UNIT 5 NORMAL SUBGROUPS

Structure

5.1 INTRODUCTION

In Block 1 you studied subgroups and cosets. We start this unit by discussing a special class of subgroups, called normal subgroups. You will see that the cosets of such a subgroup form a group with respect to a suitably defined operation. These groups are called quotient groups. We will discuss them in some detail in Sec. 5.3.

Once you are comfortable with normal subgroups and quotient groups, you will find it easier to understand the concepts and results that are presented in the next unit. So, make sure that you have met the following objectives before going to the next unit.

Objectives

After reading this unit, you should be able to

- verify whetlier a subgroup is normal or not,
- obtain a quotient group corresponding to a given normal subgroup.

5.2 NORMAL SUBGROUPS

In E 1 of Unit 4 you saw that a left coset of a subgroup H, aH, need not be the same as the right coset Ha. But, there are certain subgroups for which the right and left cosets represented by the same element coincide. This type of subgroup is very important in group theory, and we give it a special name.

Definition : A subgroup N of a group G is called a **normal** subgroup of G if $Nx = xN \ \forall \ x \in G$, and we write this as $N \trianglelefteq G$.

For example, any group G has two normal subgroups, namely, {e} and G itself. Can you see why? Well, $\{e\}x = \{x\} = x\{e\}$, for any $x \in G$, and $Gx = G = xG$, for any $x \in G$.

Let us consider another example.

Example 1 : Show that every subgroup of Z is normal in Z.

Solution : From Example 4 of Unit 3, you know that if H is a subgroup of Z, then $H = mZ$, for some $m \in Z$. Now, for any $z \in Z$,

$H + z = \{..., -3m + z, -2m + z, -m + z, z, m + z, 2m + z, ...\}$

$= \{..., z - 3m, z - 2m, z - m, z, z + m, z + 2m, ...\}$ (since + is commutative)

$= z + H.$

$\therefore H \trianglelefteq Z.$

Example 1 is a special case of the fact that every subgroup of a commutative group is a normal subgroup. We will prove this fact later (in Theorem 2).

Try the following exercise now.

E 1) Show that $A_3 \trianglelefteq S_3$ (see Example 3 of Unit 4).

Let us now prove a result that gives equivalent conditions for a subgroup to be normal.

$g^{-1}Hg = \{g^{-1}hg \mid h \in H\}$

Theorem 1 : Let H be a subgroup of a group G. The following statements are equivalent.

a) H is normal in G.

b) $g^{-1}Hg \subseteq H \ \forall\ g \in G$.

c) $g^{-1}Hg = H \ \forall\ g \in G$.

Proof: We will show that (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (a). This will show that the three statements are equivalent.

(a) $\Longrightarrow$ (b): Since (a) is true, $Hg = gH \ \forall\ g \in G$. We want to prove (b). For this, consider $g^{-1}Hg$ for $g \in G$. Let $g^{-1}hg \in g^{-1}Hg$.

Since $hg \in Hg = gH$, $\exists\ h_1 \in H$ such that $hg = gh_1$.

$\therefore g^{-1}hg = g^{-1}gh_1 = h_1 \in H$

$\therefore$ (b) holds.

(b) $\Longrightarrow$ (c): Now, we know that (b) holds, i.e., for $g \in G$, $g^{-1}Hg \subseteq H$. We want to show that $H \subseteq g^{-1}Hg$. Let $h \in H$. Then

$h = ehe = (g^{-1}g)\,h\,(g^{-1}g)$

$= g^{-1}(ghg^{-1})g$

$= g^{-1}\{(g^{-1})^{-1}hg^{-1}\}g \in g^{-1}Hg$, since $(g^{-1})^{-1}hg^{-1} \in (g^{-1})^{-1}H(g^{-1}) \subseteq H$.

$\therefore H \subseteq g^{-1}Hg$.

$\therefore g^{-1}Hg = H \ \forall\ g \in G$.

$Ha = Hb \Longleftrightarrow Hac = Hbc$ for any $a, b, c \in G$.

