:

A function f: R → R is called even if f(− x) = f(x), ∀ x ∈ R.
It is called odd **if f(− x) = − f(x),** ∀ x ∈ R

EXAMPLE 7: Verify whether the function **f:** R → **R defined by**

(i) $f(x) = \sin^2 x + \cos^3 2x$

$f(x) = \sqrt{a^2 + ax + x^2} - \sqrt{a^2 - ax + x^2}$

are even or odd.

SOLUTION: (i) $f(x) = \sin^2 x + \cos^3 2x, \forall x \in R$

$\Longrightarrow f(-x) = \sin^2(-x) + \cos^3 2(-x)$

$= \sin^2 x + \cos^3 2x = f(x), \forall x \in R$

$\Longrightarrow$ f is an even function.

(ii) $f(x) = \sqrt{a^2 + ax + x^2} - \sqrt{a^2 - ax + x^2}, \ \forall x \in R$

$\Longrightarrow f(-x) = \sqrt{a^2 - ax + x^2} - \sqrt{a^2 + ax + x^2}$

$= -f(x), \forall x \in R$

$\Longrightarrow$ **f** is an odd function.

EXERCISE 11)

Determine which of the following functions are even or odd or neither:

(i) $f(x) = x$ **(ii) a constant function**

(iii) sin x, cos x, tan x,

(iv) $f(x) = \dfrac{x-4}{x^2-9}, \forall x \in R, x \notin \{-3, 3\}$

7. Bounded Functions

In Unit 2, you were introduced to the notion of a bounded set, upper and lower bounds of a set. **Let** us now extend these **notions** to a function.

You know that if **f:** S → R is a function, (S ⊂ R), then

$\{f(x) : x \in S\},$

is called the range set or simply the range of the function **f.**

A **function is said to be bounded if its range is bounded.**

Let f: S → R be a function. It is said to **be bounded above if there exists a** real number K such that

$f(x) \le K, \forall x \in S$

The number K is called an upper bound of f. The function f is said to be bounded below if there exists a **number k** such that

$f(x) \ge k \ \forall x \in S$

The number **k is called** a lower bound of **f.**

A function $f: S \to R$, which is bounded above as well as bounded below, is said to be bounded. This implies that there exist two real numbers k and K such that
$k \leq f(x) \leq K \; \forall \, x \in S$.

This is equivalent to say that a function $f: S \to R$ is bounded if there exists a real number M such that

$$|f(x)| \leq M, \; \forall \, x \in S.$$

A function may be bounded above only or may be bounded below only or neither bounded above nor bounded below.

Recall that sin x and cos x are both bounded functions. Can you say why? It is because of the reason that the range of each of these functions is $[-1, 1]$.

EXAMPLE 8: A function $f: R \to R$ defined by

(i) $f(x) = -x^2, \; \forall \, x \in R$ is bounded above with 0 as an upper bound

(ii) $f(x) = x, \; \forall \, x \geq 0$ is bounded below with 0 as a lower bound

(i) $f(x) =$ for $|x| \leq 1$ is bounded because $|f(x)| \leq 1$ for $|x| \leq 1$.

Try the following'exercise.

EXERCISE 12)

Test which of the following functions with domain and co-domain as R are bounded and unbounded:

(i) $f(x) = \tan x$

(ii) $f(x) = [x]$

(iii) $f(x) = e^x$

(iv) $f(x) = \log x$

EXERCISE 13)

Suppose $f: S \to R$ and $g: S \to R$ are any bounded functions on S. Prove that $f + g$ and $f \cdot g$ are also bounded functions on S.

4.5 SUMMARY

In this unit, we have discussed various types of real functions. We shall frequently use these functions in the concepts and examples to be discussed in the subsequent units throughout the course).

