UNIT 7 MATRICES - I

Structure

7.1 INTRODUCTION

You have studied linear transformations in Units 5 and 6. We will now study a simple means of representing them, namely, by matrices (the plural form of 'matrix'). We will show that, given a linear transformation, we can obtain a matrix associated to it, and vice versa. Then, as you will see, certain properties of a linear transformation can be studied more easily if we study the associated matrix instead. For example, you will see in Block 3, that it is often easier to obtain the characteristic roots of a matrix than of a linear transformation.

Matrices were introduced by the English mathematician, Arthur Cayley, in 1858. He came upon this notion in connection with linear substitutions. Matrix theory now occupies an important position in pure as well as applied mathematics. In physics one comes across such terms as matrix mechanics, scattering matrix, spin matrix, annihilation and creation matrices. In economics we have the input-output matrix and the pay off matrix; in statistics we have the transition matrix; and, in engineering, the stress matrix, strain matrix, and many other matrices.

Matrices are intimately connected with linear transformations. In this unit we will bring out this link. We will first define matrices and derive algebraic operations on matrices from the corresponding operations on linear transformations. We will also discuss some special types of matrices. One type, a triangular matrix, will be used often in Unit 8. You will also study invertible matrices in some detail, and their connection with change of bases. In Block 3 we will often refer to the material on change of bases, so do spend some time on Sec. 7.6.

To realise the deep connection between matrices and linear transformations, you should go back to the exact spot in Units 5 and 6 to which frequent references are made.

This unit may take you a little longer to study, than previous ones, but don't let that worry you. The material in it is actually very simple.

Objectives

After studying this unit, you should be able to

- define and give examples of various types of matrices;
- obtain a matrix associated to a given linear transformation;

- define a linear transformation, if you know its associated matrix;
- evaluate the sum, difference, product and scalar multiples of matrices;
- obtain the transpose and conjugate of a matrix;
- determine if a given matrix is invertible;
- obtain the inverse of a matrix;
- discuss the effect that the change of basis has on the matrix of a linear transformation.

7.2 VECTOR SPACE OF MATRICES

Consider the following system of three simultaneous equations in four unknowns:

$$\begin{aligned} x - 2y + 4z + t &= 0 \\ x + \tfrac{1}{2}y \qquad + 11t &= 0 \\ 3y - 5z \qquad &= 0 \end{aligned}$$

The coefficients of the unknowns, x, y, z and t, can be arranged in rows and columns to form a rectangular array as follows:

1	–2	4	1	(coefficients of the first equation)
1	1/2	0	11	(coefficients of the second equation)
0	3	–5	0	(coefficients of the third equation)

Such a rectangular array (or arrangement) of numbers is called a matrix. A matrix is usually enclosed within square brackets [ ] or round brackets ( ) as

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 1 & \frac{1}{2} & 0 & 11 \\ 0 & 3 & -5 & 0 \end{bmatrix} \text{ or } \begin{pmatrix} 1 & -2 & 4 & 1 \\ 1 & \frac{1}{2} & 0 & 11 \\ 0 & 3 & -5 & 0 \end{pmatrix}$$

The numbers appearing in the various positions of a matrix are called the **entries** (or **elements) of the matrix**. Note that the same number may appear at two or more different positions of a matrix. For example, 1 appears in 3 different positions in the matrix given above.

In the matrix above, the three horizontal rows of entries have 4 elements each. These are called the **rows** of this matrix. The four vertical rows of entries in the matrix, having 3 elements each, are called its **columns**. Thus, this matrix has three rows and four columns. We describe this by saying that this is a matrix of size 3×4 ("3 by 4" or "3 cross 4"), or that this is a 3×4 matrix. The rows are counted from top to bottom and the columns are counted from left to right. Thus, the first row is $(1, -2, 4, 1)$, the second row is $(1, \frac{1}{2}, 0, 11)$, and so on. Similarly,

the first column is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, the second column is $\begin{bmatrix} -2 \\ \frac{1}{2} \\ 3 \end{bmatrix}$, and so on.

Note that each row is a 1×4 matrix and each column is a 3×1 matrix.

We will now define a matrix of any size.

7.2.1 Definition of a Matrix

Let us see what we mean by a matrix of size $m \times n$, where m and n are any two natural numbers.

Let **F** be a field.

A rectangular array

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

of mn elements of **F** arranged in m rows and n columns is called a **matrix of size m × n**, or an **m × n matrix**, over **F**. You must remember that the mn entries need not be distinct.

The element at the intersection of the ith row and the jth column is called the (i, j) th element. For example, in the m × n matrix above, the (2, n) th element is a_{2n}, which is the intersection of the 2nd row and the nth column.

A brief notation for this matrix is $[a_{ij}]_{m \times n}$, or simply $[a_{ij}]$, if m and n need not be stressed. We also denote matrices by capital letters A, B, C, ..., etc. The set of all m × n matrices over **F** is denoted by $\mathbf{M}_{m\times n}$ (**F**).

Thus, $[1, \sqrt{2}] \in \mathbf{M}_{1\times 2}(\mathbf{R})$.

If m = n, then the matrix is called a **square matrix**. The set of all n × n matrices over **F** is denoted by $\mathbf{M}_n$ (**F**).

In an m × n matrix each row is a 1 × n matrix and is also called a **row vector**. Similarly, each column is an m × 1 matrix and is also called a **column vector**.

Let us look at a situation in which a matrix can arise.

Example 1: There are 20 male and 5 female students in the B.Sc. (Math. Hon's) I year class in a certain college, 15 male and 10 female students in B.Sc. (Math. Hon's) II year and 12 male and 10 female students in B.Sc. (Math. Hon's) III year. How does this information give rise to a matrix?

Solution: One of the ways in which we can arrange this information in the form of a matrix is as follows:

	B.Sc. I	B.Sc. II	B.Sc. III
Male	20	15	12
Female	5	10	10

This is a 2 × 3 matrix.

Another way could be the 3 × 2 matrix

	Female	Male
B.Sc I	5	20
B.Sc. II	10	15
B.Sc III	10	12

Either of these matrix representations immediately shows us how many male/female students there are in any class.

To get used to matrices and their elements, you can try the following exercises

E E 1) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 & 3 & 2 \\ 5 & 4 & 1 & 5 \\ 0 & 3 & 2 & 0 \end{bmatrix}$

Give the

a) (1, 2)th elements of A and B.

b) third row of A.

c) second column of A and the first column of B.

d) fourth row of B.

E E2) Write two different 4 × 2 matrices.

How did you solve E 2? Did the (i, j)th entry of one differ from the (i,j)th entry of the other for some i and j? If not, then they were equal. For example, the two 1 × 1 matrices [2] and [2] are equal. But [2] ≠ [3], since their entries at the (1, 1) position differ.

Definition: Two matrices are said to be **equal** if

i) they have the same size, that is, they have the same number of rows as well as the same number of columns, and

ii) their elements, at all the corresponding positions, are the same.

The following example will clarify what we mean by equal matrices.

Example 2: If $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} x & y \\ z & 3 \end{bmatrix}$, then what are x, y and z?

Solution: Firstly, both matrices are of the same size, namely, 2 x 2. Now, for these matrices to be equal the (i, j) th elements of both must be equal $\forall$ i, j. Therefore, we must have x = 1, y = 0, z = 2.

E E 3) Are [1] and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ equal? Why?

Now that you are familiar with the concept of a matrix, we will link it up with linear transformations.

7.2.2 Matrix of a Linear Transformation

We will now obtain a matrix that corresponds to a given linear transformation. You will see how easy it is to go from matrices to linear transformations, and back.
Let U and V be vector spaces over a field **F**, of dimensions n and m, respectively. Let

$B_1 = \{e_1, ..., e_n\}$ be an ordered basis of U, and

$B_2 = \{f_1,...... , f_m\}$ be an ordered basis of V. (By an **ordered basis** we mean that the order in which the elements of the basis are written is fixed. Thus, an ordered basis $\{e_1, e_2\}$ is not equal to an ordered basis $\{e_2, e_1\}$.)

Given a linear transformation T:U → V, we will associate a matrix to it. For this, we consider $T(e_1), ..., T(e_n)$, which are all elements of V and hence, they are linear combinations of $f_1,..., f_m$. Thus, there exist mn scalars α_{ij}, such that

$T(e_1) = \alpha_{11} f_1 + \alpha_{21} f_2 + ... + \alpha_{m1} f_m$

....

$T(e_j) = \alpha_{1j} f_1 + \alpha_{2j} f_2 + ... + \alpha_{mj} f_m$

....

$T(e_n) = \alpha_{1n} f_1 + \alpha_{2n} f_2 + ... + \alpha_{mn} f_m$

From these n equations we form an m × n matrix whose first column consists of the coefficients of the first equation, second column consists of the coefficients of the second equation, and so on. This matrix.

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{bmatrix}$$

is called the **matrix of T with respect to the bases B_1 and B_2**. Notice that **the coordinate vector of $T(e_j)$ is the jth column of A.**

We use the notation $[T]_{B_1, B_2}$ for this matrix. Thus, to obtain $[T]_{B_1, B_2}$ we consider $T(e_i)\ \forall e_i \in B_1$, and write them as linear combinations of the elements of B_2.

If T ∈ L (V, V), B is a basis of V and we take $B_1 = B_2 = B$, then $[T]_{B, B}$ is called the matrix of T with respect to the basis B, and can also be written as $[T]_B$.

Remark : Why do we insist on ordered bases? What happens if we interchange the order of the elements in B_1 to $\{e_n, e_1, ..., e_{n-1}\}$? The matrix $[T]_{B_1,B_2}$ also changes, the last column becoming the first column now. Similarly, if we change the positions of the f_i's in B_2, the rows of $[T]_{B_1,B_2}$ will get interchanged.

Thus, to obtain a unique matrix corresponding to T, we must insist on B_1 and B_2 being ordered bases. **Henceforth, while discussing the matrix of a linear mapping, we will always assume that our bases are ordered bases.**

We will now give an example, followed by some exercises.