(c) $\Longrightarrow$ (a): For any $g \in G$, we know that $g^{-1}Hg = H$.

$\therefore g(g^{-1}Hg) = gH$, that a, $Hg = gH$.

$\therefore H \trianglelefteq G$, that is, (a) holds.

We would like to make the following remark about Theorem 1.

Remark : Theorem 1 says that $H \trianglelefteq G \Longleftrightarrow g^{-1}Hg = H \ \forall\ g \in G$. **This** does **not** mean that $g^{-1}hg = h \ \forall\ h \in H$ and $g \in G$.

For example, in E 1 you have shown that $A_3 \trianglelefteq S_3$. Therefore, by Theorem 1, $(1\ 2)^{-1}A_3(1\ 2) = A_3$. But, $(1\ 2)^{-1}(1\ 3\ 2)(1\ 2) \neq (1\ 3\ 2)$. In fact, it is $(1\ 2\ 3)$.

Try the following exercise now.

E 2) Consider the subgroup $SL_2(\mathbf{R}) = \{A \in GL_2(\mathbf{R}) \mid \det(A) = 1\}$ of $GL_2(\mathbf{R})$ (see Example 5 of Unit 2). Using the facts that

$$\det(AB) = \det(A)\det(B) \text{ and } \det(A^{-1}) = \frac{1}{\det(A)},$$

prove that $SL_2(\mathbf{R}) \trianglelefteq GL_2(\mathbf{R})$.

We now prove a simple result that we stated after Example 1. It is actually a corollary to Theorem 1.

Theorem 2 : Every subgroup of a commutative group is normal.

Proof : Let G be an abelian group, and $H \leq G$. For any $g \in G$ and $h \in H$, $g^{-1}hg = (g^{-1}g)h = h \in H$. $\therefore g^{-1}Hg \subseteq H$. Thus, $H \trianglelefteq G$.

Theorem 2 says that if G is abelian, then all its subgroups are normal. Unfortunately, the converse of this is not true. That is, there are non-commutative groups whose subgroups are all normal. We will give you an example after doing Theorem 3. Let us first look at another example of a normal subgroup.

Example 2 : Consider the Klein 4-group, K_4, given in Example 7 of Unit 3. Show that both its subgroups $\langle a \rangle$ and $\langle b \rangle$ are normal.

Solution : Consider the table of the operation given in Example 7 of Unit **3.** Note that a and b are of order 2. Therefore, $a = a^{-1}$ and $b = b^{-1}$. Also note that $ba = ab$.

Now, let $H = \langle a \rangle = \{e, a\}$. We will check that $H \trianglelefteq K_4$, that is, $g^{-1}hg \in H \ \forall\ g \in K_4$ and $h \in H$.

Now, $g^{-1}eg = e \in H \ \forall\ g \in K_4$.

Further, $e^{-1}ae = a \in H$, $a^{-1}aa = a \in H$, $b^{-1}ab = bab = a \in H$ and $(ab)^{-1}a(ab) = b^{-1}(a^{-1}aa)b = bab = a \in H$.

$\therefore\ H \trianglelefteq K_4$.

By a similar proof we can show that $\langle b \rangle \trianglelefteq K_4$.

In Example 2, both $\langle a \rangle$ and $\langle b \rangle$ are of index 2 in K_4. We have the following result about such subgroups.

Theorem **3** : Every subgroup of a group G of index **2** is normal in G.

Proof : Let $N \leq G$ such that $|G : N| = 2$. Let the two right cosets of N be N and Nx, and the two left cosets be N and yN.

Now, $G = N \cup yN$, and $x \in G$. $\therefore$ $x \in N$ or $x \in yN$.
Since $N \cap Nx = \phi$, $x \notin N$. $\therefore$ $x \in yN$. $\therefore$ $xN = yN$.
To show that $N \trianglelefteq G$, we need to show that $Nx = xN$.
Now, for any $n \in N$, $nx \in G = N \cup xN$. Therefore, $nx \in N$ or $nx \in xN$.
But $nx \notin N$, since $x \notin N$. $\therefore$ $nx \in xN$.

