UNIT 2 STRUCTURE OF REAL NUMBERS

STRUCTURE

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2.1 INTRODUCTION

In Unit 1 we have discussed the construction of real numbers from the rational numbers which, in turn, were constructed from integers. In this unit, we show that the set of real numbers has an additional property which the set of rational numbers does not have, namely it is a complete ordered field. The questions, therefore, that arise are: What is a field? What is an ordered field? What does it mean for an ordered field to be complete? In order to answers these questions we need a few concepts and definitions e.g. those of order inequalities and intervals in R. We shall discuss these concepts in Section 2.2. Also in this section, we shall explain the extended real number system.

You know that a given set is either finite or infinite. In fact a set is finite, if it contains just n elements where n is some natural number. A set which is not finite is called an infinite set. The elements of a finite set can be counted as one, two, three and so on, while those of an infinite set can not be counted in this way. Can you count the elements of the set of natural numbers? Try it. In Section 2.4, we shall show that this notion of counting can be extended in certain sense to even infinite sets.

OBJECTIVES

In this unit, you should, therefore, be able to

- identify the order relation in the set of real numbers and extended real number system,
- describe the field structure of the set of real numbers,
- discuss the order-completeness of the set of real numbers
- apply the notion of countability to various infinite sets.

2.2 ORDER RELATIONS IN REAL NUMBERS

In Section 1.3, we have demonstrated that every real number can be represented as a unique point on a line and every point on a line represents a unique real number. This helps us to introduce the notion of inequalities and intervals on the real line which we shall frequently use in our subsequent discussion through out the course.

You know that a real number x is said to be positive if it lies on the right side of O, the point which corresponds to the number 0 (zero) on the real line. We write it as $x > 0$. Similarly, a real number x is negative, if it lies on the left side of O. This is written as $x < 0$. If $x \geq 0$, then x is a non-negative real number. The real number x is said to be non-positive if $x \leq 0$.

Let x and y be any two real numbers. Then, we say that x is greater than y if $x-y > 0$. We express this by writing $x > y$. Similarly x is less than y if $x - y < 0$ and we write $x < y$. Also x is greater than or equal to y ($x \geq y$) if $x - y \geq 0$. Accordingly, x is less than or equal to y ($x \leq y$) if $x - y \leq 0$. Given any two real numbers x and y, exactly one of the following can hold:

either (i) $x > y$
or (ii) $x < y$
or (iii) $x = y$.

This is called the law of trichotomy. The order relation $\leq$ has the following properties:

PROPERTY 1: For any x, y, z in R,

(i) If $x \leq y$ and y I x, then $x = y$,
(ii) If $x \leq y$ an $y \leq z$, then $x \leq z$,
(iii) If x I y then x + z I y + z,
(iv) If $x \leq y$ and $o \leq z$, then $x z \leq y z$.

The relation satisfying (i) is called **anti-symmetric.** It is called transitive if it satisfies (ii). The property (iii), shows that the inequality remains unchanged under addition of a real number. The property (iv) implies that the inequality also remains unchanged under multiplication by a non-negative real number. However, in this case the inequality gets reversed under multiplication by a non-positive real number. Thus, if x I y and $z \leq 0$, then $x z \geq yz$. For instance, if $z = -1$, we see that

$$-2 \leq 4 \Longrightarrow 2(-1) \geq 4(-1) \Longrightarrow -2 \geq -4.$$

EXERCISE 1)

State the properties of order relation in the set R of real numbers with respect to the relation 3 (is greater than or equal to) and illustratq the inequality under multiplication by a negative real number.

We state the following results without proof:

There lie an infinite number of rational numbers **between** any two distinct rational numbers.

As a matter of fact, something more is true.

Between any two real numbers, there lie infinitely many rational (irrational) numbers. Thus there lie an infinite number or real numbers between any two given real numbers.

2.2.1 INTERVALS

Now that the notion of an order has been introduced in R, we can talk of some special subsets of R defined in terms of the order relation. Before we formally define subset, we first introduce the notion of 'betweenness', which we have already used intuitively in the previous results. If 1, 2, 3 are three real numbers, then we say that 2 lies between 1 and 3. Thus, in general, if a, b and c are any three real numbers such that $a \leq b \leq c$ then we say that b lies 'between' a and c. Closely related to notion of betweeness is the concept of an interval.

DEFINITION 1: INTERVAL

An interval in R is an nonempty subset of R which has the property that, whenever two numbers a and b belong to it, all numbers between a and b also belong to it.

The set N of natural numbers is not an interval because while 1 and 2 belong to N, but 1.5 which lies between 1 and 2, does not belong to N.

We now discuss various forms of an interval.

Let $a, b \in \mathbf{R}$ with $a \leq b$.

(i) Consider the set $\{x \in R : a \leq x \leq b\}$. This set is denoted by [a, b], and is called a closed interval. Note that the-end points a and b are included in it.

(ii) Consider the set $\{x \in R : a < x < b\}$, This set is denoted by]a, b[, and is called an open interval. In this case the end points a and b are not included in it,

(iii) The interval $\{x \in \mathbf{R} : a \leq x < b\}$ is denoted by $[a, b[$.

(iv) The interval $\{x \in \mathbf{R} : a < x \leq b\}$ is denoted by $]a, b]$.

You can see the graph of all the four intervals in the Figure 1.

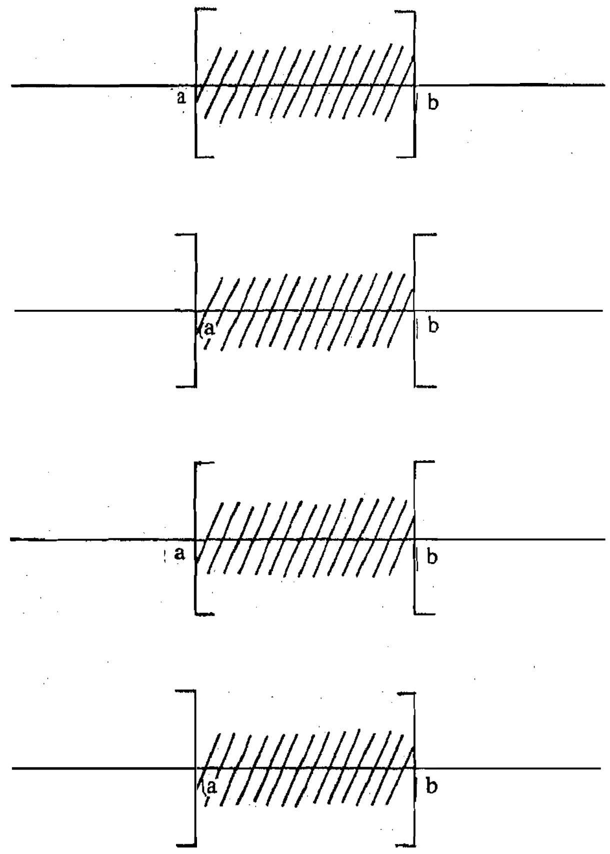


Fig. 1

Intervals of these types are called bounded **intervals.** Some authors also call them **finite** intervals. But remember that these are not finite sets. In fact these are **infinite** sets except for the case $[a, a] = \{a\}$.

You can **easily** verify that an open interval $]a, b[$ as well as $]a, b]$ and $[a, b[$ are **always** contained in the closed interval $[a, b]$.

EXAMPLE 1: **Test** whether or not the union of any **two** intervals is an interval.

SOLUTION: Let $[2, 5]$ and $[7, 12]$ be two intervals. Then $[2, 5] \cup [7, 12]$ is not an interval as can be seen on the line in Figure 2.