Example 3: Consider the linear operator
$T : \mathbf{R}^3 \to \mathbf{R}^2$: T (x, y, z) = (x, y). Choose bases B_1 and B_2 of $\mathbf{R}^3$ and $\mathbf{R}^2$, respectively. Then obtain $[T]_{B_1,B_2}$.

Solution: Let $B_1 = \{e_1, e_2, e_3\}$, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. Let $B_2 = \{f_1, f_2\}$, where $f_1 = (1, 0)$, $f_2 = (0, 1)$. Note that B_1 and B_2 are the standard bases of $\mathbf{R}^3$ and $\mathbf{R}^2$, respectively.

$T(e_1) = (1, 0) = f_1 = 1.f_1 + 0.f_2$

$T(e_2) = (0, 1) = f_2 = 0.f_1 + 1.f_2$

$T(e_3) = (0, 0) = 0f_1 + 0f_2$.

Thus, $[T]_{B_1,B_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

E E 4) Choose two other bases B_1' and B_2' of $\mathbf{R}^3$ and $\mathbf{R}^2$, respectively. (In Unit 4 you came across a lot of bases of both these vector spaces.) For T in the example above, give the matrix $[T]_{B_1',B_2'}$

What E4 shows us is that the matrix of a transformation depends on the bases that we use for obtaining it. The next two exercises also bring out the same fact.

E E 5) Write the matrix of the linear transformation $T: \mathbf{R}^3 \to \mathbf{R}^2$: T (x, y, z) = (x+2y+2z, 2x+3y+4z) with respect to the standard bases of $\mathbf{R}^3$ and $\mathbf{R}^2$.

E E 6) What is the matrix of T, in E 5, with respect to the bases

$B_1' = \{(1, 0, 0), (0, 1, 0), (1,-2, 1)\}$ and

$B_2' = \{(1, 2), (2, 3)\}$?

The next exercise is about an operator that you have come across often.

E E 7) Let V be the vector space of polynomials over **R** of degree ≤ 3, in the variable t. Let D: V → V be the differential operator given in Unit 5 (E6, when n = 3). Show that the matrix of D with respect to the basis $\{1, t, t^2, t^3\}$ is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So far, given a linear transformation, we have obtained a matrix from it. This works the other way also. That is, given a matrix we can define a linear transformation corresponding to it.

Example 4 : Describe $T:\mathbf{R}^3 \rightarrow \mathbf{R}^3$ such that

$$[T]_B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$, where B is the standard basis of $\mathbf{R}^3$.

Solution: Let $B = \{e_1, e_2, e_3\}$. Now, we are given that

$T(e_1) = 1.e_1 + 2.e_2 + 3.e_3$

$T(e_2) = 2.e_1 + 3.e_2 + 1.e_3$

$T(e_3) = 4.e_1 + 1.e_2 + 2.e_3$

You know that any element of $\mathbf{R}^3$ is $(x, y, z) = xe_1 + ye_2 + ze_3$.

Therefore, $T(x, y, z) = T(xe_1 + ye_2 + ze_3)$

$= xT(e_1) + yT(e_2) + zT(e_3)$, since T is linear.

$= x(e_1 + 2e_2 + 3e_3) + y(2e_1 + 3e_2 + e_3) + z(4e_1 + e_2 + 2e_3)$

$= (x+2y+4z)e_1 + (2x + 3y + z)e_2 + (3x + y + 2z)e_3$

$= (x+2y+4z, 2x+3y+z, 3x+y+2z)$

$\therefore T: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is defined by $T(x, y, z) = (x + 2y + 4z, 2x + 3y + z, 3x + y + 2z)$

Try the following exercises now.

E E 8) Describe $T: \mathbf{R}^3 \rightarrow \mathbf{R}^2$ such that

$[T]_{B_1, B_2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, where B_1 and B_2 are the standard bases of $\mathbf{R}^3$ and $\mathbf{R}^2$, respectively.

E E 9) Find the linear operator T: **C** → **C** whose matrix, with respect to the basis {1, i} is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (Note that **C**, the field of complex numbers, is a vector space over **R**, of dimension 2.)

Now we are in a position to define the sum of matrices and multiplication of a matrix by a scalar.

7.2.3 Sum and Multiplication by Scalars

In Unit 5 you studied about the sum and scalar multiples of linear transformations. In the following theorem we will see what happens to the matrices associated with the linear transformations that are sums or scalar multiples of given linear transformations.

Theorem 1: Let U and V be vector spaces over **F**, of dimensions n and m, respectively. Let B_1 and B_2 be arbitrary bases of U and V, respectively. (Let us abbreviate $[T]_{B_1, B_2}$ to [T] during this theorem.) Let S, T ∈ L(U, V) and α ∈ **F**. Suppose $[S] = [a_{ij}]$, $[T] = [b_{ij}]$. Then

$$[S + T] = [a_{ij} + b_{ij}], \text{ and}$$

$$[\alpha S] = [\alpha a_{ij}]$$

Proof: Suppose $B_1 = \{e_1, e_2, ..., e_n\}$ and $B_2 = \{f_1, f_2, ..., f_m\}$. Then all the matrices to be considered here will be of size m × n.

Now, by our hypothesis,

$$S(e_j) = \sum_{i=1}^{m} a_{ij} f_i \ \forall j = 1, \ldots, n \text{ and}$$

$$T(e_j) = \sum_{i=1}^{m} b_{ij} f_i \ \forall j = 1, \ldots, n$$

$\therefore$, $(S + T)(e_j) = S(e_j) + T(e_j)$ (by definition of S + T)

$$= \sum_{i=1}^{m} a_{ij} f_i + \sum_{i=1}^{m} b_{ij} f_i$$

$$= \sum_{i=1}^{m} (a_{ij} + b_{ij}) f_i$$

Thus, by definition of the matrix with respect to B_1 and B_2, we get $[S + T] = [a_{ij} + b_{ij}]$.

Now, $(\alpha S)(e_j) = \alpha (S(e_j))$ (by definition of αS)

$$= \alpha\left(\sum_{i=1}^{m} a_{ij} f_i\right)$$

$$= \sum_{i=1}^{m} (\alpha a_{ij}) f_i$$

Thus, $[\alpha S] = [\alpha a_{ij}]$

Theorem 1 motivates us to define the sum of 2 matrices in the following way.

Definition: Let A and B be the following two $m \times n$ matrices.

Two matrices can be added if and only if they are of the same size

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

Then the **sum of A and B** is defined to be the matrix

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

In other words, $A + B$ is the $m \times n$ matrix whose (i, j) th element is the sum of the (i, j) th element of A and the (i, j)th element of B.

Let us see an example of how two matrices are added.

Example 5: What is the sum of $\begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 5 \end{bmatrix}$?

Solution: Firstly, notice that both the matrices are of the same size (otherwise, we can't add them). Their sum is

$$\begin{bmatrix} 1+0 & 4+1 & 5+0 \\ 0+1 & 1+4 & 0+5 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 5 \\ 1 & 5 & 5 \end{bmatrix}$$

E E 10) What is the sum of

a) $[1 \quad 2]$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$?

b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$?

Now, let us define the scalar multiple of a matrix, again motivated by Theorem 1.

Definition: Let α be a scalar, i.e., $\alpha \in$ **F**, and let $A = [a_{ij}]_{m\times n}$. Then we define the **scalar multiple of the matrix A by the scalar** α to be the matrix

$$\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{bmatrix}$$

In other words, αA is the $m \times n$ matrix whose (i, j) th element is α times the (i, j) th element of A.

Example 6: What is 2A, where $A = \begin{bmatrix} 1/2 & 1/4 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$?

Solution: We must multiply each entry of A by 2 to get 2A.
Thus,

$$2A = \begin{bmatrix} 1 & 1/2 & 2/3 \\ 0 & 0 & 0 \end{bmatrix}.$$

E E 11) Calculate $3\begin{bmatrix} 1 \\ 2 \end{bmatrix}, 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $3\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

Remark: The way we have defined the sum and scalar multiple of matrices allows us to write Theorem 1 as follows:

$$[S + T]_{B_1, B_2} = [S]_{B_1, B_2} + [T]_{B_1, B_2}$$

$$[\alpha S]_{B_1, B_2} = \alpha[S]_{B_1, B_2}$$

The following exercise will help you in checking if you have understood the contents of Sections 7.2.2 and 7.2.3.

E E 12) Define $S: \mathbf{R}^2 \to \mathbf{R}^3 : S(x,y) = (x,0,y)$ and $T : \mathbf{R}^2 \to \mathbf{R}^3 : T(x,y) = (0,x,y)$. Let B_1 and B_2 be the standard bases for $\mathbf{R}^2$ and $\mathbf{R}^3$, respectively.

Then what are $[S]_{B_1, B_2}, [T]_{B_1, B_2}, [S + T]_{B_1, B_2}, [\alpha S]_{B_1, B_2}$, for any $\alpha \in \mathbf{R}$.

We now want to show that the set of all $m \times n$ matrices over **F** is actually a vector space over **F**.

7.2.4 $M_{m\times n}$ (F) is a Vector Space

After having defined the sum and scalar multiplication of matrices, we enumerate the properties of these operations. This will ultimately lead us to prove that the set of all $m\times n$ matrices over **F** is a vector space over **F**. Do keep the properties VS1-VS10 (of Unit 3) in mind.

For any $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}] \in M_{m\times n}$ (F) and $\alpha, \beta \in$ **F**, we have

i) Matrix addition is associative:

$(A + B) + C = A + (B + C)$, since

$(a_{ij} + b_{ij}) + c_{ij} = a_{ij} + (b_{ij} + c_{ij}) \forall i, j$, as they are elements of a field.

ii) Additive identity: The matrix of the zero transformation (see Unit 5), with respect to any basis, will have 0 as all its entries. This is called the **zero matrix**. Consider the zero matrix **0**, of size $m \times n$. Then, for any $A \in M_{m\times n}$ (F),

$A + \mathbf{0} = \mathbf{0} + A = A$,

since $a_{ij} + \mathbf{0} = \mathbf{0} + a_{ij} = a_{ij} \forall i, j$.