Thus, $Nx \subseteq xN$.

By a similar argument we can show that $xN \subseteq Nx$.

$\therefore$ $Nx = xN$, and $N \trianglelefteq G$.

We will use this theorem in Unit 7 to show that, for any $n \geq 2$, the alternating group A_n is a normal subgroup of S_n.

In fact, if you go back to the end of Sec. 4.3, you can see that $A_4 \trianglelefteq S_4$, since Lagrange's theorem implies that

$$|S_4 : A_4| = \frac{o(S_4)}{o(A_4)} = \frac{4!}{12} = 2.$$

Now let us look at an example to show that the converse of Theorem 2 is not true.

Consider the quaternion group Q_8, which we discussed in Example 4 of Unit 4. It has the following 6 subgroups:

$H_0 = \{I\}$, $H_1 = \{I, -I\}$, $H_2 = \{I, -I, A, -A\}$, $H_3 = \{I, -I, B, -B\}$,
$H_4 = \{I, -I, C, -C\}$, $H_5 = Q_8$.

You know that H_0 and H_5 are normal in Q_8. Using Theorem **3,** you can see that H_2, H_3 and H_4 are normal in Q_8.

By actual multiplication you can see that

$g^{-1}H_1g \subseteq H_1 \ \forall\ g \in Q_8$. $\therefore$ $H_1 \trianglelefteq Q_8$.

Therefore, all the subgroups of Q_8 are normal.
But, you know that Q_8 is non-abelian (for instance, $AB = -BA$).

So far we'have given examples of normal subgroups. Let us look at an example of a subgroup that isn't normal.

Example 3 : Show that the subgroup $\langle (1\ 2) \rangle$ of S_3 is not normal.

Solution : We have to find $g \in S_3$ such that $g^{-1}(1\ 2)g \notin \langle (1\ 2) \rangle$.

Let us try $g = (1\ 2\ 3)$.
Then, $g^{-1}(1\ 2)g = (3\ 2\ 1)\ (1\ 2)\ (1\ 2\ 3)$
$= (3\ 2\ 1)\ (2\ 3) = (1\ 3) \notin \langle (1\ 2) \rangle$
Therefore, $\langle (1\ 2) \rangle$ is not normal in S_3.

Try the following exercises now.

E 3) Cdnsider the group of all 2 X 2 diagonal matrices over $\mathbf{R}^*$, with respect to multiplication. How many of its subgroups are normal?

E 4) Show that $Z(G)$, the centre of G, is normal in G. (Remember that $Z(G) = \{x \in G \mid xg = gx \ \forall \ g \in G\}$.)

E 5) Show that $\langle (2\ 3) \rangle$ is not normal in S_3.

In Unit 3 we proved that if $H \leq G$ and $K \leq H$, then $K \leq G$. That is, '$\leq$' is a transitive relation. But '$\trianglelefteq$' is not a transitive relation. That is, if $H \trianglelefteq N$ and $N \trianglelefteq G$, it is not necessary that $H \trianglelefteq G$. We'll give you an example in Unit 7. But, corresponding to the property of subgroups given in Theorem 4 of Unit 3, we have the following result.

Theorem 4 : Let H and K be normal subgroups of a group G. Then $H \cap K \trianglelefteq G$.

Proof : From Theorem 4 of Unit 3, you know that $H \cap K \leq G$. We have to show that $g^{-1}xg \in H \cap K \ \forall \ x \in H \cap K$ and $g \in G$.

Now, let $x \in H \cap K$ and $g \in G$. Then $x \in H$ and $H \trianglelefteq G$. $\therefore g^{-1}xg \in H$.
Similarly, $g^{-1}xg \in K$. $\therefore g^{-1}xg \in H \cap K$.
Thus, $H \cap K \trianglelefteq G$.

In the following exercise we ask you to prove an important property of normal subgroups.

E 6) a) Prove that if $H \trianglelefteq G$ and $K \leq G$, then $HK \leq G$.
(**Hint** : Use Theorem 5 of Unit 3.)

b) Prove that if $H \trianglelefteq G$, $K \trianglelefteq G$, then $HK \trianglelefteq G$.