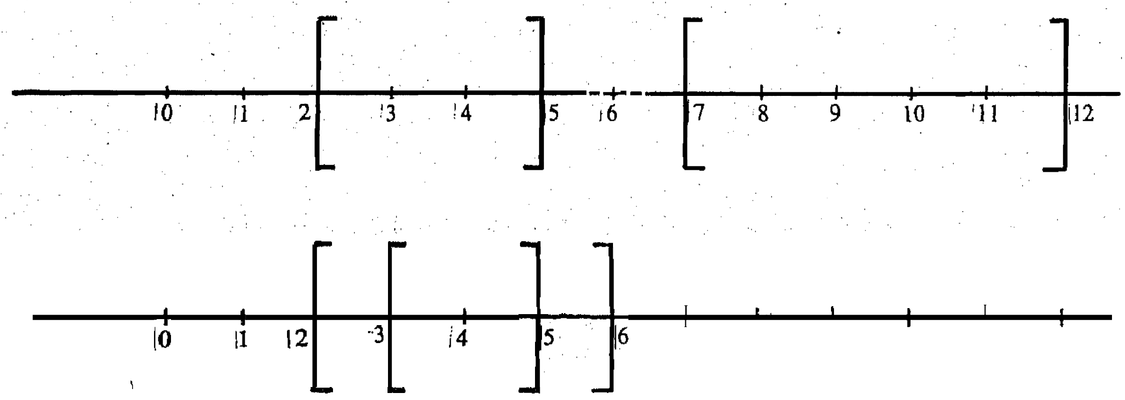


Fig. 2

However, if you take the intervals which are not disjoint, then the union is an interval. For example, the union of $[2, 5]$ and $[3, 6]$ is $[2, 6]$ which is an interval. **Thus** the union of **any** two **intervals is an** interval provided the **intervals** are not disjoint.

Now try the following exercise:

EXERCISE 2)
Give examples to show that the intersection of any two intervals may not be an interval. What happens, if the two intervals are not disjoint? Justify your answer by an example.

2.2.2 EXTENDED REAL NUMBERS

The notion of the extended real number system is important since we need it in this unit as well as in the subsequent units.

You are quite familiar with the symbols $+\infty$ and $-\infty$. You often call these symbols are 'plus infinity' and 'minus infinity', respectively. The symbols $+\infty$ and $-$ m are extremely useful. Note that these are **not** real numbers.,

Let us construct a new set **R*** by adjoining $-\infty$ and $+$m to the set **R** and write it as

$$\mathbf{R}^* = \mathbf{R} \cup \{-\infty, +\infty\}.$$

Let us extend the order structure to **R*** by a relation $<$ as $-\infty < x < +\infty$, for every $x \in \mathbf{R}$. Since the symbols $-\infty$ and $+\infty$ do not represent any real numbers, you should, therefore, not apply any result stated for real numbers, to the symbols $+\infty$ and $-\infty$. The only purpose of using these symbols is that it becomes convenient to extend the notion of (bounded) intervals to unbounded intervals which are as follows :

Let a and b be any two real numbers. Then we adopt the following notations :

$[a, \infty] = \{x \in \mathbf{R}: x \geq a)$
$[a, \infty] = \{x \in \mathbf{R}: x > a)$
$[-\infty, b] = \{x \in \mathbf{R}: x \leq b)$
$[-\infty, b] = \{x \in \mathbf{R}: x < b)$
$[-\infty, \infty] = \{x \in \mathbf{R}: -\infty < x < \infty\}.$

You can see the geometric representation of these intervals in Figure 3.

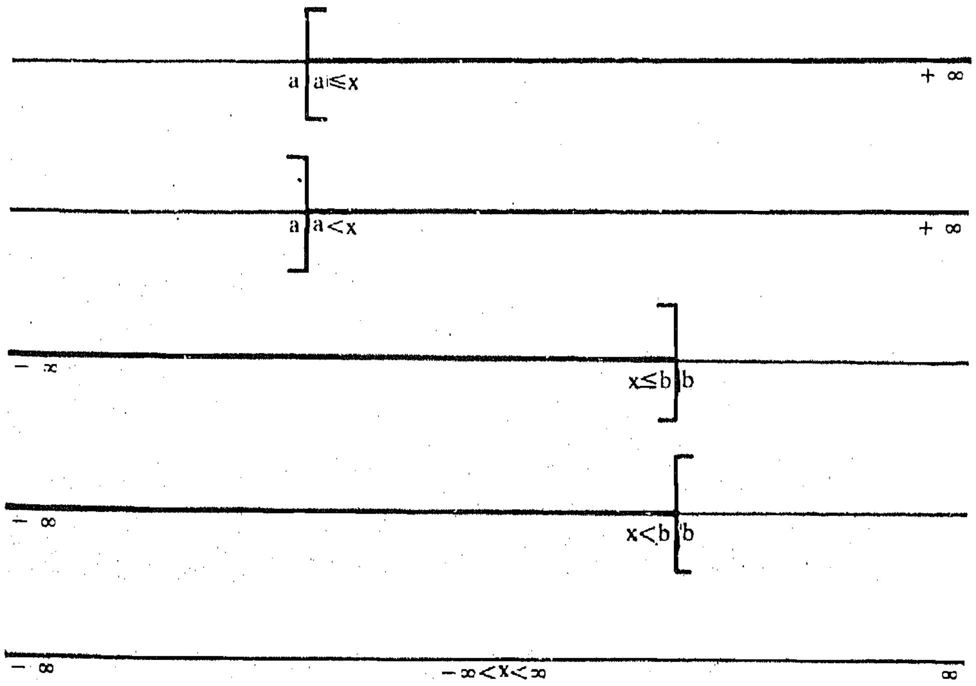


Fig. 3

All these unbounded intervals are also sometimes called infinite intervals.

You can perform the operations of addition and multiplication involving $-\infty$ and $+\infty$ in the following way: For any $x \in \mathbf{R}$, we have

$x + (+\infty) = +\infty,$
$x + (-\infty) = -\infty,$
$x.(+\infty) = +\infty$, if $x > 0$
$x.(+\infty) = -\infty$ if $x < 0$
$x.(-\infty) = -\infty$, if $x > 0$

$x.(-\infty) = +\infty$, if $x < 0$

$\infty + \infty = +\infty$, $-\infty - \infty = -\infty$

$\infty.(-\infty) = -\infty$, $(-\infty).(-\infty) = +\infty$.

Note that the operations $\infty - \infty$, $0.\infty$, $\frac{\infty}{\infty}$ are not defined.

2.3 ALGEBRAIC STRUCTURE

Richard Dedekind

During the 19th Century, a new trend emerged in Mathematics to use algebraic structures in order to provide a solid foundation for Calculus and Analysis. In this quest, several methods were used to characterise the red numbers. One of the methods was related to the least upper bound principle used by Richard Dedekind. which we discuss in this section.

This leads us to the description of the real numbers as a complete ordered field. In order to define a complete ordered field. we need some definitions and concepts.

You are quite familiar with the operations of addition and multiplication on numbers, union and intersection on the subsets of a universal set. For example if you add or multiply any two natural numbers, the sum or the product is a natural number. These operations of addition or multiplications on the sets of numbers are examples of a binary operation on a set. In general, we can define a binary operation on a set in the following way:

DEFINITION 2: BINARY OPERATION

Given a non-empty set S, a binary operation on S is a rule which associates with each pair of elements of S, a unique element of S.

We denote this rule by symbols such as ., *, +. etc.