Thus, **0** is the additive identity for $M_{m\times n}$ (F).

iii) Additive inverse: Given $A \in M_{m\times n}$ (F) we consider the matrix $(-1)A$. Then

$A + (-1)A = (-1)A + A = \mathbf{0}$

This is because the (i,j)th element of $(-1)A$ is $-a_{ij}$, and $a_{ij} + (-a_{ij}) = 0 = (-a_{ij}) + a_{ij} \forall i,j$.

Thus, $(-1)A$ is the **additive inverse** of A. We denote $(-1)A$ by $-A$.

iv) Matrix addition is commutative:

$A + B = B + A$

This is true because $a_{ij} + b_{ij} = b_{ij} + a_{ij} \forall i,j$.

v) $\alpha(A + B) = \alpha A + \alpha B$.

vi) $(\alpha + \beta)A = \alpha A + \beta A$

vii) $(\alpha\beta)A = \alpha(\beta A)$

viii) $1.A = A$

E E 13) Write out the formal proofs of the properties (v) – (viii), given above.

These eight properties imply that $M_{m\times n}$ (F) is a vector space over **F**.

Now that we have shown that $M_{m\times n}$ (F) is a vector space over **F**, we know it must have a dimension.

7.2.5 Dimension of $M_{m\times n}$ (F) over F

What is the dimension of $M_{m\times n}$ (F) over **F**? To answer this question we prove the following theorem. But, before you go further, check whether you remember the definition of a vector space isomorphism (Unit 5).

Theorem 2: Let U and V be vector spaces over **F** of dimensions n and m, respectively. Let B_1 and B_2 be a pair of bases of U and V, respectively. The mapping $\phi : L(U,V) \to M_{m\times n}(F)$, given by $\phi(T) = [T]_{B_1, B_2}$ is a vector space isomorphism.

Proof: The fact that ϕ is a linear transformation follows from Theorem 1. We proceed to show that the map is also 1 – 1 and onto. For the rest of the proof we shall denote $[S]_{B_1, B_2}$ by $[S]$ only, and take $B_1 = \{e_1, ..., e_n\}$, $B_2 = \{f_1, f_2, ..., f_m\}$.

ϕ is 1 – 1: Suppose S, T $\in$ L(U,V) be such that $\phi(S) = \phi(T)$.

Then $[S] = [T]$. Therefore, $S(e_j) = T(e_j) \forall e_j \in B_1$.

Thus, by Unit 5 (Theorem 1), we have S = T.

ϕ is on o: If A$\in$ $\mathbf{M}_{m\times n}$ **(F)** we want to construct T $\in$ L(U,V)

such that ϕ (T) = A. Suppose $A = [a_{ij}]$. Let $v_1, ..., v_n \in V$ such that

$$v_j = \sum_{i=1}^{m} a_{ij} f_i \text{ for } j = 1, ..., n.$$

Then, by Theorem 3 of Unit 5, there exists a linear transformation T $\in$ L(U,V) such that

$$T(e_j) = v_j = \sum_{i=1}^{n} a_{ij} f_i.$$

Thus, by definition, $\phi(T) = A$.

Therefore, ϕ is a vector space isomorphism.

A corollary to this theorem gives us the dimension of $\mathbf{M}_{m\times n}$ **(F)**.

Corollary: Dimension of $\mathbf{M}_{m\times n}$ (F)= mn.

Proof: Theorem 2 tells us that $\mathbf{M}_{m\times n}$ (F) is isomorphic to L(U,V). Therefore, $\dim_F \mathbf{M}_{m\times n}(F) = \dim_F L(U,V)$ (by Theorem 12 of Unit 5) = mn, from Unit 6 (Theorem 1).

Why do you think we chose such a roundabout way for obtaining dim$\mathbf{M}_{m\times n}$ (F)? We could as well have tried to obtain mn linearly independent m × n matrices and show that they generate $\mathbf{M}_{m\times n}$ (F). But that would be quite tedious (see E16). Also, we have done so much work on L(U,V) so why not use that! And, doesn't the way we have used seem neat?

Now for some exercises related to Theorem 2.

E E 14) At most, how many matrices can there be in any linearly independent subset of $\mathbf{M}_{2\times 3}$(F)?

E E 15) Are the matrices [1, 0] and [1, -1] linearly independent over **R**?

E E 16) Let E_{ij} be an m×n matrix whose (i,j) th element is 1 and the other elements are 0. Show that $\{E_{ij}: 1 \le i \le m, 1 \le j \le n\}$ is a basis of $\mathbf{M}_{m\times n}$ **(F)** over **F**. Conclude that $\dim_F \mathbf{M}_{m\times n}$ **(F)** = mn.

Now we move on to the next section, where we see some ways of getting new matrices from given ones.

7.3 NEW MATRICES FROM OLD

Given any matrix we can obtain new matrices from them in different ways. Let us see three of these ways.

7.3.1 Transpose

Suppose $A = \begin{bmatrix} 1 & 0 & 9 \\ 2 & 5 & 9 \end{bmatrix}$

From this we form a matrix whose first and second columns are the first and second rows of A, respectively. That is, we obtain

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 5 \\ 9 & 9 \end{bmatrix}.$$

Then B is called the transpose of A. Note that A is also the transpose of B, since the rows of B are the columns of A. Here A is a 2×3 matrix and B is a 3×2 matrix.

In general, if $A = [a_{ij}]$ is an $m \times n$ matrix. Then the $n \times m$ matrix whose ith column is the ith row of A, is called the transpose of A. The transpose of A is denoted by A^t (The notation and A' is also widely used.)

Note that, if $A = [a_{ij}]_{m\times n}$, then $A^t = [b_{ij}]_{m\times n}$ where b_{ij} is the intersection of the ith row and the jth column of A^t. $\therefore$ b_{ij} is the intersection of the jth row and ith column of A, i.e., a_{ji}.
$\therefore b_{ij} = a_{ji}$.

E E 17) Find A^t, where $a = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$.

We now give theorem that lists some properties of the transpose.

Theorem 3: Let $A, B \in M_{m\times n}(F)$ and $\alpha \in F$. Then,

a) $(A + B)^t = A^t + B^t$

b) $(\alpha A)^t = \alpha A^t$.

c) $(A^t)^t = A$

Proof: a) Let $A = [a_{ij}]$ and $B = [b_{ij}]$. Then $A + B = [a_{ij} + b_{ij}]$. Therefore, $(A + B)^t = [c_{ij}]$, where

c_{ij} = the (j,i)th element of $A+B = a_{ji} + b_{ji}$

= sum of the (j,i)th elements of A and B

= sum of the (i,j)th elements of A^t and B^t.

= (i,j)th element of $A^t + B^t$.

Thus, $\mathbf{(A + B)^t = A^t + B^t}$

We leave you to complete the proof of this theorem. In fact that is what E 18 says!

E E 18) Prove (b) and (c) of Theorem 3.

E E 19) Show that, if $A = \mathbf{A^t}$, then A must be a square matrix.

E 19 leads us to some definitions.

Definitions : A square matrix A such that $A^t = A$ is called a **symmetric matrix**. A square matrix A such that $A^t = -A$, is called a **skew-symmetric matrix**.
For example, the matrix in E 17, and

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, are both symmetric matrices.

$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ is an example of a skew-symmetric matrix, since

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}^t = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

E E 20) Take a 2×2 matrix A. Calculate $A + A^t$ and $A - A^t$. Which of these is symmetric and which is skew-symmetric?

Every square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix.

What you have shown in E20 is true for a square matrix of any size, namely, for any $A \in M_n(F)$, $A + A^t$ is symmetric and $A - A^t$ is skew-symmetric.

We now give another way of getting a new matrix from a given matrix over the complex field.

7.3.2 Conjugate

If A is a matrix over **C**, then the matrix obtained by replacing each entry of A by its complex conjugate is called the **conjugate of A**, and is denoted by $\bar{A}$.

The complex conjugate of $a+ib \in \mathbf{C}$ is $a-ib$.

Three properties of conjugates, which are similar to those of the transpose, are

a) $\overline{A+B} = \bar{A} + \bar{B}$, for $A, B \in M_{m\times n}(C)$

b) $\overline{\alpha A} = \bar{\alpha}\,\bar{A}$, for $\alpha \in$ C and $A \in M_{m\times n}(C)$

c) $\bar{\bar{A}} = A$, for $A \in M_{m\times n}(C)$

Let us see an example of obtaining the conjugate of a matrix.

Example 7: Find the conjugate of $\begin{bmatrix} 1 & i \\ 2+i & -3-2i \end{bmatrix}$

Solution: By definition, the required matrix will be

$$\begin{bmatrix} 1 & -i \\ 2-i & -3+2i \end{bmatrix}$$

Example 8: What is the conjugate of $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$?

Solution: Note that this matrix has only real entries. Thus, the complex conjugate of each entry is itself. This means that the conjugate of this matrix is itself.

This example leads us to make the following observation.

Remark: $\overline{A} = \bar{A}$ if and only if A is a real matrix.

Try the following exercise now.

E E 21) Calculate the conjugate of $\begin{bmatrix} i & 2 \\ 3 & i \end{bmatrix}$.

We combine what we have learnt in the previous two sub-sections now.

7.3.3 Conjugate Transpose

Given a matrix $A \in M_{m\times n}(F)$ we form a matrix B by taking the conjugate of A^t. Then $B = \bar{A}^t$; is called the **conjugate transpose of A**.