Now consider an important group which is the product of two subgroups, of which only one is normal.

Example 4 : Let G be the group generated by
$\{x, y \mid x^2 = e, y^4 = e, xy = y^{-1}x\}$.

Let $H = \langle x \rangle$ and $K = \langle y \rangle$.
Then show that $K \trianglelefteq G$, $H \ntrianglelefteq G$ and $G = HK$.

'$\ntrianglelefteq$' denotes 'is not a normal subgroup of'.

Solution : Note that the elements of G are of the form x^iy^j, where $i = 0, 1$ and $j = 0, 1, 2, 3$
$\therefore G = \{e, x, xy, xy^2, xy^3, y, y^2, y^3\}$.
$\therefore |G : K| = 2$. Thus, by Theorem 3, $K \trianglelefteq G$.

Note that we can't apply Theorem 2, since G is non-abelian (as $xy = y^{-1}x$ and $y \neq y^{-1}$).

Now let us see if $H \trianglelefteq G$.

Consider $y^{-1}xy$. Now $y^{-1}xy = xy^2$, because $y^{-1}x = xy$.

If $xy^2 \in H$, then $xy^2 = e$ or $xy^2 = x$. (Remember $o(x) = 2$, so that $x^{-1} = x$.)
Now, $xy^2 = e \Rightarrow y^2 = x^{-1} = x$
$\Rightarrow y^3 = xy = y^{-1}x$
$\Rightarrow y^4 = x$
$\Rightarrow e = x$, a contradiction..
Again $xy^2 = x \Rightarrow y^2 = e$, a contradiction.
$\therefore y^{-1}xy = xy^2 \notin H$, and hence, $H \ntrianglelefteq G$.

Finally, from the definition of G you see that $G = HK$.

The group G is of order 8 and is called the **dihedral group, D_8**. It is the group of symmetries of a square, that is, its elements represent the different ways in which two copies of a square can be placed so that one covers the other. A geometric interpretation of its generators is the following (see Fig. 1) :

Take y to be a rotation of the Euclidean plane about the origin through $\frac{\pi}{2}$, and x the reflection about the vertical axis.

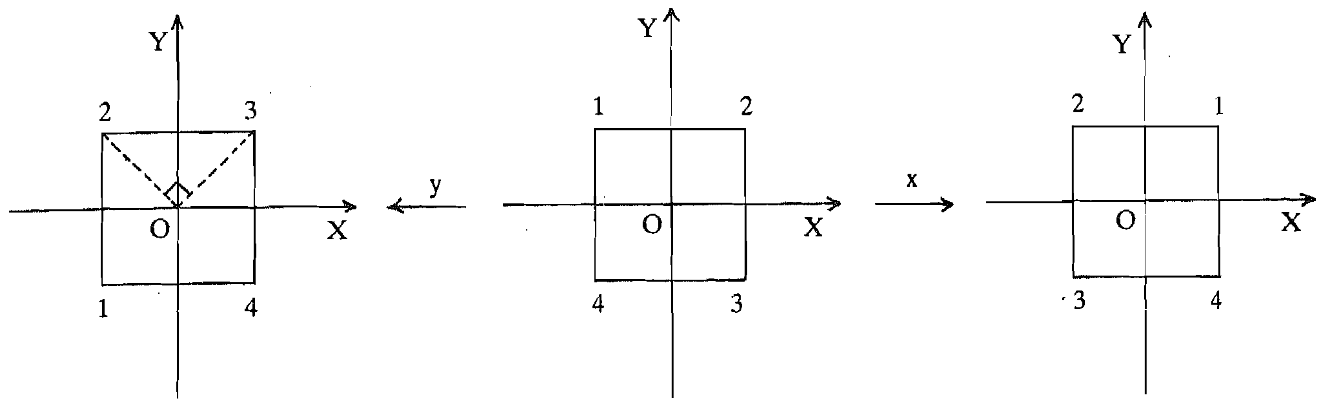

Fig. 1 : Geometric representation of the generators of D_8

We can generalise D_8 to the dihedral group
$D_{2n} = < \{x, y \mid x^2 = e, y^n = e, xy = y^{-1}x\} >$, for $n > 2$.