By an Algebraic Structure, we mean a non-empty set together with one or more binary operations defined on it. A field is an algebraic structure which we define as follows:

DEFINITION 3: FIELD STRUCTURE

A field consists of a non-empty set F together with two binary operations defined on it, denoted by the symbols '+'(addition) and '.'(multiplication) and satisfying the following axioms for any elements x, y, z of the set F.,

A_1: $x+y \in F$ (Additive Closure)

A_2: $x+(y+z) = (x+y)+z$ (Addition is Associative)

A_3: $x+y = y+x$ (Addition is **Commutative)**

A_4: There exists an element in F, denoted by '0' and called the zero or the zero element of F such that $x+0 = 0+x = x \ \forall \ x \in F$ (Additive Identity)

A_5: For each $x \in F$, there exists an element $-x \in F$ with the property $x+(-x) = (-x)+x = 0$ (Additive Inverse)
The element $-x$ is called additive **inverse** of x.

M_1 $x.y \in F$ (Multiplicative Closure)

M_2 $(x.y).z = x.(y.z)$ (Multiplication is Associative)

M_3 $x.y = y.x$ (Multiplication is **Commutative)**

M_4 There exists an element 1 different from 0 called the unity of F, such that $1.x = x.1 = x \ \forall \ x \in F$ (Multiplicative Identity)

M_5 For each $x \in F$, $x \neq 0$, there exists an element $x^{-1} \in F$ such that $x.x^{-1} = x^{-1}.x = 1$. (Multiplicative Inverse)

The, element x^{-1} is called the multiplicative inverse of x.

D: $x.(y+z) = x.y+x.z$
$(x+y).z = x.z+y.z.$
(Multiplication is distributive over Addition).

Since the unity is not equal to the zero **i.e.** $1 \neq 0$ in a field, therefore any field must contain at least two elements. Note that the axioms A_1 (closure under addition) and

M_1 (closure under multiplication) are unnecessary because the closures are implied in the definition of a binary operation. However, we include them, for the sake of emphasis. Now try the following exercises:

EXERCISE 3)
Show that the set {0, 1} forms a field under the operations '+' and '.' defined by the following tables:

+	0	1
0	0	1
1	1	0

.	0	1
0	0	0
1	0	1

EXERCISE 4)
Show that the zero and the unity are unique in a field.

Now, you can easily verify that all the eleven axioms are satisfied by the set of rational numbers with respect to the ordinary addition and multiplication. Thus, the set Q forms a field under the operations of addition and multiplication, and so does, the set R of all the real numbers.

EXERCISE 5)
Do the sets N (of natural numbers) and Z (set of integers) form fields? Justify your answers. Also verify that the set C of complex numbers is a field.

We state (without proof) some important properties satisfied by a field. They follow from the field axioms. Can you try?

PROPERTY 1

For any x, y, z in F,
1. $x+z = y+z \Longrightarrow x = y$,
2. $x.0 = 0 = 0.x$,
3. $(-x).\ y = -x.y = x.\ (-y)$,
4. $(-x).\ (-y) = x.y$,
5. $x.z = y.z,\ z \neq 0 \Longrightarrow x = y$,
6. $x.y = 0 \Longrightarrow$ either $x = 0$ or $y = 0$.

Thus by now you know that the sets Q, R apd C form fields under the operations of addition and multiplication.

2.3.1 ORDERED FIELD

In Section 2.2, we defined the order relation $\leq$ in R. It is easy to see that this order relation satisfies the following properties:

PROPERTIES 2

Let x, y, z be any elements of R. Then

O_1: For any two elements x and y of R, one and only of the following holds:
(i) $x < y$, (ii) $y < x$, (iii) $x = y$,
O_2: $x \leq y,\ y \leq x \Longrightarrow x \leq z$,
O_3: $x \leq y \Longrightarrow x + z \leq y + z$,
O_4: $x \leq y,\ 0 < z \Longrightarrow x.z \leq y.z$

We express this observation by saying that the field R is an ordered field (i.e. it satisfies the properties O_1 – O_4). It is easy to see that these properties are also satisfied by the field Q of rational numbers. Therefore, Q is also an ordered field.. What about the field C of Complex numbers? Try it yourself as an exercise,

EXERCISE 6)

Show that the field C of Complex numbers is not an ordered field.

2.3.2 COMPLETE ORDERED FIELD

Although R and Q are both ordered fields, yet there is a property associated with the order relation which is satisfied by R but not by Q. This property is known as the **Order-Completeness,** introduced for the first time by Richard Dedekind. To explain this situation more precisely, we need a few more mathematical concepts which are discussed as follows:

Consider set S = {1, 3, 5, 7). You can see that each element of **S** is less than or equal to 7. That is $x \leq 7$, for each $x \in S$. Take another set S, where $S = \{x \in \mathbf{R} : x \leq 17)$.
Once again, you see that each element of S is less than 18. That is, $x < 18$, for each $x \in S$. In both the examples, the sets have a special property namely that every element of the set is less than or equal lto some number. This number is called an upper bound of the corresponding set and such a set is said to be bounded above. Thus, we have the following definition:

DEFINITION 4: UPPER BOUND OF A SET

Let $S \subset \mathbf{R}$. If there is a number $u \in \mathbf{R}$ **such** that $x \leq u$, for every $x \in S$, then S is said to b bounded above. **The** number u is called an upper bound of S.

EXAMPLE 2: Verify whether the following sets are bounded above. Find an upper bound of the set, if it exists.

(i) The set of negative integers {–1, –2, –3, ...}.

(ii) The set N of natural numbers.

(iii) The sets **Z,** Q and R.

SOLUTION : (i) The set is bounded above with -1 as an upper bound,

(ii) The set N is not bounded above.

(iii) All these sets are not bounded above.

EXERCISE 7)

(i) Define a set which is bounded below. Also define a lower bound of a set.

(ii) Give at least two **examples** of a set (one of an infinite set) which is bounded below and mention a lower bound in each case.

(iii) Is the set of negative integers bounded below? Justify your answers.

Now consider a set S = {2, 3, 4, 5, 6, 7). You can easily see that this set is bounded above because 7 is an upper bound of S. Again this set is also bounded below because 2 is a lower bound of S. Thus S is both bounded above as well as bounded below. Such a set is called a bounded set. Consider the following sets:

s , = {... –3, –2, –1, 0, 1, 2, ...},
S_2 = {0, 1, 2, ...},
S_3 = (0, –1, –2, ...}.

You can easily see that S, is neither bounded above nor bounded below. The set S_2 is not bounded above while S, is not bounded below. Such sets are known as Unbounded Sets.

Thus, we can have the following definition.

DEFLNITION 5: BOUNDED SETS

A set S is bounded if it is both bounded above and bounded below.

In other words, S has an upper bound as well as a lower bound. Thus, if S is bounded, then there exist **numbers u** (an upper bound) and v (a lower bound) such that $v \leq x \leq u$, for every $x \in S$.

If a set S is **not** bounded then S is called an unbounded set. Thus S is unbounded if either it is not bounded above or it is not bounded below.

EXAMPLE 3 : (i) Any finite set is bounded.

(ii) The set Q of **rational** numbers is unbounded.

(iii) The set R of real numbers is unbounded

(iv) The set P = {sin x, sin 2x, **sin 3x,......**, sin nx, ...} is bounded because $-1 \leq \sin nx \leq 1$, for every n and x.

EXERCISE 8)

Test which of the fallowing sets are bounded above, bounded below, bounded and unbounded.

(i) The intervals **]a, b[,** [a, b],]a, **b]** and [a, **b[,** where a and b are any two real numbers.

(U) **The intervals** $[2, \infty[$, $]-\infty, 3[$, $]5, \infty[$ and $]-\infty, 4]$.
(iii) **The set** $\{\cos \theta, \cos 2\theta, \cos 3\theta, \ldots\ldots\}$.
(iv) $S = \{x \in R: -a \leq x \leq a\}$ **for some** $a \in R$.

You can easily verify that a subset of a bounded set is always bounded since the bounds of the given set will become the bounds of the subset.