Example 9: Find $\bar{A}^t$ where $A = \begin{bmatrix} 1 & i \\ 2+i & -3-2i \end{bmatrix}$

Solution: Firstly,

$$A^t = \begin{bmatrix} 1 & 2+i \\ i & -3-2i \end{bmatrix}. \text{ Then}$$

$$\bar{A}^t = \begin{bmatrix} 1 & 2-i \\ -i & -3+2i \end{bmatrix}.$$

Now, note a peculiar occurrence. If we first calculate $\bar{A}$ and then take its transpose, we get the same matrix, namely, $\bar{A}^t$. That is, $(\bar{A})^t = \bar{A}^t$.

In general, $(\bar{A})^t = \bar{A}^t \; \forall A \in M_{m\times n}(C)$.

E E 22) Show that $A = \bar{A}^t \Rightarrow A$ is a square matrix.

$\bar{A}^t$ is also denoted by A^θ or A^*.

E 22 leads us to the following definitions.

Definitions: A square matrix A for which $\bar{A}^t = A$ is called a **Hermitian matrix**. A square matrix A is called a **skew-Hermitian matrix** if $\bar{A}^t = -A$.

For example, the matrix $\begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$ is Hermitian, whereas the matrix $\begin{bmatrix} i & 1+i \\ -1+i & 0 \end{bmatrix}$ is a skew–Hermitian matrix.

Note: If $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$, then $A = A^t = \bar{A}^t$ (since the entries are all real). $\therefore$ A is symmetric as well as Hermitian. In fact, for a real matrix A, A is Hermitian if A is symmetric. Similarly, A is skew-Hermitian iff A is skew-symmetric.

We will now discuss two important, and often-used, types of square matrices.

7.4 SOME TYPES OF MATRICES

In this section we will define a diagonal matrix and a triangular matrix.

7.4.1 Diagonal Matrix

Let U and V be vector spaces over **F** of dimension n. Let $B_1 = \{e_1,\ldots,e_n\}$ and $B_2 = \{f_1,\ldots\ldots,f_n\}$ be based of U and V, respectively. Let $d_1,\ldots, d_n \in$ **F**. Consider the transformation $T: U \to V: T(a_1e_1 + \ldots + a_ne_n) = a_1d_1f_1 + \ldots + a_nd_nf_n$.

Then $T(e_1) = d_1f_1$, $T(e_2) = d_2f_2,\ldots,T(e_n) = d_nf_n$.

$$\therefore [T]_{B_1,B_2} = \begin{bmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ . & . & \ldots & . \\ 0 & 0 & \ldots & d_n \end{bmatrix}$$

Such a matrix is called a diagonal matrix. Let us see what this means.

Let $A = [a_{ij}]$ be a square matrix. The entries $a_{11}, a_{22},\ldots, a_{nn}$ are called the **diagonal entries** of A. This is because they lie along the diagonal, from left to right, of the matrix. All the other entries of A are called the **off-diagonal entries** of A.

A square matrix whose off-diagonal entries are zero (i.e., $a_{ij} = 0 \ \forall i \neq j$) is called a **diagonal matrix**. The diagonal matrix

$$\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

is denoted by **diag (d_1,d_2, ... , d_n)**.

Note: The d_i's may or may not be zero. What happens if all the d_i's are zero? Well, we get the $n \times n$ zero matrix, which corresponds to the zero operator.

If $d_i = 1 \ \forall i = 1, \ldots\ldots, n$, we get the **identity matrix**, I_n (or I, when the size is understood).

E E 23) Show that I_n is the matrix associated to the identity operator from $\mathbf{R}^n$ to $\mathbf{R}^n$.

If $\alpha \in$ **F**, the linear operator $\alpha I: \mathbf{R}^n \longrightarrow \mathbf{R}^n : \alpha I(v) = \alpha v$, for all $v \in \mathbf{R}^n$, is called a **scalar operator**. Its matrix with respect to any basis is $\alpha I = \text{diag}(\alpha, \alpha, \ldots\ldots, \alpha)$. Such a matrix is called a **scalar matrix**. It is a diagonal matrix whose diagonal entries are all equal.

With this much discussion on diagonal matrices, we move onto describe triangular matrices.

7.4.2 Triangular Matrix

Let $B = \{e_1, e_2, \ldots, e_n\}$ be a basis of a vector space V. Let $S \in L(V,V)$ be an operator such that

$S(e_1) = a_{11}e_1$

$S(e_2) = a_{12}e_1 + a_{22}e_2$

$\vdots \quad \vdots \quad \vdots$

$S(e_n) = a_{1n}\, e_1 + a_{2n}\, e_2 + \ldots + a_{nn}\, e_n,$

Then, the matrix of S with respect to B is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Note that $a_{ij} = 0 \; \forall \, i > j$

A square matrix A such that $a_{ij} = 0 \; \forall \, i > j$ is called an **upper triangular matrix.** If $a_{ij} = 0 \; \forall \, i \geq j$, then A is called **strictly upper triangular.**

For example, $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are all upper triangular, while $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ is strictly upper triangular.

Note that every strictly upper triangular matrix is an upper triangular matrix.

Now let $T : V \to V$ be an operator such that $T(e_j)$ is a linear combination of $e_j, e_{j+1}, \ldots, e_n \; \forall j$.
The matrix of T with respect to B is

$$[T]_B = \begin{bmatrix} b_{11} & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{bmatrix}$$

Note that $b_{ij} = 0 \; \forall \, i < j$

Such a matrix is called a **lower triangular matrix.** If $b_{ij} = 0$ for all $i \leq j$, **then B is said to be a strictly lower triangular matrix.**

The matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & 5 & 0 \end{bmatrix}$$

is a strictly lower triangular matrix. Of course, it is also lower triangular!

Remark: If A is an upper triangular 3 x 3 matrix, say

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \text{ then } A^t = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}, \text{ a lower triangular matrix.}$$

In fact, for any $n \times n$ upper triangular matrix A, its transpose is lower triangular, and vice versa.

E E 24) If an upper triangular matrix A is symmetric, then show that it must be a diagonal matrix.

E 25) Show that the diagonal entries of a skew-symmetric matrix are all zero, but the converse is not true.

Let us now see how to define the product of two or more matrices.

7.5 MATRIX MULTIPLICATION

We have already discussed scalar multiplication. Now we see how to multiply two matrices. Again, the motivation for this operation comes from linear transformations.

7.5.1 Matrix of the Composition of Linear Transformations

Let U, V and W be vector spaces over F, of dimensions p, n and m, respectively. Let B_1, B_2 and B_3 be bases of these respective spaces. Let $T \in L(U,V)$ and $S \in L(V,W)$. Then $ST (= S \circ T) \in L(U, W)$ (see Sec. 6.4).

Suppose $[T]_{B_1, B_2} = B = [b_{jk}]_{n \times p}$

and $[S]_{B_2, B_3} = A = [a_{ij}]_{m \times n}$

We ask : What is the matrix $[ST]_{B_1, B_3}$?

To answer this we suppose

$B_1 = \{e_1, e_2, \ldots\ldots, e_p\}$

$B_2 = \{f_1, f_2, \ldots\ldots, f_n\}$

$B_3 = \{g_1, g_2, \ldots\ldots, g_m\}$

Then, we know that $T(e_k) = \sum_{j=1}^{n} b_{jk} f_j \; \forall \; k = 1, 2, \ldots, p$,

and $S(f_j) = \sum_{i=1}^{m} a_{ij} \, g_i \; \forall \; j = 1, 2, \ldots, n$.

Therefore, $S \circ T(e_k) = S(T(e_k)) = S\left(\sum_{j=1}^{n} b_{jk} f_j\right) = b_{1k} S(f_1) + b_{2k} S(f_2) + \cdots + b_{nk} S(f_n)$

$$= b_{1k}\left(\sum_{i=1}^{m} a_{i1} g_i\right) + b_{2k}\left(\sum_{i=1}^{m} a_{i2} g_i\right) + \cdots + b_{nk}\left(\sum_{i=1}^{m} a_{in} g_i\right)$$

$$= \sum_{i=1}^{m} (a_{i1} b_{1k} + a_{i2} b_{2k} + \cdots + a_{in} b_{nk}) g_i,$$ on collecting the coefficients of g_i.

Thus, $[ST]_{B_1, B_3} = [c_{ik}]_{m \times p}$, where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$.

We define the matrix $[c_{ik}]$ to be the product AB.

So, let us see how we obtain AB from A and B.

Let $A = [a_{ij}]_{m \times n}$, $B = [b_{jk}]_{n \times p}$ be two matrices over F of sizes $m \times n$ and $n \times p$, respectively. We define AB to be the $m \times p$ matrix C whose (i,k)th entry is

The product of an $m \times n$ and an $n \times p$ matrix is an $m \times p$ matrix.

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} = a_{i1} b_{1k} + a_{i2} b_{2k} + \cdots + a_{in} b_{nk}$$

In order to obtain the (i,k) th element of AB, take the ith row of A and the kth column of B

They are both n-tuples. Multiply their corresponding elements and add up all these products.

For example, if the 2nd row of A = [1 2 3], and the 3rd column of

$B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, then the **(2, 3)** entry of AB = $1 \times 4 + 2 \times 5 + 3 \times 6 = 32$.

Note that two matrices A and B can only be multiplied if the number of columns of A = the number of rows of B. The following illustration may help in explaining what we do to obtain the product of two matrices.

$$\overset{\textbf{A}}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \overset{\textbf{B}}{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2k} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nk} & \cdots & b_{np} \end{bmatrix}} = \overset{\textbf{AB}}{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2k} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ik} & \cdots & c_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mk} & \cdots & c_{mp} \end{bmatrix}}$$

where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$.

Note: This is a very new kind of operation so take your time in trying to understand it.

To get you used to matrix multiplication we consider the product of a row and a column matrix.

Let $A = [a_1, a_2, ..., a_n]$ be a 1×n matrix and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be an n×1 matrix. Then AB is the **1×1** matrix

$[a_1 b_1 + a_2 b_2 + \cdots + a_n b_n]$.

E 26) What is $[1 \quad 0 \quad 0] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$?

Now for another example.

