UNIT 8 GEOMETRICAL PROPERTIES OF CURVES

Structure

8.1 INTRODUCTION

We started our study of Calculus by stating two problems. One of them was the problem of finding a tangent to a curve at a given point. In Unit 3 we have seen that the solution of this problem was instrumental in the development of differential calculus. Now having studied various techniques of differentiation, we shall once again take up this problem. The study of the tangents of a curve will then lead us to normals and asymptotes of curves, which we shall study in Sec.2 and Sec.5, respectively. In the last unit we discussed some other geometric features of functions, like maxima, minima, points of inflection and curvature. You will see that all these will prove very useful when we tackle curve tracing in the next unit.

Objectives

After studying this unit you should be able to

- obtain the equations of the tangent and the normal to a given curve at a given point
- calculate the angle of intersection of two curves at a given point of intersection
- obtain the angle between the radius vector and the tangent at a point on a given curve
- define and identify a singular point, a node and a cusp
- define asymptotes and obtain their equations.

8.2 EQUATIONS OF TANGENTS AND NORMALS

In Unit 3 we have seen how a tangent can be defined precisely with the help of derivatives. We have noted that the slope of a tangent to the curve $y = f(x)$ at (x_0, y_0) is given by $f'(x_0)$, whenever it exists. In fact, we had also obtained the equations of the tangents of some simple curves. Once we know how to find the equation of a tangent, it is easy to find one for a normal too. A normal to a curve, $y = f(x)$ at (x_0, y_0) is a line which passes through (x_0, y_0) and is perpendicular to the tangent at that point. This means that the slope of this normal will be

$$-\frac{1}{f'(x_0)}, \text{ if } f'(x_0) \neq 0.$$

A line L_1 is perpendicular to a line L_2 iff $m_1m_2 = -1$, where m_1 and m_2 are the slopes of L_1 and L_2, respectively.

Now, what happens when $f'(x_0) = 0$? $f'(x_0) = 0$ implies that the slope of the tangent at (x_0, y_0) is zero, that is, this tangent is parallel to the x-axis. In this case, the normal (which is perpendicular to the tangent) would be parallel to the y-axis. The equation of this normal, would then be $x = x_0$.

Now let us look at various curves and try to obtain the equations of their tangents and normals.

Example 1 Consider the curve $y = 2\sqrt{ax}$, $a \neq 0$, shown in Fig. 1.

Here, $\frac{dy}{dx} = \sqrt{\frac{a}{x}} = \frac{2a}{y}$. Thus, $\frac{dy}{dx}$ exists and is non-zero for all $y \neq 0$. Now y will be zero only if x is zero. Thus, we can find the equations of tangents and normals to this curve at any point, except the origin (0, 0). We know that the slope of the tangent at any point (x_0, y_0) will be $2a/y_0$. The slope of the normal will, therefore, be $-y_0/2a$. Thus, the equation of the tangent at (x_0, y_0) is

a = 0 gives us the trivial case when the curve is the line y = 0, or the x-axis.

$$y - y_0 = \frac{2a}{y_0}(x - x_0)$$

$$\Longrightarrow yy_0 - y_0^2 = 2ax - 2ax_0$$

$$\Longrightarrow yy_0 = 2ax + y_0^2 - 2ax_0$$

$$\Longrightarrow yy_0 = 2a(x + x_0), \text{ since } y_0^2 = 4ax_0.$$

Recall that the equation of a line through (x_0, y_0) having a slope m is $y - y_0 = m(x - x_0)$.

The equation of the normal at (x_0, y_0) is

$$y - y_0 = \frac{-y_0}{2a}(x - x_0)$$

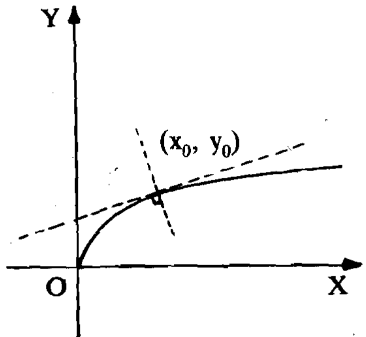


Fig. 1

Now let us see an example where the equation of the curve is given in the parametric form. In Unit 4 we have already seen what a parameter is.