Now consider any two bounded sets say $S = \{1, 2, 5, 7\}$ and $T = \{2, 3, 4, 6, 7, 8\}$. Their union and intersection are given by

$$S \cup T = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

and

$$S \cap T = \{2, 7\}.$$

Obviously $S \cup T$ and $S \cap T$ are both bounded sets. You can prove this assertion in general for any two bounded sets.

EXERCISE 9)
Prove that the union and the intersection of any two bounded sets are bounded.

Now consider the set of negative integers namely

$$S = \{-1, -3, -2, -4, \ldots.\}.$$

You know that -1 is an upper bound of S. Is it the only upper bound of St Can you think of some other upper bound of S? Yes, certainly, you can. What about 0? The number 0 is also an upper bound of S. Rather, any real number greater than -1 is an upper bound of S. You can find infinitely many upper bounds of S. However, you can not find an upper bound less than -1. Thus -1 is the least upper bound of S.

It is quite obvious that if a set **S** is bounded above, then it has an infinite number of upper bounds. Choose the least of these upper bounds. This is called the least upper bound of the set **S** and is known as the Supremum of the set S. (The word 'Supremum' is a Latin word). We formulate the definition of the Supremum of a set in the following way:

DEFINITION 6: THE SUPREMUM OF A SET

Let **S** be a set bounded above. The least of all the upper bounds of S is called the least upper bound or the Supremum of S. Thus, if a set **S** is bounded above, then a real number m is the supremum of S if the following two conditions are satisfied:

(i) m is an upper bound of S,
(ii) if k is another upper bound of S, then $m \leq k$.

EXERCISE 10)

Give an example of an infinite set which is bounded below. Show that it has an infinite number of lower boudds and hence develop the concept of the greatest lawer bound of the set.

The greatest lower bound, in Latin terminology, is called the Infimum of a set.

Let us now discuss a few examples of sets having the supremum and the infimum:

EXAMPLE 4: Each of the intervals $]a, b[$, $[a, b]$ $]a, b]$, $[a, b[$ has both the supremum and the infimum. The number a is the infimum and b is the'supremum in each case. In case of $[a, b]$ the supremum and the infimum both belong to the set whereas this is not the case for the set $]a, b[$. In case of the set $]a, b]$, the infimum does not belong to it and the supremum belongs to it. Similarly, the infimum belongs ta $[a, b[$ but the supremum does not belong to it.

Very often in our discussion, we have used the expressions 'the supremum', rather than a .supremum. What does it mean? It simply means that the supremum of a set, if it exists, is unique i.e. a set can not have more than one supremum. Let us prove it in the form of the following theorem:

THEOREM 1: Prove that the supremum of a set, if it exists, is unique.

PROOF : If possible, let there be two supremums (Suprema) say m and m' of a set S.

Since m is the least upper bound of **S,** therefore by definition, we have

$$m \leq m'.$$

Similarly, since m' the least upper bound of S, therefore, we must have

$$m' \leq m.$$

This shows that $m = m'$ which proves the theorem.
You can now similarly prove the following result:

EXERCISE 11)
Prove that the **infimum** of a set, if it exists, is unique.

In example 3, you have seen that supremum or the infimum **of a** set may or may not belong to the set. If the supremum of a set belongs to the set, then it is called the greatest member of the set. Similarly, if the **infimum** of a set belongs to it, then it is called the least member of the set.

EXAMPLE 5: **(i)** Every **finite** set has the **greatest as well as the least** member.

(ii) **The** set **N has** the least member but not the **greatest.** Determine that number.

(iii) The set of negative **integers** has the **greatest member** but **not** the least member. What is that number?

Try the following exercise:

EXERCISE 12)
Check which of the following sets have the greatest and the least member:

(i) $\{x: a \leq x \leq b\}$.

(ii) $\{x: a < x \leq b\}$.

(iii) $\{x: a \leq x < b\}$.

(iv) $\{x: a < x < b\}$.

(v) $[a, \infty[$, $]a, \infty[$.

(vi) $]-\infty, b]$, $]-\infty, b[$.

You have seen that whenever a set S is bounded above, then **S** has the supremum. In fact. this is true in general. Thus, we have the following property of R without proof:

PROPERTY 3: COMPLETENESS PROPERTY

Every non empty subset S of R which is bounded above, has the supremum.

Similarly, we have

Every non-empty **subset** S of R that is bounded below, has the infimum

In fact, it can be easily shown that the above two statements are equivalent.

Now, if you consider a non-empty subset S of **Q,** then **S** considered'as a subset of R must have, by property 2, a supremum. However, this supremum may not be in Q. This fact is expressed by saying that Q considered as a field in its dwn right is not Order-Complete. We illustrate this observation as follows;

Construct a subset **S** of Q consisting of all those positive rational numbers whose squares are less than 2 i.e.

$$S = \{x \in Q: x > 0, x^2 < 2\}.$$

Since the number $1 \in S$, therefore S is non empty. Also, 2 is an upper **bound of S** because every element of **S** is less than 2. Thus the set S is non-empty and bounded, above. According to the Axiom of Completeness of R, the subset S must have the supremum in R. We claim that this supremum does not belong to Q.

Suppose m is the supremum of the set **S.** If possible, let m belong to Q. Obviously, then $m > 0$. Now either $m^2 < 2$ or $m^2 = 2$ or $m^2 > 2$.

Case (i) When $m^2 < 2$. Then a number y defined as

$$y = \frac{4+3m}{3+2m}$$

is a positive rational number and

$$y - m = \frac{2(2-m^2)}{3+2m}.$$

Since $m^2 < 2$, therefore $2 - m^2 > 0$. Hence

$$y - m = \frac{3(2-m^2)}{3+2m} > 0$$

which implies that $y > m$.

Again,

$$y^2 - 2 = \left(\frac{4+3m}{3+2m}\right)^2 - 2$$

$$= \frac{m^2-2}{(3+2m)^2}.$$

Since $m^2 < 2$, therefore

$$y^2 - 2 < 0 \text{ i.e. } y^2 < 2.$$

This shows that $y \in S$ and also it is greater than m (the supremum of S). This is absurd. Thus the case $m^2 < 2$ is not possible.

Case (ii) When $m^2 = 2$.

This means there exists a rational number whose square is equal to 2 which is again not possible, since you have already proved this in Section 1.3.

Case (iii) When $m^2 > 2$

In this case consider the positive rational number y defined in case (i). Accordingly, we have

$$y - m = \frac{2(2-m^2)}{3+2m} < 0 \text{ (check yourself)}$$

i.e. $y < m$.

Also $2 - y^2 = 2 - \left(\frac{4+3m}{3+2m}\right) 2 = \frac{2-m^2}{(3+2m)^2}$

i.e. $2 - y^2 < 0$ or $y^2 > 2$,

which shows that y is an upper bounh of S.

Thus y is an upper bound of S which does not belong to S. At the same time y is less than the supremum of S. This is again absurd. Thus $m^2 > 2$ is also not possible. Hence none of three possibilities is true. This means there is something wrong with our supposition. In other words, our supposition is false and therefore the set, S does pot possess the suprernum in Q.

This justifies that the field Q of rational numbers is not order-complete.

Now you can also try a similar exercise.

EXERCISE 13)

Let S be a subsdt of all those positive rational numbers whose squares are less than 3 i.e.

$$S = \{x \in Q : x > 0 \text{ and } x^2 < 3\}.$$

Show that S is nonempty and bounded above but it does not have the least upper bound in Q.

2.4 COUNTABILITY

In Section 1.2, we recalled the notion of a set and certain related concepts. Subsequently, we discussed certain properties of the sets of numbers N, Z, Q, R and

George Cantor.

C. A few more important properties and related aspects concerning these sets are yrt to be examined. One such significant aspect is the countability of these sets: The concept of Countability of sets was introduced by George Cantor which, forms a corner stone of Modern Mathematics.