Example 10: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 0 & 8 \\ 0 & 0 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \\ 4 & 0 \end{bmatrix}$

Find AB, if it is defined.

Solution: AB is defined because the number of columns of A = 3 = number of rows of B.

$$AB = \begin{bmatrix} 1.2 + 0.3 + 0.4 & 1.1 + 0.5 + 0.0 \\ 7.2 + 0.3 + 8.4 & 7.1 + 0.5 + 8.0 \\ 0.2 + 0.3 + 9.4 & 0.1 + 0.5 + 9.0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 46 & 7 \\ 36 & 0 \end{bmatrix}$$

Notice that BA is not defined because the number of columns of B = 2 ≠ number of rows of A. Thus, **if AB is defined then BA may not be defined.**

In fact, even if AB and BA are both defined **it is possible that AB ≠ BA.** Consider the following example.

Example 11: Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$. Is AB = BA?

Solution: AB is a 2×2 matrix. BA is a 3×3 matrix.

So AB and BA are both defined, but they are of different sizes. Thus, $AB \neq BA$.

Another point of difference between multiplication of numbers and matrix multiplication is that **$A \neq 0$, $B \neq 0$, but AB can be zero.**

For example, if $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$,

then $AB = \begin{bmatrix} 1 \times 1 + 1(-1) & 1 \times 0 + 1 \times 0 \\ 1 \times 1 + 1(-1) & 1 \times 0 + 1 \times 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

So you see, the product of two non-zero matrices can be zero.

The following exercises will give you some practice in matrix multiplication.

E E 27) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Write AB and BA, if defined.

E E 28) Let $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Write C + D, CD and DC, if defined. Is CD = DC?

E E 29) With A, B as in E 27, calculate $(A + B)^2$ and $A^2 + 2AB + B^2$. Are they equal? (Here A^2 means A.A.)

E E 30) Let $A = \begin{bmatrix} -bd & b \\ -d^2 b & db \end{bmatrix}$, b, d $\in$ **F**. Find A^2.

E E 31) Calculate $\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $[x\ y\ z]\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}$

E E 32) Take a 3×2 matrix A whose end row consists of zeros only. Multiply it by any 2×4 matrix B. Show that the 2nd row of AB consists of zeros only. (In fact, for any two matrices A and B such that AB is defined, if the ith row of A is the zero vector, then the ith row of AB is also the zero vector. Similarly, if the jth column of B is the zero vector, then the jth column of AB is the zero vector.)

We now make an observation.

Remark: If $T \in L(U, V)$ and $S \in L(V, W)$, then

$[ST]_{B_1,B_3} = [S]_{B_2,B_3}\ [T]_{B_1,B_2}$, where B_1, B_2, B_3 are the bases of U, V, W, respectively.

Let us illustrate this remark.

Example 12: Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be a linear transformation such that $T(x, y) = (2x + y, x + 2y, x + y)$. Let $S : \mathbf{R}^3 \to \mathbf{R}^2$ be defined by $S(x, y, z) = (-y + 2z, y-z)$. Obtain the matrices $[T]_{B_1,B_2}$, $[S]_{B_2,B_1}$, and $[S{\circ}T]_{B_1}$, and verify that $[S{\circ}T]_{B_1} = [S]_{B_2,B_1}\ [T]_{B_1,B_2}$, where B_1 and B_2 are the standard bases in $\mathbf{R}^2$ and $\mathbf{R}^3$, respectively.

Solution : Let $B_1 = \{e_1, e_2\}$, $B_2 = \{f_1, f_2, f_3\}$.

Then $T(e_1) = T(1,0) = (2,1,1) = 2f_1 + f_2 + f_3$

$T(e_2) = T(0,1) = (1,2,1) = f_1 + 2f_2 + f_3$

Thus,

$$[T]_{B_1,B_2} = \begin{bmatrix}2 & 1\\1 & 2\\1 & 1\end{bmatrix}$$

Also,

$S(f_1) = S(1,0,0) = (0,0) = 0.e_1 + 0.e_2$

$S(f_2) = S(0,1,0) = (-1,1) = -e_1 + e_2$

$S(f_3) = S(0,0,1) = (2,-1) = 2e_1 - e_2$

Thus,

$$[S]_{B_2,B_1} = \begin{bmatrix}0 & -1 & 2\\0 & 1 & -1\end{bmatrix}$$

$$\text{So, } [S]_{B_2,B_1}[T]_{B_1,B_2} = \begin{bmatrix}0 & -1 & 2\\0 & 1 & -1\end{bmatrix}\begin{bmatrix}2 & 1\\1 & 2\\1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I_2$$

Also, $S{\circ}T(x,y) = S(2x + y, x + 2y, x + y)$

$= (-x - 2y + 2x + 2y, x + 2y - x - y)$

$= (x,y)$

Thus, $S \circ T = I$, the identity map.

This means $[S \circ T]_{B_1} = I_2$

Hence, $[S \circ T]_{B_1} = [S]_{B_2, B_1} [T]_{B_1, B_2}$.

E E 33) Let $S : \mathbf{R}^3 \to \mathbf{R}^3 : S(x, y, z) = (0, x, y)$, and $T : \mathbf{R}^3 \to \mathbf{R}^3 : T(x, y, z) = (x, 0, y)$
Show that $[S \circ T]_B = [S]_B [T]_B$, where B is the standard basis of $\mathbf{R}^3$.

We will now look a little closer at matrix multiplication.

7.5.2 Properties of a Matrix Product

We will now state 5 properties concerning matrix multiplication. (Their proofs could get a little technical, and we prefer not to give them here.)

(1) **Associative Law:** If, A, B, C are $m \times n$, $n \times p$ and $p \times q$ matrices, respectively, over **F**, then (AB) C = A (BC), i.e., matrix multiplication is associative.

(2) **Distributive Law:** If A is an $m \times n$ matrix and B, C are $n \times p$ matrices, then A (B + C) = AB + AC.

Similarly if, A and B are $m \times n$ matrices, and C is an $n \times p$ matrix, then (A + B)C = AC + BC.

(3) **Multiplicative Identity:** In Sec. 7.4.1, we defined the identity matrix I_n. This acts as the multiplicative identity for matrix multiplication. We have $AI_n = A$, $I_m A = A$, for every $m \times n$ matrix A.

(4) If $\alpha \in \mathbf{F}$, and A, B are $m \times n$ and $n \times p$ matrices over **F**, respectively, then $\alpha(\mathbf{AB}) = (\alpha A)\mathbf{B} = \mathbf{A}(\alpha\mathbf{B})$.

(5) If A, B are $m \times n$, $n \times p$ matrices over **F**, respectively, then $(\mathbf{AB})^t = \mathbf{B}^t \mathbf{A}^t$. (This says that the operation of taking the transpose of a matrix is anti-commutative.)

These properties can help you in solving the following exercises.

E E 34) Show that $(A + B)^2 = A^2 + AB + BA + B^2$, for any two $n \times n$ matrices A and B.

E E 35) For $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$, show that 2 (AB) = (2A) B.

E E 36) Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -4 & 0 \\ 2 & -1 & 3 \\ 4 & 0 & -2 \end{bmatrix}$

Find $(AB)^t$ and $B^t A^t$. Are they equal?

E E 37) Let A, B be two symmetric $n \times n$ matrices over **F**. Show that AB is symmetric if and only if AB = BA.

The following exercise is a nice property of the product of diagonal matrices.

E E 38) Let A, B be two diagonal $n \times n$ matrices over **F**. Show that AB is also a diagonal matrix.

Now we shall go on to introduce you to the concept of an invertible matrix.

7.6 INVERTIBLE MATRICES

In this section we will first explain what invertible matrices are. Then we will see what we mean by the matrix of a change of basis. Finally, we will show you that such a matrix must be invertible.

7.6.1 Inverse of a Matrix

Just as we defined the operations on matrices by considering them on linear operators first, we give a definition of invertibility for matrices based on considerations of invertibility of linear operators.

It may help you to recall what we mean by an invertible linear transformation. A linear transformation $T : U \to V$ is invertible if

a) T is 1 – 1 and onto, or, equivalently,

b) there exists a linear transformation $S : V \to U$ such that $S \circ T = I_U$, $T \circ S = I_V$.

In particular, $T \in L(V, V)$ is said to be invertible if $\exists\, S \in L(V, V)$ such that $ST = TS = I$.

We have the following theorem involving the matrix of an invertible linear operator.

Theorem 4: Let V be an n-dimensional vector space over a field **F**, and B be a basis of V. Let $T \in L(V, V)$. T is invertible iff there exists $A \in \mathbf{M}_n(\mathbf{F})$ such that $[T]_B\, A = I_n = A\, [T]_B$

Proof: Suppose T is invertible. Then $\exists$ $S \in L(V, V)$ such that $TS = ST = I$. Then, by Theorem 2, $[TS]_B = [ST]_B = I$. That is, $[T]_B [S]_B = [S]_B [T]_B = I$. Take $A = [S]_B$. Then $[T]_B A = I = A [T]_B$.

Conversely, suppose $\exists$ a matrix A such that $[T]_B A = A [T]_B = I$.
Let $S \in L(V, V)$ be such that $[S]_B = A$. (S exists because of Theorem 2.) Then $[T]_B [S]_B = [S]_B [T]_B = I = [I]_B$. Thus, $[TS]_B = [ST]_B = [I]_B$.

So, by Theorem 2, TS = ST = I. That is, T is invertible.

Theorem 4 motivates us to give the following definition.

Definition : A matrix $A \in \mathbf{M}_n(\mathbf{F})$ is said to be **invertible** if $\exists$ $B \in \mathbf{M}_n(\mathbf{F})$ such that $AB = BA = I_n$.
Remember, only a square matrix can be invertible.

I_n is an example of an invertible matrix, since $I_n \cdot I_n = I_n$. On the other hand, the n × n zero matrix **0** is not invertible, since $\mathbf{0}A = \mathbf{0} \neq I_n$, for any A.