Try the following exercise now.

E 7) Describe D_6 and give its geometric interpretation.

Let us now utilise normal subgroups to form new algebraic structures.

5.3 QUOTIENT GROUPS

In this section we will use a property of normal subgroups to create a new group. This group is analogous to the concept of quotient spaces given in the Linear Algebra course.

Let H be a normal subgroup of a group G. Then $gH = Hg$ for every $g \in G$. Consider the collection of all cosets of H in G. (Note that since $H \trianglelefteq G$, we need not write 'left coset' or 'right coset; simply 'coset' is enough.) We denote this set by G/H. Now, for $x, y \in H$, we have

$(Hx)(Hy) = H(xH)y$, using associativity,
$= HHxy$, using normality of H,
$= Hxy$, since $HH = H$ because H is a subgroup.

Now, we define the product of two cosets Hx and Hy and G/H by
$(Hx)(Hy) = Hxy$ for all x, y in G.

Our definition seems to depend on the way in which we represent a coset. Let us explain this. Suppose C_1 and C_2 are two cosets, say $C_1 = Hx$ and $C_2 = Hy$. Then $C_1C_2 = Hxy$. But C_1 and C_2 can be written in the form Hx and Hy in several ways. So, you may ask : Does C_1C_2 depend on the particular way of writing C_1 and C_2?

In other words, if $C_1 = Hx = Hx_1$ and $C_2 = Hy = Hy_1$, then is $C_1C_2 = Hxy$ or is $C_1C_2 = Hx_1y_1$? Actually, we will show you that $Hxy = Hx_1y_1$, that is, the **product of cosets is well-defined.**

Since $Hx = Hx_1$ and $Hy = Hy_1$, $xx_1^{-1} \in H$, $yy_1^{-1} \in H$.

$\therefore (xy)(x_1y_1)^{-1} = (xy)(y_1^{-1}x_1^{-1}) = x(yy_1^{-1})x_1^{-1}$

$= x(yy_1^{-1})x^{-1}(xx_1^{-1}) \in H$, since $xx_1^{-1} \in H$ and $H \trianglelefteq G$

i.e., $(xy)(x_1y_1)^{-1} \in H$.
$\therefore Hxy = Hx_1y_1$.

So, we have shown you that multiplication is a well-defined binary operation on G/H.

We will now show that $(G/H,.)$ is a group.

Theorem 5 : Let H be a normal subgroup of a group G and G/H denote the set of all cosets of H in G. Then G/H becomes a group under multiplication defined by $Hx.Hy = Hxy$, $x, y \in G$. The coset $H = He$ is the identity of G/H and the inverse of Hx is the coset Hx^{-1}.

Proof: We have already observed that the product of two cosets is a coset.

This multiplication is also associative, since

$$((Hx)(Hy))(Hz) = (Hxy)(Hz)$$
$$= Hxyz, \text{ as the product in } G \text{ is associative,}$$
$$= Hx(yz)$$
$$= (Hx)(Hyz)$$
$$= (Hx)((Hy)(Hz)) \text{ for } x, y, z \in G.$$

Now, if e is the identity of G, then $Hx.He = Hxe = Hx$ and $He.Hx = Hex = Hx$ for every $x \in G$. Thus, $He = H$ is the identity element of G/H.

Also, for any $x \in G$, $Hx\,Hx^{-1} = Hxx^{-1} = He = Hx^{-1}x = Hx^{-1}.Hx$.

Thus, the inverse of Hx is Hx^{-1}.

So, we have proved that G/H, the set of all cosets of a normal subgroup H in G, forms a group with respect to the multiplication defined by $Hx.Hy = Hxy$. Phis group is called the **quotient group** (or **factor group)** of G by H.