2.4.1 COUNTABLE SETS

You can easily count the elements of a finite set. For example, you very frequently use the term **'one hundred rupees'** or **'fifty boxes', 'two dozen eggs',** etc. These figures pertain to the **number** of elements of a set. Denote the number of elements in a finite set S by n (S). If **S** = {a, b, c, d}, then n (S) = 4. Similarly n (S) = 26, if **S** is the set of the letters of English alphabet. Obviously, then n (ϕ) = 0, where ϕ is the null set.

You can make another interesting observation when you count the number of elements of a finite set. While you are counting these elements, you are indirectly and perhaps unconsciously, using a very important concept of the one-one correspondence between two sets. Recall the concept of **one-one correspondence** from Section 1.2. Here one of the sets is a finite subset of the set of natural numbers and the other set is the set consisting of the articles/objects like rupees, boxes, eggs, etc. Suppose you have a basket of oranges. While counting the oranges, you are associating a natural number to each of the oranges. This, as you know, is a one-one correspondence between'the set of oranges and a subset of natural members. Similarly, when you count the fingers of your hands, you are in fact showing a one-one correspondence between the set of the fingers with a subset, say N_{10} = (1, 2, ... 10) of N.

Although, we have an intuitive idea of finite and infinite sets, yet we give a mathematical definition of these sets in the following way:

DEFINITION 7: FINITE AND INFINITE SETS

A set S is said to be finite if it is empty or if there is a positive integer k such that there is one-one correspondence between the elements of the set S and the set N_K = {1, 2, 3 ...., k]. A set is said to be infinite if it is not finite.

The advantage of using the concept of one-one correspondence is that it helps in studying the countability of infinite sets. Let E = {2, 4, 6, ...} be the set of even natural numbers. If we define a mapping f: **N** $\rightarrow$ **E** as

$$f(n) = 2n, \forall n \in N,$$

then we find that **f** is a one-one correspondence between **N** and E.

Consider another example. Suppose S = {1, 2, ... n) and **T** = {x_1, x_2, ... x_n}. Define a mapping **f:** S $\rightarrow$ **T** as

$$f(n) = x_n \; \forall n \in S.$$

Then again **f** is a one-one correspondence between S and **T.**

Such sets are known as equivalent sets. We define the equivalent sets in the following way:

DEFINITION 8: EQUIVALENT SETS

Any two sets **are equivalent** if **there** is **one-one correspondence between them.**

Thus if two sets S and T are equivalent, we write,as S $\sim$ T

You can easily show that S, **T** and **P** are any three sets such that S $\sim$ **T** and **T** $\sim$ P, then S $\sim$ P.

The notion of the equivalent sets is very important because it forms the basis of the 'counting' of the infinite sets.

Now, consider any two line segments AB and CD.

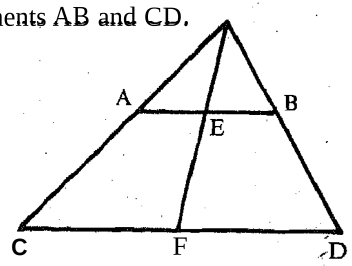


Fig. 4

Let M denote the set of points on AB and N the set of points on CD. Let us check whether M and N are equivalent.

Join CA and DB to meet in the point P. Let a line through P meet AB in E and CD in F. Define $f:M \to N$ as $f(x) = y$ where x is any point on AB and y is any point on CD. The construction shows that f is a one-one correspondence. Thus M and N are equivalent sets.

The following are some examples of equivalent sets : Let I be an interval with end points a and b, and J be an interval with end points c and d. Also, we assume that I and J are intervals of the same type. Define $f: I \to J$, by

$$f(t) = \left(\frac{t-a}{b-a}\right) d + \left(\frac{b-t}{b-a}\right) c, \text{ for } t \in I.$$

Then, it is not difficult to see that f is a one-to-one correspondence between intervals I and J. Hence, all the intervals of same type are equivalent to each another.

Now, we introduce the following definition:

DEFINITION 9: DENUMERABLE AND **COUNTABLE SETS**

A **set which is equivalent to the set of natural numbers is called a denumerable set. Any set which is either finite or denumerable, is called a Countable set.**

Any set which is not countable is said to be an uncountable set.

EXAMPLE *6* : (i) A **mapping $f:Z \to N$ defined by**

$$f(n) = \begin{cases} -2n, & \text{if } n \text{ is a negative integer} \\ 2n+1, & \text{if } n \text{ is non-negative integer} \end{cases}$$

is a one-to-one correspondence. Hence $Z \sim N$. Thus **the set of integers is a denumerable set and hence a countable set.**

(ii) Let **E** denote the set of all even natural numbers. Then the mapping $f:N \to E$ defined as $f(n) = 2n$ is a one-one correspondence. Hence the set E of **even natural numbers is a denumerable set and hence a countable set.**

(iii) Let D denote the set of all odd integers and E the set of even integers. Then the, mapping $f: E \to D$, defined as $f(n) = n + 1$ is a one-one correspondence.Thus $E \sim D$. But, $E \sim N$, therefore $D \sim N$. Hence D is a denumerable set and hence a countable set.

We observe that a set S is denumerable if and only if it is of the form $\{a_1, a_2, a_3, ...\}$ for distinct elements $a_1, a_2, a_3,.....$ For, in this case the mapping $f(a_n) = n$ is one-one mapping of S onto N i.e. the sets $\{a_1, a_2, a_3,.....\}$ and the set N are equivalent.

If we consider the set $S_2 = \{2, 3, 4,.....\}$, we find that the mapping $f:N \to S_2$ defined as $f(n) = n+1$ is one-one and onto. Thus S_2 is denumerable. Similarly if we consider $S_3 = \{3, 4.....\}$ or $S_k = \{k, k+1,......\}$, then we find that all these are denumerable sets and hence are countable sets.

We have seen that the set of integers is countable.

Now' we discuss the countability of the rational and real numbers. Here is an interesting theorem.

THEOREM 2: Every **infinite subset of a denumerable set is denumerable.**

PROOF : Let **S** be a denumerable set. Then S can be written as

$$S = \{a_1, a_2, a_3,.....\}.$$

Let **A** be an infinite subset of S. We want to show that **A** is also denumerable.

You can see that the elements of S are designated by subscripts 1, 2, 3,.... Let n_1 be the smallest subscript for which $a_{n_1} \in A$. Then consider the set $A - \{a_{n_1}\}$. Again, in this new set, let n_2 be the smallest subscript such that $a_{n_2} \in A - \{a_{n_1}\}$.

Let n_k be the smallest subscript such that

$$a_{n_k} \in A - \{a_{n_1}, a_{n_2}, ..., a_{n_{k-1}}\}.$$

Note that such an element a_{n_k} always exists for each $k \in N$ as A is infinite. For, then

$$A - \{a_{n_1}, a_{n_2}, ..., a_{n_k}\} \neq \emptyset$$

for each $k \in N$. Thus, we can write

$$A = \{a_{n_1}, a_{n_2}, a_{n_3}, ..., a_{n_k}, ...\}.$$

Define **f:** $N \rightarrow A$ by $f(k) = a_{n_k}$. Then it can be verified that f is a one-one correspondence. Hence A is denumerable. This completes the proof of the theorem.

EXERCISE 14)
Every subset **of** a countable set is countable.

Now consider the sets $S = \{6, 8, 10, 12, ...\}$ and $T = \{3, 5, 7, 9, 11, ...\}$, which are both denumerable. Their union $S \cup T = \{3, 5, 6, 7, 8, 9, ...\}$ is an infinite subset of N and hence is denumerable. Again, if $S = \{-1, 0, 1, 2\}$ and $T = \{20, 40, 60, 80, ...\}$, then we see that $S \cup T = \{-1, 0, 1, 2, 20, 40, 60, ...\}$ is a denumerable set. Note that in each case $S \cap T = \emptyset$. In fact, you can prove a general result in the following exercise.