Note that Theorem 4 says that **T is invertible iff $[T]_B$ is invertible.**

We give another example of an invertible matrix now.

Example 13: Is $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ invertible?

Solution: Suppose A were invertible. Then $\exists$ $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that AB = I = BA. Now,

$$AB = I \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow c = 0,\ d = 1,\ a = 1,\ b = -1$$

$$\therefore B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$ Now you can also check that BA = I.

Therefore A is invertible.

We now show that if an inverse of a matrix exists, it must be unique.

Theorem 5 : Suppose $A \in \mathbf{M}_n(\mathbf{F})$ is invertible. There exists a **unique** matrix $B \in \mathbf{M}_n(\mathbf{F})$ such that AB = BA = I.

Proof: Suppose $B, C \in \mathbf{M}_n(\mathbf{F})$ are two matrices such that AB = BA = I, and AC = CA = I.
Then B = BI = B (AC) = (BA) C = IC = C.

Because of Theorem 5 we can make the following definition.

Definition : Let A be an invertible matrix. The unique matrix B such that AB = BA = I is called the **inverse of A** and is denoted by A^{-1}.

Let us take an example.

Example 14: Calculate the product AB, where

$$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}.$$

Use this to calculate A^{-1}.

Solution : Now $AB = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix}$.

Now, how can we use this to obtain A^{-1}? Well, if AB = I, then a+b = 0. So, if we take

$$B = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix},$$

we get AB = BA = I. Thus, $A^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$

E E 39) Is the matrix $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ invertible? If so, find its inverse.

We will now make a few observations about the matrix inverse, in the form of a theorem.

Theorem 6: a) If A is invertible, then

i) A^{-1} is invertible and $(A^{-1})^{-1} = A$.

ii) A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

b) If $A, B \in M_n(F)$ are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

Proof: (a) By definition,

$A A^{-1} = A^{-1} A = I$ ....(1)

i) Equation (1) shows that A^{-1} is invertible and $(A^{-1})^{-1} = A$.

ii) If we take transposes in Equation (1) and use the property that $(AB)^t = B^t A^t$, we get $(A^{-1})^t A^t = A^t (A^{-1})^t = I^t = I$.

So A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

(b) To prove this we will use the associativity of matrix multiplication. Now

$(AB)(B^{-1}A^{-1}) = [A(BB^{-1})]A^{-1} = AA^{-1} = I$.

$(B^{-1}A^{-1})(AB) = B^{-1}[(A^{-1}A)B] = B^{-1}B = I$.

So AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

We now relate matrix invertibility with the linear independence of its rows or columns. When we say that the m rows of $A = [a_{ij}] \in M_{m\times n}(F)$ are linearly independent, what do we mean? Let $R_1, \ldots, R_m$ be the m row vectors $[a_{11}, a_{12}, \ldots, a_{1n}], [a_{21}, \ldots, a_{2n}], \ldots, [a_{m1}, \ldots, a_{mn}]$, respectively. We say that they are linearly independent if, whenever $\exists\, a_1, \ldots, a_m \in F$ such that

$a_1R_1 + \ldots + a_m R_m = \mathbf{0}$,

then $a_1 = 0, \ldots\ldots, a_m = 0$.

Similarly, the n columns $C_1, \ldots, C_n$ of A are linearly independent if $b_1C_1 + \ldots b_n C_n = \mathbf{0}$ $\Rightarrow b_1 = 0, b_2 = 0, \ldots, b_n = 0$, where $b_1, \ldots, b_n \in F$.

We have the following result.

Theorem 7: Let $A \in M_n(F)$. Then the following conditions are equivalent.

a) A is invertible.

b) The columns of A are linearly independent.

c) The rows of A are linearly independent.

Proof: We first prove (a) $\Leftrightarrow$ (b), using Theorem 4. Let V be an n-dimensional vector space over F and $B = \{e_1, \ldots, e_n\}$ be a basis of V. Let $T \in L(V,V)$ be such that $[T]_B = A$. Then A is invertible iff T is invertible iff $T(e_1), T(e_2), \ldots, T(e_n)$ are linearly independent (see Unit 5, Theorem 9). Now we define the map

$$\theta: V \to M_{n\times 1}(F): \theta(a_1 e_1 + \cdots + a_n e_n) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Before continuing the proof we give an exercise.

E 40) Show that θ is a well-defined isomorphism.

Now let us go on with proving Theorem 7.

Let $C_1, C_2, \ldots, C_n$ be the columns of A. Then $\theta(T(e_i)) = C_i$ for all $i = 1, \ldots, n$. Since θ is an isomorphism, $T(e_1), \ldots, T(e_n)$ are linearly independent iff $C_1, C_2, \ldots, C_n$ are linearly independent. Thus, A is invertible iff $C_1, \ldots, C_n$ are linearly independent. Thus, we have proved (a) $\Leftrightarrow$ (b).

Now, the equivalence of (a) and (c) follows because A is invertible $\Leftrightarrow$ A^t is invertible
$\Leftrightarrow$ the columns of A^t are linearly independent (as we have just shown)
$\Leftrightarrow$ the rows of A are linearly independent (since the columns of A^t are the rows of A).

So we have shown that (a) $\Leftrightarrow$ (c).

Thus, the theorem is proved.

From the following example you can see how Theorem 7 can be useful.

Example 15:

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in M_3(\mathbf{R})$.

Determine whether or not A is invertible.

Solution: Let R_1, R_2, R_3 be the rows of A. We will show that they are linearly independent.

Suppose $xR_1 + yR_2 + zR_3 = \mathbf{0}$, where $x, y, z \in \mathbf{R}$. Then,

$x(1,0,1) + y(0,1,1) + z(1,1,1) = (0, 0, 0)$. This gives us the following equations.

$x + z = 0$

$y + z = 0$

$x + y + z = 0$

On solving these we get $x = 0, y = 0, z = 0$.

Thus, by Theorem 7, A is invertible.

E 41) Check if

$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix} \in M_3(\mathbf{Q})$ is invertible.

We will now see how we associate a matrix to a change of basis. This association will be made use of very often in the next block.

7.6.2 Matrix of Change of Basis

Let V be an n-dimensional vector space over **F**. Let B = {$e_1, e_2, \ldots, e_n$} and B′ = {$e_1', e_2', \ldots, e_n'$} be two bases of V. Since $e_j' \in V$, for every j, it is a linear combination of the elements of B. Suppose,

$$e_j' = \sum_{i=1}^{n} a_{ij} e_i \ \forall j = 1, \ldots, n.$$

The n×n matrix A = [a_{ij}] is called the **matrix of the change of basis from B to B′**. It is denoted by $M_B^{B'}$.

Note that A is the matrix of the transformation T ∈ L (V, V) such that T (e_j) = e_j' ∀j = 1, ..., n, with respect to the basis B. Since {$e_1', \ldots, e_n'$} is a basis of V, from Unit 5 we see that T is 1–1 and onto. Thus T is invertible. So A is invertible. Thus, **the matrix of the change of basis** from B to B′ is invertible.

Note: a) $M_B^B = I_n$. This is because, in this case $e_i' = e_i \forall i = 1, 2, \ldots, n$.

b) $M_B^{B'} = [I]_{B', B}$. This is because

$$I(e_j') = e_j' = \sum_{i=1} a_{ij} e_i \forall j = 1, 2, \ldots, n$$

Now suppose A is any invertible matrix. By Theorem 2, ∃ T ∈ L (V, V) such that $[T]_B = A$. Since A is invertible, T is invertible. Thus, T is 1–1 and onto. Let $f_i = T(e_i) \forall i = 1, 2 \ldots n$, Then B′ = {$f_1, f_2, \ldots, f_n$} is also a basis of V, and the matrix of change of basis from B to B′ is A.

In the above discussion, we have just proved the following theorem

Theorem 8: Let B = {$e_1, e_2, \ldots, e_n$} be a fixed basis of V. The mapping B′ → $M_B^{B'}$ is a 1–1 and onto correspondence between the set of all bases of V and the set of invertible n×n matrices over **F**.

Let us see an example of how to obtain $M_B^{B'}$

Example 16: In $\mathbf{R}^2$, B = {e_1, e_2} is the standard basis. Let B′ be the basis obtained by rotating B through an angle θ in the anti-clockwise direction (see Fig. 1). Then B′ = (e_1', e_2') where e_1' = (cos θ, sin θ), e_2' = (–sin θ, cos θ). Find $M_B^{B'}$

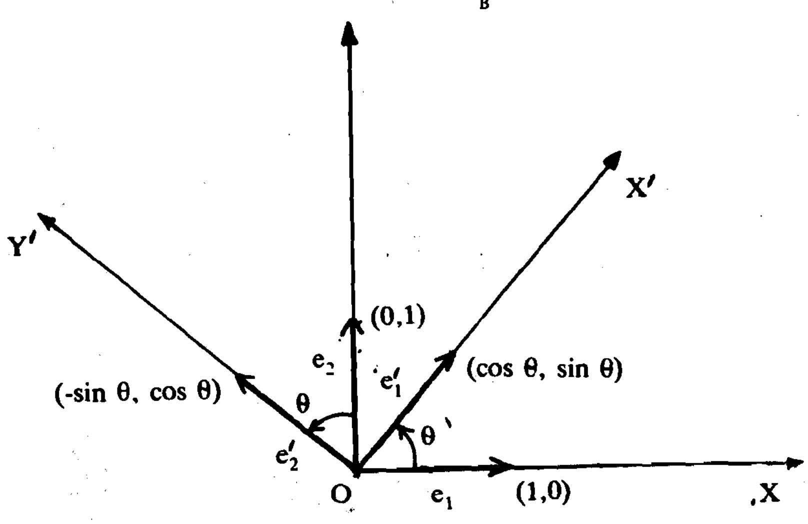

Fig. 1: Change of basis.