Example 2 To find the equations of the tangent and the normal at the point $\theta = \pi/4$ to the curve given by $x = a\cos^3\theta$, $y = a\sin^3\theta$, (see Fig. 2), we first note that

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

Hence, the slope of the tangent at $\theta = \pi/4$ is $-\tan\pi/4 = -1$. The slope of the normal at this point thus comes out to be 1. Now, if $\theta = \pi/4$, $\cos\theta = 1/\sqrt{2}$ and $\sin\theta = 1/\sqrt{2}$

Thus, $x = a/2\sqrt{2}$ and $y = a/2\sqrt{2}$.

The equation of the tangent at $(a/2\sqrt{2}, a/2\sqrt{2})$ is

$$y - \frac{a}{2\sqrt{2}} = -1\left(x - \frac{a}{2\sqrt{2}}\right)$$

That is, $x + y = \frac{a}{\sqrt{2}}$ or $\sqrt{2}(x + y) = a$

The equation of the normal at $(a/2\sqrt{2}, a/2\sqrt{2})$ is

$$y - \frac{a}{2\sqrt{2}} = 1\left(x - \frac{a}{2\sqrt{2}}\right)$$

or $y = x$.

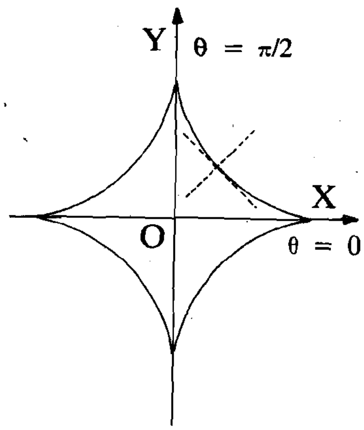


Fig. 2

Example 3 illustrates the method of finding the equations of tangents and normals when the equation of the curve is given in the implicit form.

Example 3 Let us find the equations of the tangent and the normal to the curve defined by $x^3 + y^3 - 6xy = 0$ at a point (x_0, y_0) on it.

Fig. 3 shows this curve.

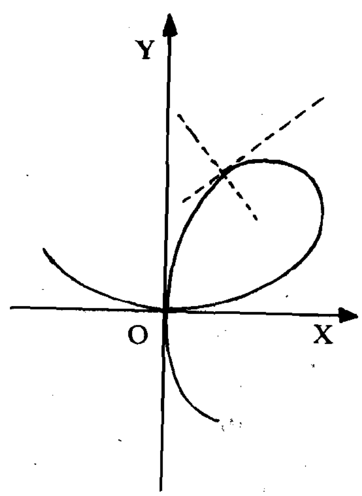


Fig. 3

In Unit 4 we have seen how to calculate the derivative when the relation between x and y is expressed implicitly. We shall follow the same procedure again. Differentiating the given equation throughout with respect to x, we get

$$3x^2 + 3y^2 \frac{dy}{dx} - 6y - 6x \frac{dy}{dx} = 0, \text{ which means}$$

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

Thus, the slope at the point (x_0, y_0) is $\frac{2y_0 - x_0^2}{y_0^2 - 2x_0}$.

Hence, the equation of the tangent at (x_0, y_0) is

$$y - y_0 = \frac{2y_0 - x_0^2}{y_0^2 - 2x_0} (x - x_0)$$

Simplifying, and using the relation $x_0^3 + y_0^3 = 6x_0y_0$, this reduces to

$$(2y_0 - x_0^2)x + (2x_0 - y_0^2)y + 2x_0y_0 = 0$$

Now the normal at (x_0, y_0) is a line passing through (x_0, y_0) and having slope $\frac{-(y_0^2 - 2x_0)}{2y_0 - x_0^2}$. Hence, the equation of the normal at (x_0, y_0) is

$$y - y_0 = -\frac{y_0^2 - 2x_0}{2y_0 - x_0^2} (x - x_0)$$

Simplifying, we get

$$(y_0^2 - 2x_0)x + (2y_0 - x_0^2)y + (x_0 - y_0)(2x_0 + x_0y_0 + 2y_0) = 0$$

If you have followed these examples, you should have no problem in solving the following exercises.