EXERCISE 15)

(i) **If S** and **T** arc two denumerable sets, such that $S \cap T = \emptyset$, then $S \cup T$ is denumerable.

(ii) If S is denumerable and T is finite such that $S \cap T = \emptyset$, then also $S \cup T$ is denumerable.

(iii) The condition $S \cap T = \emptyset$ can be relaxed in (i) and (ii).

Thus, it follows that the union of any two countable sets is countable.

Indeed, let S and T be any two countable sets. Then S and T are either finite or denumerable.

If S and T are both finite, then $S \cup T$ is also a finite set and hence $S \cup T$ is countable.

If S is denumerable and T is finite, then also we know that $S \cup T$ is denumerable. Hence $S \cup T$ is countable. Again, if S is finite and T is denumerable, then again $S \cup T$ is denumerable and countable.

Finally, if both S and T are denumerable, then $S \cup T$ is also denumerable1 and hence countable. In fact, this result can be extended to countably many countable sets. We prove this in the following theorem.

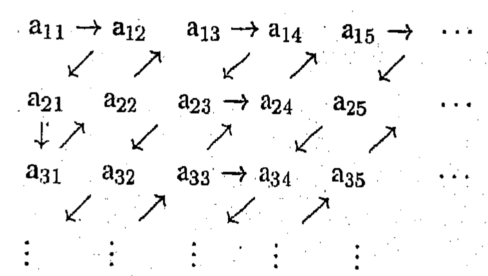
THEOREM 3 : The union **of** a countable number **of** countable sets is countable.

PROOF : Let the given sets be $A_1, A_2, A_3, ...$ Denote the elements of these sets, using double subscripts, as follows:

$$A_1 = \{a_{11}, a_{12}, a_{13}, ...\}$$
$$A_2 = \{a_{21}, a_{22}, a_{23}, ...\}$$
$$A_3 = \{a_{31}, a_{32}, a_{33}, ...\},$$

and so on. Note that the double subscripts have been used for the sake of convenience only. Thus a_{ij} is the jth element in the set A_i. Now let us try to form a single list of all elements of the union of these given sets.

One method of doing this is by using Cantor's diagonalised counting as indicated by arrows in the following table.



Diagonalised Counting of $\bigcup_{i=1}^{\infty} A_i$.

elist the elements as indicated through the arrows. This is a scheme for making a single st of all the elements.

ollowing the arrows in above table, you can easily arrive at the new single list:

$$a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13}, a_{41}, a_{32}, \ldots\ldots$$

ote that while doing so, you must omit the duplicates, if any.

ow, if any of the sets $A_1, A_2, \ldots\ldots$, are finite, then this will merely shorten the final list. hus, we have

$$\cup_i A_i = \cup_i \{a_{i1}, a_{i2}, \ldots\}, \; i = 1, 2, 3, \ldots\ldots$$

which each element appears only once. This set is countable and, so, complete! the oof of the theorem.

/e are now in a position to discuss the countability lof the sets of rational and real umbers.

.4.2 COUNTABILITY OF REAL NUMBERS.

/e have already established that the sets N and Z are countable. Let us now consider e case of the set Q of rational numbers. For this we need the following theorems:

HEOREM 4 : The set of all rational numbers between [0, 1] is countable.

ROOF : Make a systematic scheme in an order for listing the rational numbers x where $\leq x \leq 1$, (without duplicates) of the following sets

$$A_1 = \{0, 1\}$$

$$A_2 = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$$

$$A_3 = \{\frac{2}{3}, \frac{2}{5}, \frac{2}{7} \ldots\ldots\}$$

$$A_4 = \{\frac{3}{4}, \frac{3}{5}, \frac{3}{7}, \frac{3}{8}, \ldots\}$$

....

....

'ou can see that each of the above sets is countable. Their union is given by

$$\cup_i A_i = \{0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \ldots\} = [0, 1] \cap Q,$$

'hich is countable by Theorem 3.

'HEOREM 5 : The set of all positive rational numbers is countable.

ROOF: Let Q_+ denote the set of all positive rational numbers. To prove that Q_+ is ountable, consider the following sets :

$$A_1 = \{1, 2, 3, \ldots\ldots\}$$

$$A_2 = \{\frac{1}{2}, \frac{2}{2}, \frac{5}{2}, \ldots\ldots\}$$

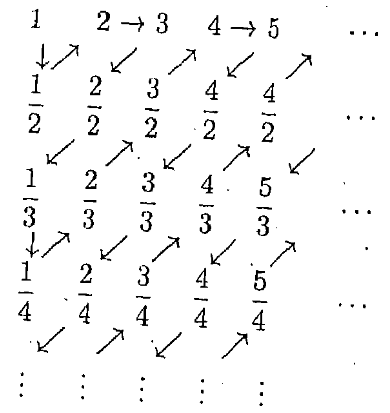
$$A_3 = \{\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \ldots\ldots\}$$

$$A_4 = \{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \ldots\ldots\}$$

....

....

elist the elements of these sets in a manner as you have done in Theorem 3 or as own below:



You may follow the method of indicating by arrows for making a single list or you may follow another path as indicated here. Accordingly, write down the elements of Q_+ as they appear in the figure by the arrows, while omitting those numbers which are already listed to avoid the duplicates. We will have the following list:

$$Q^+ = \{ 1, \frac{1}{2}, 2, 3, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, 4, ...\}$$

$$= \cup_i A_i \quad (i = 1, 2, 3, ...),$$

which is countable by Theorem 3. Thus $\mathbf{Q}_+$ is countable.

Now let $\mathbf{Q}_-$ denote the set of all negative rational numbers. But $\mathbf{Q}_+$ and $\mathbf{Q}_-$ are equivalent sets because there is one-one correspondence between Q_+ and Q_-, $f: \mathbf{Q}_+ \rightarrow \mathbf{Q}_-$, given by

$$f(x) = -x, \ \forall \ x \in \mathbf{Q}_+.$$

Therefore $\mathbf{Q}_-$ is also countable. Further $\{0\}$ being a finite set is countable. Hence,

$$\mathbf{Q} = \mathbf{Q}_+ \cup \{0\} \cup \mathbf{Q}_-$$

is a countable set. Thus, in fact, we have proved the following theorem:

THEOREM 6: The set **Q** of all rational numbers is countable.

You may start thinking that perhaps every finite set is denumerable. This is not true. We have not yet discussed the countability of the set of real numbers or of the set of irrational numbers. To do so, we first discuss the countability of the set of real numbers in an interval with end points 0 and I, which may be closed or open or semi-closed.

Consider the real numbers in the interval]0, 1[.

Each real number in]0, 1[can be expressed in the decimal expansion. This expansion may be non-terminating or may be terminating e.g.

$$\frac{1}{3} = .333 ...$$

is an example of non-terminating decimal expansion, whereas

$$\frac{1}{4} = .25, \ \frac{1}{2} = .5, ...,$$

are terminating decimal expansions. Even the terminating expansion can also be expressed as non-terminating expansion in the sense that you can write

$$\frac{1}{4} = .25 = .24999 ....$$

Thus, we agree to say that each real number (rational of irrational) in the]0, 1[can be expressed as a non-terminating decimal expansion in terms of the digits from 0 to **9.**

Suppose $x \in]0, 1[$. Then it can be written as

$$x = .c_1c_2c_3 .....$$

where c_1, c_2 ,.... take their values from the set {0, 1, 2, **3, 4, 5, 6, 7,** 8, 9} of ten digits. Similarly, let y be another real number in (0, 1). Then y can also be expressed as

$$y = .d_1d_2d_3.....$$

Ve say that $x = y$ iff the digits in their corresponding position in the expansions of x nd y are identical. Thus, if there is even a single decimal places, say, 10th place such nat $d_{10} \neq c_{10}$, then $x \neq y$.