In Section 4.2, we have introduced the notion of an algebraic function and its various types. A function $f: S \to R$ $(S \subset R)$ defined as $y = f(x), \; \forall \, x \in S$ is said to be algebraic if it satisfies identically an equation of the form
$p_0(x)\, y^n + p_1(x)\, y^{n-1} + p_2(x)\, y^{n-2} + ... + p(x)\, y + p_n(x) = 0$,
where $p_0(x), p_1(x), ..., p_n(x)$ are polynomials in x for all
$x \in S$ and n is a positive integer. In fact, any function constructed by a finite number of algebraic operations—addition, subtraction, multiplication, division and root extraction—is an algebraic function. Some of the examples of algebraic functions are the polynomial functions, rational functions and irrational functions.

But not all functions are algebraic. The functions which are not algebraic, are called transcendental functions. These have been discussed in Section 4.3. Some important examples of the transcendental functions are trigonometric functions, logarithmic functions and exponential functions which have been defined in this section. We have defined the monotonic functions also in this section.

In Section 4.4, we have discussed some special functions. These are the identity function, the periodic functions, the modulus function, the signum function, the greatest integer function, even and odd functions. Lastly, we have introduced the bounded functions and discussed a few examples.

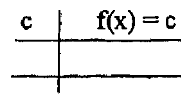
4.6 ANSWERS/HINTS/SOLUTIONS

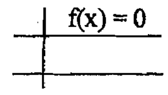
E1) $(f + g)(x)\ ax^2 + bx$
$(f - g)(x) = ax^2 - bx$
$(f.g)(x) = ax^2.bx = abx^3$

$(f/g)(x) = \frac{ax^2}{bx} = \frac{ax}{b}$ provided $b \neq 0$, $x \neq 0$.

$kf = k\ ax^2$

E2)





constant function   Zero function (x-axis)

E3) $y = f(x) = \sqrt{x} \Rightarrow y^2 = x$. Now draw the graph.

E4) $y = f(x) \Rightarrow y = \sqrt{x + \sqrt{x}}$

$\Rightarrow y^2 = x + \sqrt{x}$
$\Rightarrow y^2 - x = \sqrt{x}$
$\Rightarrow (y^2 - x)^2 = x$
$\Rightarrow y^4 - 2y^2 x + x^2 - x = 0,$

which shows that $y = f(x)$ is an algebraic **function.**

E5) (i) Let $x_1, x_2 \in]0, \pi[$. Then
$f(x_1) = f(x_2) \Rightarrow \cos x_1 = \cos x_2$
$\Rightarrow \cos x_1 - \cos x_2 = 0$

$\Rightarrow 2 \sin\left(\frac{x_1 + x_2}{2}\right) \sin \frac{(x_2 - x_1)}{2} = 0$

$\Rightarrow$ either $\sin\left(\frac{x_1 + x_2}{2}\right) = 0$ or $\sin \frac{(x_2 - x_1)}{2} = 0$

Now, $\sin \frac{x_2 - x_1}{2} = 0 \Rightarrow \frac{x_2 - x_1}{2} = 0, \pm \pi, \pm 2\pi, \ldots$, and

$\sin \frac{x_1 + x_2}{2} = 0 \Rightarrow \frac{x_1 + x_2}{2} = 0, \pm \pi, \pm 2\pi, \ldots$

Now, $\frac{x_2 - x_1}{2} = 0 \Rightarrow x_1 = x_2$

$\frac{x_2 - x_1}{2} = \pm n\pi\ (n \geq 1) \Rightarrow x_2 = \pm 2n\pi + x_1,$

which, is not possible.

Hence $\frac{x_2 - x_1}{2} = \pm n\pi$ is not possible.

Thus, the only possibility is

$\frac{x_2 - x_1}{2} = 0$ which means $x_2 = x_1$.

Since, $x_1, x_2 \in]0, \pi[$, we cannot have the case

$\frac{x_1 + x_2}{2} = 0, \pm \pi, \pm 2\pi, \ldots$

Hence

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$

In other words, $f(x) = \cos x$ is one-one in $[0, \pi]$.

Now, the range of cos x is $[-1, 1] \neq R$. Therefore cos x, defined from $[0, \pi]$ to R is not onto. But, if cos x is defiend from [0,] to [–1, I], then it is certainly one-one and onto.

(i) Do it yourself.