Note that the order of the quotient group G/H is the index of H in G. Thus, by Lagrange's theorem you know that if G is a finite group, then

$$o(G/H) = \frac{o(G)}{o(H)}$$

Also note that if $(G, +)$ is an abelian group and $H \leq G$, then $H \trianglelefteq G$. Further, the operation on G/H is defined by $(H + x) + (H + y) = H + (x + y)$.

Let us look at a few examples of quotient groups.

Example 5 : Obtain the group G/H, where $G = S_3$ and $H = A_3 = \{I, (1\ 2\ 3), (1\ 3\ 2)\}$.

Solution : Firstly, note that $A_3 \trianglelefteq S_3$, since $|S_3 : A_3| = 2$.

From Example 3 of Unit 4 you know that G/H is a group of order 2 whose elements are H and (1 2) H.

Example 6 : Show that the group $\mathbf{Z}/n\mathbf{Z}$ is of order n.

Solution : The elements of $\mathbf{Z}/n\mathbf{Z}$ are of the form $a + n\mathbf{Z} = \{a + kn \mid k \in Z\}$.
Thus, the elements of $\mathbf{Z}/n\mathbf{Z}$ are precisely the congruence classes modulo n, that is, the elements of $\mathbf{Z}_n$ (see Sec. 2.5.1).

Thus, $\mathbf{Z}/n\mathbf{Z} = \{\bar{0}, \bar{1}, \bar{2}, ..., \overline{n-1}\}$.

$\therefore\ o(\mathbf{Z}/n\mathbf{Z}) = n$.

Note that addition in $\mathbf{Z}/n\mathbf{Z}$ is given by $\bar{a} + \bar{b} = \overline{a+b}$.

Try these simple exercises now.

E 8) For any group G, determine the quotient groups corresponding to $\{e\}$ and G.

E 9) Show that the quotient group of a cyclic group is cyclic.
(Hint : If $G = \langle x \rangle$, then show that $G/H = \langle Hx \rangle$.)

Now, do G and G/H always have the same algebraic properties?

On solving the following exercises you will see that if G is abelian, then so is G/H; but the converse need not be true. That is, if G/H is abelian, G may not be so. Thus, G and G/H need not have the same algebraic properties.

E 10) Show that if a group G is commutative, then so is G/H, for any $H \trianglelefteq G$.

E 11) Take the group D_8 of Example 4. Show that D_8/K is abelian, even though D_8 is non-abelian.

You may be surprised to know that given a group G, we can always define a normal subgroup H, such that G/H is abelian. This subgroup is the commutator subgroup.

Definition : Let G be a group and x, y ∈ G. Then $x^{-1}y^{-1}xy$ is called the **commutator** of x and y. It is denoted by **[x, y].**

The subgroup of G generated by the set of all commutators is called the **commutator subgroup** of **G.** It is denoted by *[G, G]*.

For example, if G is a commutative group, then
$x^{-1}y^{-1}xy = x^{-1}xy^{-1}y = e \ \forall \ x, y \in G$. ∴ [G, G] = {e}.

Try this exercise now.

E 12) Obtain [G, G], where G is cyclic.

Now, let us prove the commutativity of the factor group corresponding to the commutator subgroup.

Theorem 6 : Let G be a group. Then [G, G] is a normal subgroup of G. Further, G/[G, G] is commutative.

Proof: We must show that, for any commutator $x^{-1}y^{-1}xy$ and for any g ∈ G, $g^{-1}(x^{-1}y^{-1}xy)g \in [G, G]$.

Now $g^{-1}(x^{-1}y^{-1}xy)g = (g^{-1}xg)^{-1}(g^{-1}yg)^{-1}(g^{-1}xg)(g^{-1}yg) \in [G, G]$.
∴ $[G, G] \trianglelefteq G$.

For the rest of the proof let us denote [G, G] by H, for convenience.

Now, for x, y ∈ G,

$$HxHy = HyHx \Leftrightarrow Hxy = Hyx \Leftrightarrow (xy)(yx)^{-1} \in H$$

$$\Leftrightarrow xyx^{-1}y^{-1} \in H.$$

Thus, since $xyx^{-1}y^{-1} \in H \ \forall \ x, y \in G$, $HxHy = HyHx \ \forall \ x, y \in G$. That is, G/H is abelian.