Ve now discuss the following result due to George Cantor.

HEOREM 7: The set of real numbers in the interval]0, 1[is not countable.

ROOF : Since the set of numbers in]0, 1[is an infinite set, therefore, it is enough to how that the set of real numbers in]0, 1[is not denumerable.

f possible, suppose the set of real numbers in]0, 1[is denumerable. Then there is a ne-one correspondence between N and the elements of]0, 1[i.e. there is a functio : $N \to$]0, 1[which is one-one and onto. Thus, if

$f(1) = x_1, f(2) = x_2, \ldots\ldots, f(k) = x_k, \ldots$, then
]0, 1[= $\{x_1, x_2, \ldots, x_k, \ldots\}$.

Ve shall show that there is at least one real number]0, 1[which is not an image of any lement of N under f. In other words, there is an element of]0, 1[which is not in the list $x_1, x_2 \ldots$

et $x_1, x_2, \ldots$ be written as

$$x_1 = 0.\ a_{11}\ a_{12}\ a_{13}\ a_{14} \ldots$$
$$x_2 = 0.\ a_{21}\ a_{22}\ a_{23}\ a_{24} \ldots$$
$$x_3 = 0.\ a_{31}\ a_{32}\ a_{33}\ a_{34} \ldots$$
$$x_4 = 0.\ a_{41}\ a_{42}\ a_{43}\ a_{44} \ldots$$

....
....
....

rom this we construct a real number

$$z = .b_1\ b_2\ b_3\ b_4 \ldots,$$

vhere $b_1, b_2, \ldots$ can take any digits from {0, 1, 2,......., 9} in such a way that $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, $b_3 = a_{33}$, ... Thus

$$z = .b_1\ b_2\ b_3 \ldots$$

s a real number in]0, 1[such that $z \neq x_1$ because $b_1 \neq a_{11}$, $z \neq x_2$ because $b_2 \neq a_{22}$. In eneral $z \neq x_n$ because $a_{nn} \neq b_n$. Therefore z is not in the list $\{x_1, x_2, x_3, \ldots\}$.

Ience]0, 1[is not countable.

Ve have already mentioned that the intervals [0, 1], [0, 1[,]0, 1] and]0, 1[are equivalent ets. Since the set of real numbers in]0, 1[is not countable, therefore none of the ntervals is a countable set of real numbers.

Iow you can easily conclude that the set of irrational numbers in]0, 1[is not countable. f possible, suppose that the set of irrational numbers in]0, 1[is countable. Also you now that the set of rational numbers in]0, 1[is countable and that the union of two ountable sets is countable. Therefore, the union of the set of rational numbers and the et of irrational numbers]0, 1[is countable i.e. the set of all real numbers in]0, 1[is ountable which by above theorem is not so. Hence the set of irrational numbers in]0, 1[s not countable.

n fact, every interval]a, b[or [a, b],]a, b], [a, b[is an uncountable set of real numbers.

Iow what about the countability of the set **R** of real numbers?

uppose that **R** is countable. Then an interval]0, 1[, being an infinite subset of **R**, must e countable. But then, we have already proved that the set]0, 1[is not countable. Ience **R** can not be countable.

'hus even by the method of countability of sets, we have **established the much desired istinction between Q and R in the sense that Q is countable but R is not countable.**

Also, we observe that although **R** is not countable, yet it contains subsets which are countable. For example **R** has subsets as **Q**, **Z** and **N** which are countable. At the same time **R** is an infinite set. We sum upthis observation in the form of the following theorem:

THEOREM 8 : Every infinite set contains a denumerable set.

PROOF : Let S be an infinite set. Consider some element of S. Denote it by n_1. Consider the set $S - \{a_1\}$. Now pick up an element from the new set and denote it by a_2.

Consider the set

$$S - \{a_1, a_k\}.$$

Proceeding in this way, having chosen a_{k-1}, you can have the set

$$S - \{a_1, a_2, ... a_{k-1}\}.$$

This set is always non-empty because **S** is an infinite set. Hence, we can choose an element in this set. Denote the element by a_k. This can be done for each $k \in N$. Thus the set

$$\{a_1, a_2, ..., a_k ....\}$$

is a denumerable subset of S and hence a countable subset of S. This proves the theorem.

The importance of this theorem is that it leads us to an interesting area of Cardinality of sets by which we can determine and compare the relative sizes of various infinite sets.

This, however, is beyond the scope of this course.

2.5 SUMMARY

In Section 2.2, we have discussed the order-relations (inequalities) in the set **R** of real numbers. Given any two real numbers x and y, either $x > y$ or $x = y$ or $x < y$.

This is known as the law of Trichotomy. Then we have stated a few properties with respect to the inequality '$\leq$'. The first property states that the inequality $\leq$ is antisymmetric. The second states the transitivity of $\leq$. The third allows us to add or subtract across the inequality, while preserving the inequality. The last property gives the situation in which the inequality is preserved if multiplied by a positive real number, while it is reversed if multiplied by a negative real number.

We have also defined the bounded and unbounded intervals. The bounded intervals are classified as open intervals, closed intervals, semi-openor semi-closed intervals. The unbounded intervals are introduced with the help of the extended real number system $\mathbf{R} \cup \{-\infty, \infty)$ using the symbols $+\infty$ (called plus infinity) and $-\infty$ (called minus infinity).

Section 2.3 deals with three important aspects of the real numbers: algebraic, order and the completeness. To describe these aspects, we have specified a number of axioms in each case. In the algebraic aspect, an algebraic structure,called the field is used. A field is a non-empty set F having at least two distinct elements 0 and 1 together with two binary operations + (addition) and . (multiplication) defined on F such that both + and . are commulative, associative, 0 is the additive identity, 1 is the multiplicative identity, additive inverse exists for each element of F, multiplicative inverse exists for each element other than 0 and multiplication is distributive over addition. The second aspect is concerned with the Order Structure in which, we use the axioms of the law of trichotomy, the transitivity property, the property that preserve the inequality under addition and the property that preserve the inequality under multiplication by a positive real number.

In the completeness aspect, we introduce the notions of the supremum (or infimum) of a set and state the axiom of completeness. We find that both **Q** and **R** are ordered fields but the axioms of completeness distinguishes **Q** from **R** in the sense that **Q** does not satisfy the axiom of completeness. Thus, we conclude that R is a complete-ordered Field whereas **Q** is not a complete-ordered field.

Finally in Section 2.4, we introduce the notion of the countability of sets. A set is said to be denumerable if it is in one-one correspondence with the set of natural numbers. Any set which is either finite or denumerable is called a countable set. We have shown that the sets **N, Z Q** are countable sets but the sets I (set of irrational numbers) and **R** are not countable.

Thus in this unit, we have discussed the algebraic structure, the order structure and the countability of the real numbers.

2.6 ANSWERS/HINTS/SOLUTIONS

E1) Change $\leq$ into $\geq$ in the Property 1.
Choose x, y and z to describe this property.
If $x \geq y$ and $z < 0$, then $x\,z \leq y\,z$.

E2) Take (2, 5) and (7, 12) as the intervals. Then $(2, 5) \cap (7, 12) = \emptyset$ which is not an interval. However, if you take (2, 5) and (3, 6) as the intervals, then

$$(2, 5) \cap (3, 6) = (3, 5)$$

which is an interval. Note that the intervals (2, 5) and (7, 12) are disjoint but (2, 5) and (3, 6) are not disjoint. Thus you may conclude that the intersection of the intervals is an intervals provided the intervals are not disjoint.