E6) (i) Contangent **Inverse.**
$y = \cot^{-1} x \Leftrightarrow x = \cot y$
where $0 < y < n$ and $-\infty < x < +\infty$.

(ii) Secant Inverse
$y = \sec^{-1} x \Leftrightarrow x = \sec y$,

where $0 \leq y \leq n$, $y \neq \frac{\pi}{2}$ and $|x| \geq 1$.

(iii) Cosecant inverse
$y = \text{cosec}^{-1} x \Leftrightarrow x = \text{cosec } y$,

where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$ and $|x| \geq 1$.

E7) (i) Let $f:S \rightarrow T$ be a strictly decreasing function.
Let x_1, x_2 be any two distinct elements of S. Then
$x_1 \neq x_2 \Rightarrow x_1 < x_2$ or $x_1 > x_2$
$\Rightarrow f(x_1) > f(x_2)$ or $f(x_1) < f(x_2)$
$\Rightarrow f(x_1) \neq f(x_2)$.

This shows that f is one-one.
Since f(S) = T, f is onto. Thus f is one-one and onto and hence f is invertible. In other words f^{-1} exists i.e. $f^{-1} : T \rightarrow S$ is defined.

For $y_1, y_2 \in T$, we have $y_1 = f(x_1)$, $y_2 = f(x_2)$, for some x_1, x_2 in S. If $y_1 < y_2$, we claim that $x_1 > x_2$. If not, then $x_1 \leq x_2$ which implies that $f(x_1) > f(x_2)$ i.e., $y_1 > y_2$. This is'a contradiction. Hence $y_1 < y_2 \Rightarrow x_1 > x_2 \Rightarrow f^{-1}(y_1) > f^{-1}(y_2)$.

Thus, f^{-1} is also strictly increasing.

E8) Follow the method of Example 8.

E9) Since $f(\pi + x) = |\sin^3(\pi + x)| = |-\sin^3 x| = |\sin^3 x|$,
therefore π is the period of f. You may note that π is the least such number satisfying the above relation.

E10) The function $f(x) = x - [x]$ is periodic with period 1 because 1 is the least number such that
$f(x + 1) = (x + 1) - [x + 1] = (x + 1) - [x] - 1 = x - [x]$
$= f(x)$.

E11) (i) odd (ii) even.

(iii) sin x is odd, cos x is even, tan x is odd.

$f(-x) = \frac{(-x)-4}{(-x)^2 - 9} = \frac{-x-4}{x^2 - 9}$, which shows that f is neither even nor odd.

E12) (i) It is unbounded because its range is $]-\infty, +\infty[$.
(ii) $|x|$ is bounded below with 0 as a lower bound.
(iii) e^x is bounded below because its range is $]0, \infty[$.
(iv) log x is unbounded.

E13) Since f and g are given to be bounded functions, therefore there exist numbers k_1, K_1 and k_2, K_2 such that
$k_1 \leq f(x) \leq K_1, \forall x \in S$;
$k_2 \leq g(x) \leq K_2, \forall x \in S$.
(i) Since $(f+g)(x) = f(x) + g(x), \forall x \in S$,
therefore, $k_1 + k_2 \leq f(x) + g(x) \leq K_1 + K_2, \forall x \in S$.

$\Rightarrow k \le (f+g)(x) \le K \; \forall \; x \in S$

where $k = k_1 + k_2$, $K = K_1 + K_2$ are some real numbers

Thus $f(x) + g(x)$ is a bounded function.

(ii) We know $(f.g)(x) = f(x).\, g(x) \; \forall \; x \in S$. Since f and g are bounded, therefore, we can find m_1, m_2 such that

$|f(x)| \le m_1$, $|g(x)| \le m_2$, $\forall \; x \in S$.

Then

$$|(f.g)(x)| = |f(x).\, g(x)|$$
$$= |f(x)|\,|g(x)|$$
$$\le m_1.\, m_2 \; \forall \; x \in S$$

which shows that f.g is bounded.