Solution : e_1' = cos θ (1,0) + sin θ (0,1), and

e_2' = –sin θ(1, 0) + cos θ (0,1)

Thus, $M_B^{B'} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Try the following exercise.

E E 42) Let B be the standard basis of $\mathbf{R}^3$ and B′ be another basis such that $M_B^{B'} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ What are the elements of B′?

What happens if we change the basis more than once? The following theorem tells us something about the corresponding matrices.

Theorem 9: Let B, B′, B″ be three bases of V. Then $M_B^{B'} M_{B'}^{B''} = M_B^{B''}$

Proof: Now, $M_B^{B'} M_{B'}^{B''} = [I]_{B',B}[I]_{B'',B'}$

$= [I \circ I]_{B'',B} = M_B^{B''}$

An immediate useful consequence is

Corollary : Let B, B′ be two bases of V. Then $M_B^{B'} M_{B'}^{B} = I = M_{B'}^{B} M_B^{B'}$

That is, $\left(M_B^{B'}\right)^{-1} = M_{B'}^{B}$

Proof : By Theorem 9,

$M_B^{B'} M_{B'}^{B} = M_B^{B} = I$

Similarly, $M_{B'}^{B} M_B^{B'} = M_{B'}^{B'} = I$.

But, how does the change of basis affect the matrix associated to a given linear transformation? In Sec, 7.2 we remarked that the matrix of a linear transformation depends upon the pair of bases chosen. The relation between the matrices of a transformation with respect to two pairs of bases can be described as follows.

Theorem 10: Let $T \in L(U, V)$. Let $B_1 = \{e_1, \ldots, e_n\}$ and $B_2 = \{f_1, \ldots, f_m\}$ be a pair of bases of U and V, respectively.

Let $B_1' = \{e_1', \ldots, e_n'\}$, $B_2' = \{f_1', \ldots, f_m'\}$ be another pair of bases of U and V, respectively, Then.

$[T]_{B_1', B_2'} = M_{B_2'}^{B_2} [T]_{B_1, B_2} M_{B_1}^{B_1'}$

Proof: $[T]_{B_1', B_2'} = [I_V \circ T \circ I_u]_{B_1', B_2'} = [I_V]_{B_1, B_2} [I_u]_{B_1', B_1}$

(where I_u = identity map on U and I_V = identity map on V)

$= M_{B_2}^{B_2} [T]_{B_1, B_2} M_{B_1}^{B_1'}$

Now, a corollary to Theorem 10, which will come in handy in the next block.

Corollary: Let $T \in L(V, V)$ and B, B′ be two bases of V. Then $[T]_{B'} = P^{-1}[T]_B P$, where $P = M_B^{B'}$

Proof: $[T]_B = M_B^{B'} [T]_B M_B^{B'} = P^{-1}[T]_B P$, by the corollary to Theorem 9.

Let us now recapitulate all that we have covered in this unit.

7.7 SUMMARY

We briefly sum up what has been done in this unit.

1) We defined matrices and explained the method of associating matrices with linear transformations.
2) We showed what we mean by sums of matrices and multiplication of matrices by scalars.
3) We proved that $M_{m \times n}(\mathbf{F})$ is a vector space of dimension mn over **F**.
4) We defined the transpose of a matrix, the conjugate of a complex matrix, the conjugate transpose of a complex matrix, a diagonal matrix, identity matrix, scalar matrix and lower and upper triangular matrices.

5) We defined the multiplication of matrices and showed its connection with the composition of linear transformations. Some properties of the matrix product were also listed and used.

6) The concept of an invertible matrix was explained.

7) We defined the matrix of a change of basis, and discussed the effect of change of bases on the matrix of a linear transformation.

7.8 SOLUTIONS/ANSWERS

E1) a) You want the elements in the 1st row and the 2nd column. They are 2 and 5, respectively.

b) [0 0 7]

c) The second column of A is $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

The first column of B is also $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

d) B only has 3 rows. Therefore, there is no 4th row of B.

E2) They are infinitely many answers. We give

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix} \text{and} \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \\ 4 & 0 \end{bmatrix}$$

E3) No. Because they are of different sizes.

E4) Suppose $B_1' = \{(1, 0, 1), (0, 2, -1), (1, 0, 0)\}$ and $B_2' = \{(0, 1), (1, 0)\}$

Then $T(1, 0, 1) = (1, 0) = 0.(0, 1) + 1.(1, 0)$

$T(0, 2, -1) = (0, 2) = 2.(0, 1) + 0,(1, 0)$

$T(1, 0, 0) = (1, 0) = 0.(0, 1) + 1.(1, 0)$.

$$\therefore [T]_{B_1', B_2'} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

E5) $B_1 = \{e_1, e_2, e_3\}$, $B_2 = \{f_1, f_2\}$ are the standard bases (given in Example 3).

$T(e_1) = T(1, 0, 0) = (1, 2) = f_1 + 2f_2$

$T(e_2) = T(0, 1, 0) = (2, 3) = 2f_1 + 3f_2$

$T(e_3) = T(0, 0, 1) = (2, 4) = 2f_1 + 4f_2$

$$\therefore [T]_{B_1, B_2} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$

E6) $T(1, 0, 0) = (1, 2) = 1.(1, 2) + 0.(2, 3)$

$T(0, 1, 0) = (2, 3) = 0.(1, 2) + 1.(2, 3)$

$T(1, -2, 1) = (-1, 0) = 3(1, 2) - 2(2, 3)$

$$\therefore [T]_{B_1', B_2'} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix}$$

E7) Let $B = \{1, t, t^2, t^3\}$. Then

$D(1) = 0 = 0.1 + 0.t + 0.t^2 + 0.t^3$

$D(t) = 1 = 1.1 + 0.t + 0.t^2 + 0.t^3$

$D(t^2) = 2t = 0.1 + 2.t + 0.t^2 + 0.t^3$

$D(t^3) = 3t^2 = 0.1 + 0.t + 3.t^2 + 0.t^3$.

Therefore $[D]_B$ is the given matrix.

E8) We know that

$T(e_1) = f_1$

$T(e_2) = f_1 + f_2$

$T(e_3) = f_2$

Therefore, for any $(x, y, z) \in \mathbf{R}^3$.

$T(x,y,z) = T(xe_1 + ye_2 + ze_3) = xT(e_1) + yT(e_2) + zT(e_3)$

$= xf_1 + y(f_1 + f_2) + zf_2 = (x + y) f_1 + (y + z)f_2$

$= (x + y, y + z)$

That is, $T:\mathbf{R}^3 \to \mathbf{R}^2 : T(x,y,z) = (x + y, y + z)$.

E9) We are given that

$T(1) = 0.1 + 1.i = i$

$T(i) = (-1).1 + 0.1 = -1$

$\therefore$, for any $a+ib \in \mathbf{C}$, we have

$T(a+ib) = aT(1) + bT(i) = ai - b$

E 10) a) Since [1 2] is of size 1 x 2 and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is of size 2 x 1, the sum of these matrices is not defined.

b) Both matrices are of the same size, namely, 2×2. Their sum is the matrix

$$\begin{bmatrix} 1+(-1) & 0+0 \\ 0+0 & 1+(-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

E11) $3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

and $3\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

Notice that $3\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

E12) $B_1 = \{(1,0), (0,1)\}$, $B_2 = \{(1,0,0), (0,1,0), (0,0,1)\}$

Now $S(1,0) = (1,0,0)$

$S(0,1) = (0,0,1)$

$\therefore [S]_{B_1,B_2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, a 3×2 matrix

Again, $T(1,0) = (0,1,0)$

$T(0,1) = (0,0,1)$

$\therefore [T]_{B_1,B_2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, a 3×2 matrix

$\therefore [S+T]_{B_1,B_2} = [S]_{B_1,B_2} + [T]_{B_1,B_2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$,

and $[\alpha S]_{B_1,B_2} = \alpha [S]_{B_1,B_2} = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \\ 0 & \alpha \end{bmatrix}$, for any $\alpha \in \mathbf{R}$

E13) We will prove (v) and (vi) here. You can prove (vii) and (viii) in a similar way.

v) $\alpha(A + B) = \alpha([a_{ij}] + [b_{ij}]) = \alpha[a_{ij} + b_{ij}] = [\alpha a_{ij} + \alpha b_{ij}]$

$= [\alpha a_{ij}] + [\alpha b_{ij}] = \alpha A + \alpha B$.

vi) Prove it using the fact that $(\alpha+\beta)a_{ij} = \alpha a_{ij} + \beta a_{ij}$.

E14) Since dim $\mathbf{M}_{2\times3}(\mathbf{R})$ is 6, any linearly independent subset can have 6 elements, at most.

E 15) Let $\alpha, \beta \in \mathbf{R}$ such that $\alpha [1,0] + \beta[1, -1] = [0,0]$.
Then $[\alpha+\beta, -\beta] = [0,0]$. Thus, $\beta = 0, \alpha = 0$.
$\therefore$ the matrices are linearly independent.

E16) $$E_{11} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \text{ and so on.}$$

Now any $m \times n$ matrix $A = [a_{ij}] = a_{11}E_{11} + a_{12}E_{12} + \dots + a_{mn} E_{mn}$ (For example, in the 2×2 situation,

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, $\{E_{ij} \mid i = 1, \dots, m, j = 1, \dots, n\}$ generates $\mathbf{M}_{m\times n}(\mathbf{F})$. Also, if α_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n$, be scalars such that $\alpha_{11} E_{11} + \alpha_{12}E_{12} + \dots + \alpha_{mn}E_{mn} = \mathbf{0}$.

Then,

$$\text{we get} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

Therefore, $\alpha_{ij} = 0 \; \forall i, j$.

Hence, the given set is linearly independent. $\therefore$ it is a basis of $\mathbf{M}_{m\times n}(\mathbf{F})$. The number of elements in this basis is mn.

$\therefore \dim \mathbf{M}_{m\times n}(\mathbf{R}) = mn$.

E17) $A^t = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$. In this case $A^t = A$.