$xyx^{-1}y^{-1}$
$= (x^{-1})^{-1}(y^{-1})^{-1}x^{-1}y^{-1}$

Note that we have defined the quotient group G/H only if $H \trianglelefteq G$. But if $H \ntrianglelefteq G$ we can still define G/H to be the set of all left (or right) cosets of H in G. But, in this case G/H will not be a group. The following exercise will give you an example.

E 13) For G = S_3 and H = < (1 2) >, show that the product of right cosets in G/H is not well defined.
(**Hint** : Show that H(1 2 3) = H(2 3) and H(1 3 2) = H(1 3), but H(1 2 3) (1 3 2) ≠ H(2 3) (1 3))

E 13 leads us to the following remark.

Remark : If H is a subgroup of G, then the product of cosets of H is defined only when $H \trianglelefteq G$. This is because, if $HxHy = Hxy \ \forall \ x, y \in G$, then, in particular,

$Hx^{-1}Hx = Hx^{-1}x = He = H \ \forall \ x \in G.$

Thererore, tor any $h \in H$, $x^{-1}hx = ex^{-1}hx \in Hx^{-1}Hx = H$.
That is, $x^{-1}Hx \subseteq H$ for any $x \in G$.
$\therefore H \trianglelefteq G$.

Let us now summarise what we have done in this unit.

5.4 SUMMARY

In this unit we have brought out the following points.

1. The definition and examples of a normal subgroup.
2. Every subgroup of an abelian group is normal.
3. Every subgroup of index 2 is normal.
4. If H and K are normal subgroups of a group G, then so is $H \cap K$.
5. The product of two normal subgroups is a normal subgroup.
6. If $H \trianglelefteq N$ and $N \trianglelefteq G$, then H need not be normal in G.
7. The definition and examples of a quotient group.
8. If G is abelian, then every quotient group of G is abelian. The converse is not true.
9. The quotient group corresponding to the commutator subgroup is commutative.
10. The set of left (or right) cosets of H in G is a group if and only if $H \trianglelefteq G$.

5.5 SOLUTIONS/ANSWERS

E 1) $S_3 = \{I, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$
$A_3 = \{I, (1\ 2\ 3), (1\ 3\ 2)\}$
You can check that
$A_3I = A_3 = IA_3$, $A_3(1\ 2) = (1\ 2)A_3$, and so on.
$\therefore A_3 \trianglelefteq S_3$.

E 2) For any $A \in GL_2(\mathbf{R})$ and $B \in SL_2(\mathbf{R})$,
$\det(A^{-1}BA) = \det(A^{-1})\det(B)\det(A)$

$$= \frac{1}{\det(A)}\det(A), \text{ since } \det(B) = 1$$
$$= 1$$

$\therefore A^{-1}BA \in SL_2(\mathbf{R})$.
$\therefore SL_2(\mathbf{R}) \trianglelefteq GL_2(\mathbf{R})$.

E 3) All, since this group is abelian,

E 4) Let $g \in G$ and $x \in Z(G)$. Then
$g^{-1}xg = g^{-1}gx$, since $x \in Z(G)$
$= x \in Z(G)$
$\therefore g^{-1}Z(G)g \subseteq Z(G) \ \forall \ g \in G$.
$\therefore Z(G) \trianglelefteq G$

E 5) Since $(1\ 2\ 3)^{-1}(2\ 3)(1\ 2\ 3) = (1\ 2) \notin \langle (2\ 3) \rangle$, $\langle (2\ 3) \rangle \ntrianglelefteq S_3$.

E 6) a) Take any element $hk \in HK$. Since $H \trianglelefteq G$, $k^{-1}hk \in H$. Let $k^{-1}hk = h_1$. Then $hk = kh_1 \in KH$.
$\therefore hk \in KH \ \forall \ hk \in HK$. $\therefore HK \subseteq KH$.
Again, for any $kh \in KH$, $khk^{-1} \in H$. Let $khk^{-1} = h_2$. Then $kh = h_2k \in HK$.
$\therefore kh \in HK \ \forall \ kh \in KH$.
$\therefore KH \subseteq HK$.