REVIEW

Attempt the following self-assessment questions and verify your answers given at the end:

1. Test whether the .following are rational numbers:
   (i) $\sqrt{17}$ (ii) $\sqrt{8}$ (iii) $3+\sqrt{2}$
2. The inequality $x^2 - 5x + 6 < 0$ holds for
   (i) $x < 2, x < 3$ (ii) $x > 2, x < 3$
   (ii) $x < 2, x > 3$ (iii) $x > 2, x > 3$
3. If a, b, c, d are real numbers such that
   $a^2 + b^2 = 1, c^2 + d^2 = 1,$
   then show that $ac+bd \leq 1.$
4. **Prove that** $|a + b + c| \leq |a| + |b| + |c|$
   for all a, b, c, $\in$ R.
5. Show that
   $|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$
   for $a_1, a_2, ...$ a, E R.
6. Which of the'following sets are bounded above? Write the supremum of the set if it exists.
   (i) '(n, e) (ii) $\bigcup_{n=1}^{\infty} [2n, 2n+1]$
   (ii) $\left\{n + \frac{(-1)^n}{n} : n \in N\right\}$ (iv) $\{x \in Q : x^2 < 2\}$
   (v) $\{x \in R : x < 0\}$ (vi) $\{\frac{1}{n} : n \in N$ and n is prime$\}$
   (vii) $\{x^2 : x \in R\}$ (viii) $\left\{\mathrm{Cos}\left(\frac{n\pi}{3}\right) : n \in N\right\}$
   (ix) $\{2n : n \in Z\}$ (x) $\{x \in R : x \leq 2\} \cup \{x \in R: x > 2\}$
7. Find which of the sets in question 6 are bounded below. Write the infemum if it exists.
8. Which of the sets in question 6 are bounded and unbounded.
9. Test whether the following statements are true or false :
   (i) The set Z of integers is not a NBD of any of its points.
   (ii) The interval]0, 1] is a NBD of each of its points
   (iii) The set]1, 3[U] 4, 5 [is open.
   (iv) The set $[a, \infty[\cup]-\infty, a]$ is not open.
   (v) N is a closed set.
   (vi) The derived set of Z is nonempty.
   (vii) Every real number is a limit point of the set Q of rational numbers.
   (viii) A finite bounded set has a limit point.
   (ix) [4, 5] U [7, 8] is a closed set.
   (x) Every infinite set is closed.
10. Justify the following statements:
    (i) The identity function is an odd function.
    (ii) The absolute value function is an even function.
    (iii) The greatest integer function is not onto.
    (iv) The tangent function is periodic with period π.
    (v) The function $f(x) = |x|$ for $-2 \leq x \leq 3$ is bounded.
    (vi) The function $f(x) = e^x$ is not bounded
    (vii) The function $f(x) = \sin x$, for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is monotonically increasing.
    (viii) The function $f(x) = \cos x$ for $0 \leq x \leq \pi$ is monotonically decreasing.
    (ix) The function $f(x) = \tan x$ is strictly increasing for $x \in [0, \frac{\pi}{2}]$.
    (x) $f(x) = \frac{\sqrt{2x^2-3x+2}}{3x-2}$ is an algebraic function

ANSWERS

1. None is a rational number
2. For (ii) only since $2 < x < 3$.
3. $(b-d)^2 \geq 0 \Rightarrow b^2+d^2 \geq 2bd \Rightarrow bd \leq \frac{1}{2}$
   $(a-c)^2 \geq 0 \Rightarrow a^2+c^2 \geq 2ac \Rightarrow ac \leq \frac{1}{2}$
   $\Rightarrow ac+bd \leq 1$.
4. Use the triangle inequality.
5. Use the principle of Induction.
6. (i) π; (iv) 2

   (v) 0 (vi) $\frac{1}{2}$ (viii) 1

   (ix) and (x) are unbounded.
7. (i) e
   (ii) 2
   (iii) 0
   (vi) 0
   (viii) -1
8. All the sets are unbounded except the (i)
9. (i) True (ii) False (iii) True (iv) False (v) True (vi) False (vii) True (viii) False (ix) True **(x) False.**

NOTES