E18) b) $\alpha A = [\alpha a_{ij}]$. $\therefore (\alpha A)^t = [b_{ij}]$, where
b_{ij} = (j,i)th element of $\alpha A = \alpha a_{ji}$
= α times the (j,i)th element of A
= α times the (i,j) th element of A^t
= (i,j)th element of αA^t.
$\therefore (\alpha A)^t = \alpha A^t$.

c) Let $A = [a_{ij}]$. Then $A^t = [b_{ij}]$, where $b_{ij} = a_{ji}$.
$\therefore (A^t)^t = [c_{ij}]$, where $c_{ij} = b_{ji} = a_{ij}$
$\therefore (A^t)^t = A$.

E19) Let A be an $m \times n$ matrix. Then A^t is an $n \times m$ matrix.

$\therefore$, for $A = A^t$, their sizes must be the same, that is, $m = n$.
$\therefore$ A must be a square matrix.

E20) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a square matrix over a field $\mathbf{F}$.

Then $A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$$\therefore A + A^t = \begin{bmatrix} a+a & b+c \\ c+b & d+d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}, \text{ and}$$

$$A - A^t = \begin{bmatrix} a-a & b-c \\ c-b & d-d \end{bmatrix} = \begin{bmatrix} 0 & b-c \\ -(b-c) & 0 \end{bmatrix}.$$

You can check that $(A+A^t)^t = A + A^t$ and $(A - A^t)^t = -(A - A^t)$.

$\therefore A+A^t$ is symmetric and $A - A^t$ is skew-symmetric.

E21) $\begin{bmatrix} -i & 2 \\ 3 & -i \end{bmatrix}$

E22) The size of $\bar{A}^t$ is the same as the size of A^t. $\therefore A = \bar{A}^t$ implies that the sizes of A and A^t are the same. $\therefore$ A is a square matrix.

E23) $I:\mathbf{R}^n \to \mathbf{R}^n : I(x_1,\ldots,x_n) = (x_1,\ldots, x_n)$.

Then, for any basis $B = \{e_1,\ldots, e_n\}$ of $\mathbf{R}^n$, $I(e_i) = e_i$.

$$\therefore [I]_B = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} = I_n$$

E24) Since A is upper triangular, all its elements below the diagonal are zero. Again, since $A = A^t$, a lower triangular matrix, all the entries of A above the diagonal are zero. $\therefore$, all the off-diagonal entries of A are zero. $\therefore$ A is a diagonal matrix.

E25) Let A be a skew-symmetric matrix. Then $A = -A^t$. Therefore,

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} = \begin{bmatrix} -a_{11} & -a_{21} & \ldots & -a_{n1} \\ -a_{12} & -a_{22} & \ldots & -a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \ldots & -a_{nn} \end{bmatrix}$$

$\therefore$, for any $i = 1,\ldots, n$, $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0 \Rightarrow a_{ij} = 0$.

The converse is not true. For example, the diagonal entries of $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ are zero, but this matrix is not skew-symmetric.

E26) $[1 \times 1 + 0 \times 2 + 0 \times 3] = [1]$

E27) $$AB = \begin{bmatrix} 1\times 1 + 1 \times 1 & 1 \times 0 + 1 \times 1 \\ 0 \times 1 + 1 \times 1 & 0 \times 0 + 1 \times 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

E 28) C + D is not defined.

CD is a 2×2 matrix and DC is a 3×3 matrix. $\therefore CD \neq DC$.

$$CD = \begin{bmatrix} 1\times 0 + 1 \times 1 + 0 \times 0 & 1 \times 1 + 1 \times 1 + 0 \times 0 \\ 0 \times 0 + 1 \times 1 + 0 \times 0 & 0 \times 1 + 1 \times 1 + 0 \times 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$DC = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0\times 1 + 1 \times 0 & 0 \times 1 + 1 \times 1 & 0 \times 0 + 1 \times 0 \\ 1 \times 1 + 1 \times 0 & 1 \times 1 + 1 \times 1 & 1 \times 0 + 1 \times 0 \\ 0 \times 1 + 0 \times 0 & 0 \times 1 + 0 \times 1 & 0 \times 0 + 0 \times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

E29) $$A + B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \therefore (A + B)^2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$\text{Also } A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix},$$

$$2 \cdot AB = 2\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\therefore A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix}$$

$\therefore (A + B)^2 \neq A^2 + 2AB + B^2$

E30) $$A^2 = \begin{bmatrix} -bd & b \\ d^2 b & db \end{bmatrix}\begin{bmatrix} -bd & b \\ d^2 b & db \end{bmatrix} = \begin{bmatrix} b^2 d^2 + d^2 b^2 & -b^2 d + db^2 \\ -b^2 d^3 + d^3 b^2 & d^2 b^2 + d^2 b^2 \end{bmatrix} = \begin{bmatrix} 2d^2 b^2 \\ 0 \end{bmatrix}$$

E31) $\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x\\2y\\3z\end{bmatrix}$

$[x\ y\ z]\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix} = [x \quad 2y \quad 3z]$

E32) We take $A=\begin{bmatrix}1 & 2\\0 & 0\\3 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 2 & 3 & 0\\4 & 5 & 1 & 1\end{bmatrix}$. Then

$AB=\begin{bmatrix}9 & 12 & 5 & 2\\0 & 0 & 0 & 0\\7 & 11 & 10 & 1\end{bmatrix}$, You can see that the 2nd row of AB is zero.

E33) $[S]_B=\begin{bmatrix}0 & 0 & 0\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$, $[T]_B=\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 1 & 0\end{bmatrix}$

$\therefore [S]_B[T]_B=\begin{bmatrix}0 & 0 & 0\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix}$

Also, $[SoT]_B=\begin{bmatrix}0 & 0 & 0\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix}=[S]_B[T]_B$

E34) $(A+B)^2=(A+B)(A+B)=A(A+B)+B(A+B)$ (by distributivity)

$= A^2+AB+BA+B^2$ (by distributivity)

E35) $AB=\begin{bmatrix}-1 & -8 & -10\\1 & -2 & -5\\9 & 22 & 15\end{bmatrix}$ $\therefore 2(AB)=\begin{bmatrix}-2 & -16 & -20\\2 & -4 & -10\\18 & 44 & 30\end{bmatrix}$

On the other hand, $(2A)B=\begin{bmatrix}4 & -2\\2 & 0\\-6 & 8\end{bmatrix}\begin{bmatrix}1 & -2 & -5\\3 & 4 & 0\end{bmatrix}$

$=\begin{bmatrix}-2 & -16 & -20\\2 & -4 & -10\\18 & 44 & 30\end{bmatrix}$

$\therefore 2(AB)=(2A)B$

E36) $AB=\begin{bmatrix}0 & -7 & -3\\-11 & -4 & 6\\0 & 0 & 0\end{bmatrix}$. $\therefore (AB)^t=\begin{bmatrix}0 & -11 & 0\\-7 & -4 & 0\\-3 & 6 & 0\end{bmatrix}$

Also, $B^tA^t=\begin{bmatrix}1 & 2 & 4\\-4 & -1 & 0\\0 & 3 & -2\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\-1 & 0 & 0\\0 & -3 & 0\end{bmatrix}=\begin{bmatrix}0 & -11 & 0\\-7 & -4 & 0\\-3 & 6 & 0\end{bmatrix}=(AB)^t$

E37) First, suppose AB is symmetric. Then $AB=(AB)^t=B^tA^t=BA$, since A and B are symmetric.

Conversely, suppose $AB = BA$. Then

$(AB)^t = B^tA^t = BA = AB$, so that AB is symmetric.

E38) Let $A=\text{diag}(d_1,\dots,d_n)$, $B=\text{diag}(e_1,\dots,e_n)$. Then

$$AB=\begin{bmatrix}d_1 & 0 & 0 & \dots & 0\\0 & d_2 & 0 & \dots & 0\\\vdots & \vdots & \vdots & & \vdots\\0 & 0 & \vdots & \dots & d_n\end{bmatrix}\begin{bmatrix}e_1 & 0 & 0 & \cdots & 0\\0 & e_2 & 0 & \cdots & 0\\\vdots & \vdots & \vdots & \ddots & \vdots\\0 & 0 & \vdots & \cdots & e_n\end{bmatrix}$$

$$= \begin{bmatrix} d_1 e_1 & 0 & 0 & \cdots 0 \\ 0 & d_2 e_2 & 0 & \cdots 0 \\ 0 & 0 & d_3 e_3 & \cdots 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots d_n e_n \end{bmatrix}$$

$= \text{diag}(d_1e_1, d_2e_2, \ldots, d_ne_n)$.

E39) Suppose it is invertible. Then $\exists\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

This gives us $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ which is the same as the given matrix. This shows that the given matrix is invertible and, in fact, $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

E40) Firstly, θ is a well defined map. Secondly, check that $\theta(v_1 + v_2) = \theta(v_1) + \theta(v_2)$, and $\theta(\alpha v) = \alpha\theta(v)$ for $v, v_1, v_2 \in V$ and $\alpha \in F$. Thirdly, show that $\theta(v) = \mathbf{0} \Rightarrow v = \mathbf{0}$, that is θ is $1-1$. Then, by Unit 5 (Theorem 10), you have shown that θ is an isomorphism.

E41) We will show that its columns are linearly independent over **Q**. Now, if $x, y, z \in$ **Q** such that

$$x\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} + z\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$ we get the equations

$2x + z = 0$

$z = 0$

$3y = 0$

On solving them we get $x = 0, y = 0, z = 0$.

$\therefore$ the given matrix is linearly independent.

E42) Let $B = \{e_1, e_2, e_3\}$ $B' = \{f_1, f_2, f_3\}$. Then

$f_1 = 0e_1 + 1e_2 + 0e_3 = e_2$

$f_2 = e_1 + e_2$

$f_3 = e_1 + 3e_3$

$\therefore B' = \{e_2, e_1 + e_2, e_1 + 3e_3\}$.