Thus, we have shown that HK = KH.
$\therefore$ $HK \leq G$.

b) From (a) we know that $HK \leq G$. To show that $HK \trianglelefteq G$, consider $g \in G$ and $hk \in HK$. Then
$g^{-1}hkg = g^{-1}h(gg^{-1})kg = (g^{-1}hg)(g^{-1}kg) \in HK$, since H 5 G, K 4 G.
$\therefore$ $g^{-1}HKg \subseteq HK \ \forall \ g \in G$.
$\therefore$ $HK \trianglelefteq G$.

E 7) D_6 is generated by x and y, where $x^2 = e$, $y^3 = e$ and $xy = y^{-1}x$.
$\therefore$ $D_6 = \{e, x, y, y^2, xy, xy^2\}$.

This is the group of symmetries of an equilateral triangle. Its generators are x and y, where x corresponds to the reflection about the altitude through a fixed vertex and y corresponds to a rotation about the centroid through 120° (see Fig. 2).

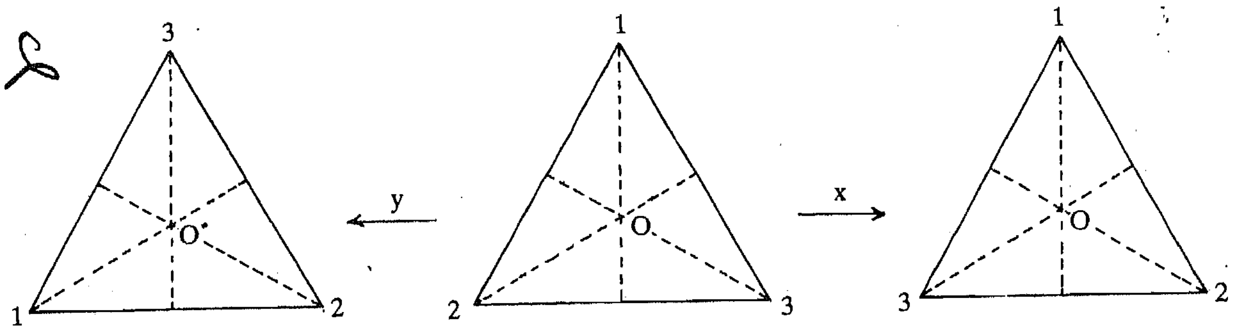

Fig. 2 : Generators of D_6

E 8) $G/\{e\} = \{\{e\}g \mid g \in G\} = \{\{g\} \mid g \in G\}$
$G/G = \{Gg \mid g \in G\} = \{G\}$, since $Gg = G \ \forall \ g \in G$.
So G/G consists of only one element, namely, the identity.

E 9) Let $G = \langle x \rangle$ and G/H be a quotient group of G. Any element of G/H is of the form $Hx^n = (Hx)^n$, since any element of G is of the form x^n. $\therefore$ $G/H = \langle Hx \rangle$.

E 10) For any two elements Hx and Hy in G/H,
(Hx) (Hy) = Hxy = Hyx, since G is abelian
= (Hy) (Hx).
$\therefore$ G/H is abelian.

E 11) $D_8/K = \{K, Kx\}$. You can check that this is abelian. You have already seen that $xy \neq yx$. $\therefore$ D_8 is not abelian.

E 12) Since G is cyclic, it is abelian. $\therefore$ $[G, G]' = \{e\}$

E 13) Now, (1 2 3) (1 3 2) = I, (2 3) (1 3) = (1 2 3).
$\therefore$ H(1 2 3) (1 3 2) = HI = H = {I, (1 2)}, and
H(2 3) (1 3) = H(1 2 3) = {(1 2 3), (2 3)}.

So, H(1 2 3) = H(2 3) and H (1 3 2) = H(1 3), but H(1 2 3) (1 3 2) $\neq$ H (2 3) (1 3).