E3) Verify that all the axioms of a field are satisfied by the elements of the set {0, 1} with respect to binary operation + and . as defined in the given tables.

E4) Suppose there are two zeros of a Field F namely 0 and 0′. Then by definition we have

$$0 + x = x, \forall\, x \in F$$

In particular, if $x = 0'$, then we get

$$0 + 0' = 0'$$

Again by definition we have

$$0' + x = x, \forall\, x \in F$$

Choose $x = 0$. Then we get

$$0' + 0 = 0.$$

It follows that $0 = 0'$.
Similarly you can prove the uniqueness of the unity.

E5) The set **N** does not form a field because its elements do not satisfy the axiom of additive inverse.

The set **Z** is not a field because the axiom of multiplicative inverse does not hold for Z.

The set $\mathbf{C} = \{z = x + iy: x \in \mathbf{R}, y \in \mathbf{R}\}$ forms a field under + and . defined as $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = [(x_1 + x_2) + i\,(y_1 + y_2)] \Rightarrow z_1 + z_2 = \in \mathbf{C}$ for any $z_1, z_2 \in \mathbf{C}$.

Again $z_1 . z_2 = (z_1 + y_1). (x_2 + iy_2)$
$= (x_1x_2 - y_1y_2) + i(x_1\,y_2 + y_2\,y_1)$

Now you can verify that all the axioms of a field are satisfied.

E6) The set C of complex numbers is a field but is not an ordered field, because order cannot be defined on C. Give an illustration. In Unit 1, we have already shown that the number $i = \sqrt{-1}$ is neither positive nor negative and also $i \neq 0$, because it is not a real number.

E7) (i) A set **S** ($\mathbf{S} \subset \mathbf{R}$) is said to be bounded below if it has a lower bound. A number $v \in \mathbf{S}$ is said to be a lower bound of **S** if
$v \leq x, \;\; \forall \in S.$

(ii) Let **S** = **N**. Then N is bounded below. The number 1 is a lower bound of N. Any finite set is bounded below. You can name a lower bound of this set depending upon the choice of the set.

(iii) No. Because it has no lower bound.

E8) (i) All are bounded sets.

(ii) [2, ∞] and]5, ∞[are **bounded below** with 2 and 5 as their respective lower bounds, whereas]−∞ ,3[and]−∞ ,4] are **bounded above,** with 3 and 4 as their respective upper bounds. Therefore, all the sets in this case are unbounded.

(iii) It is a bounded set with lower bound -1 and upper bound +1.

(iv) S = [−a, a], is a bounded set.

E9) Let **S** and **T** be any two bounded sets. Then by using definition of a bounded set, you can have the following: **S** is bounded means **S** has both a lower bound and an upper bound i.e. there exist v_1 (lower bound) and u_1 (upper bound) such that

$$v_1 \leq x \leq u_1, \forall x \in S.$$

Similarly, since **T** is bounded, there exists v_2 and u_2 such that

$$v_2 \leq x \leq u_2, \forall x \in \mathbf{T}.$$

Now you know that

$$x \in S \cup \mathbf{T} \Rightarrow x \in \mathbf{S} \text{ or } x \in \mathbf{T}$$
$$\Rightarrow v_1 \leq x \leq u_1 \text{ or } v_2 \leq x \leq u_2$$

Choose v = minimum of (v_1,v_2), and u = maximum of (u_1, u_2)..
Then

$$x \in \mathbf{S} \cup \mathbf{T} \Rightarrow v \leq x \leq u$$

$\Rightarrow$ **S** ∪ **T** is a bounded set, because x is an arbitraty element of **S** ∪ **T**.

As an illustration of this example Let **S** = (1, 5) and **T** = (2, 7). Obviously both **S** and **T** are bounded because both are open intervals i.e.

$$S = \{x: 1 \leq x \leq 5\}, T = \{x: 2 \leq x \leq 7\}$$

Obviously, then

$$S \cup T = \{x: 1 \leq x \leq 7\}$$

which is a bounded set.

Similarly, if you take the intersection of **S** and **T**, then you will have

$$S \cap T = \{x: 2 \leq x \leq 5\}$$

which is obviously a bounded set. Note that 2 is the maximum of the two lower bounds and 5 is the minimum of the upper bonds of **S** and **T**
You can similarly, prove that the intersection of any two bounded sets is a bounded set.

E10) The set **N** is bounded below only. The number 0 and negative integers are all lower bounds of **N** i.e. all the non-positive integers are lower bounds of **N.** Complete the solution.

E11) Proof is exactly similar to the proof for the uniqueness of the supremum. Do it yourself.

E12) (i) has both greatest and least

(ii) has the greatest

(iii) has the least

(iv) has none

(v) a as the least

(iv) b as the greatest.

E13) The set S is obviously non-empty and bounded above. We claim **S** has no least upper bound in **Q.**
If possible, suppose u is the least upper bound of **S** in **Q.**
Then either $u^2 < 3$ or $u^2 > 3$ or $u^2 = 3$.

(j) Suppose $u^2 < 3$. Define a rational number y as

$$y = u + \frac{1}{7}(3 - u^2).$$

Then it can be verified that $y > u$ and $y^2 < 3$. This shows that there exists a rational number y which belongs to S and is greater than the least upper bound of S. This is absurd. Hence $u^2 < 3$ is not possible.

(ii) Now suppose $u^2 > 3$. Define a rational number z as

$$z = \frac{u^2 + 3}{2u}.$$

Then it can be verified that $z < u$ and z is an upper bound of S which is again a contradiction. Thus $u^2 < 3$ is also not possible.

(iii) Finally suppose $u^2 = 3$. This means there exists a rational number whose square is 3, which is not possible.

E14) Suppose S is a countable set. Then either S is finite or S is denumerable. Let A be a subset of S..
If S is finite, then A is also finite and hence A is countable.
If S is denumerable, then **A** is also denumerable as proved in Theorem 2. Thus A is also countable. This completes the proof.

E15) (i) Let $S = \{a_1, a_2, a_3, ...\}$ and $\mathbf{T} = \{b_1, b_2, ...\}$ be any two denumerable sets such that $S \cap T = \varnothing$.
Define a function $\mathbf{f}: \mathbf{S} \cup \mathbf{T} \rightarrow \mathbf{N}$ by

$f(a_n) = 2n$, $f(b_n) = 2n-1$.

Then f is a one-one correspondence. Hence $S \cup T$ is denumerable.

Alternatively, you can actually list the elements of $S \cup T$ as $a_1, b_1, a_2, b_2, a_3, b_3$ ...., which is obviously a denumerable set.

In case $\mathbf{S} \cap \mathbf{T} \neq \varnothing$ i.e. **S** and **T** have some elements in common, then the duplicates of any element already listed would simply be omitted when the same element is encountered again in the combined list.

(ii) Now let $\mathbf{S} = \{a_1, a_2, ...\}$ and $\mathbf{T} = \{b_1, b_2, ... b_k\}$ (a finite set) by any two sets. Define $\mathbf{f}: \mathbf{S} \cup \mathbf{T} \rightarrow \mathbf{N}$ by

$f(b_i) = i$, for $1 \leq i \leq k$, and

$f(a_n) = n + k$, for each n.

Then, you can verify that f is one-one and onto i.e. f is a one-one correspondence. Hence $\mathbf{S} \cup \mathbf{T}$ is denumerable.

(iii) You may note that since.
$\mathbf{S} \cup \mathbf{T} = (\mathbf{S} - \mathbf{T}) \cup (\mathbf{T} - \mathbf{S}) \cup (\mathbf{S} \cap \mathbf{T})$, therefore, indeed, we can relax the condition $\mathbf{S} \cap \mathbf{T} = \varnothing$ in both the cases (i) and (ii).