E E 1) Find the equations of the tangent and the normal to each of the following at the specified point.

a) $y = x^2 + 2x + 1$ at $(1, 4)$

b) $x = a \cos t, y = b \sin t$ at the point given by $t = \pi/4$

c) $x^2 + y^2 = 25$ at $(-3, 4)$

Vertical Tangents

By now, you are quite familiar with the fact that f′(x) or dy/dx may not exist at some points. At such points either the tangent does not exist, or else, is parallel to the y-axis, that is, vertical. To examine the existence of vertical tangents at (x_0, y_0), we examine $\left.\frac{dx}{dy}\right|_{y = y_0}$. If $\left.\frac{dx}{dy}\right|_{y = y_0} = 0$, then, we conclude that there is a vertical tangent at (x_0, y_0). In such cases the equation of the tangent can be written as $x = x_0$.

The normal corresponding to a vertical tangent will obviously be horizontal or parallel to the x-axis. This means we can write its equation as $y = y_0$, as its passes through (x_0, y_0).

If you take the curve in Example 2, you will find that $\frac{dy}{dx}$ does not exist when $\theta = \pi/2$.

Let us examine $\frac{dx}{dy}$ at this point. $\frac{dx}{dy} = -\cot\theta = 0$ if $\theta = \pi/2$.

This means that the curve has a vertical tangent and, consequently, a horizontal normal at this point. Now, when $\theta = \pi/2$, $x = 0$ and $y = a$. Thus the equation of the tangent at $(0, a)$ is $x = 0$ and that of the normal is $y = a$.

See if you can solve this exercise now.

E E 2) Are there any points on the following curves where the tangent is parallel to either axis? If yes, find all such points.

a) $y = x^3 - x^2 - 2x$

b) $y = \sin x$

Let us now look at another example.

Example 4 To find the equations of those tangents to the curve $y = x^3$, which are parallel to the line $12x - y - 3 = 0$, we first observe that the slope of the line $12x - y - 3 = 0$ is 12. Thus, the slope of any line parallel to this line should also be 12. Now, the slope of the tangent to the curve $y = x^3$ at any point (x, y) is $f'(x) = 3x^2$.

If we equate $f'(x)$ to 12, we will get those points on the curve where the tangent is parallel to $12x - y - 3 = 0$.

Thus, $3x^2 = 12$, or $x^2 = 4$, that is, $x = \pm 2$.

If $x = 2$, $y = x^3 = 8$. If $x = -2$, $y = x^3 = -8$.

Thus, the points in question are $(2, 8)$ and $(-2, -8)$. The equations of the tangents at these points are

$y - 8 = 12(x - 2)$ and $y + 8 = 12(x + 2)$, respectively.

The following exercises will give you some more practice in applying the concepts learned in this section.

E 3) Find the equations of the tangent and the normal to each of the following curves at the point 't':

a) $x = at^2$, $y = 2at$

b) $x = a(t + \sin t)$, $y = a(1 - \cos t)$

E 4) Find the equation of the tangent to each of the following curves at the point (x_0, y_0).

a) $x^2 + y^2 + 4x + 6y - 1 = 0$

b) $xy = a$

E . E 5) Prove that the line $2x + 3y = 1$ touches the curve $3y = e^{-2x}$ at a point whose x-coordinate is zero.

E . E 6) Prove that the equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at a point $(a\sqrt{2}, b)$ is $ax + b\sqrt{2}y = (a^2 + b^2)\sqrt{2}$.

8.3 ANGLE OF INTERSECTION OF TWO CURVES

The concept of a tangent to a curve has proved very useful in describing various geometrical features of the curve. In this section we shall look at one such feature.

When two curves intersect at a point, their angle of intersection at that point can be defined with the help of their tangents there. In fact, we say that if two curves intersect at a point P, the angle of intersection of these two curves at P is an angle between the tangents to these curves at P, such that $0 \leq \theta \leq \pi/2$ (see Fig. 4).

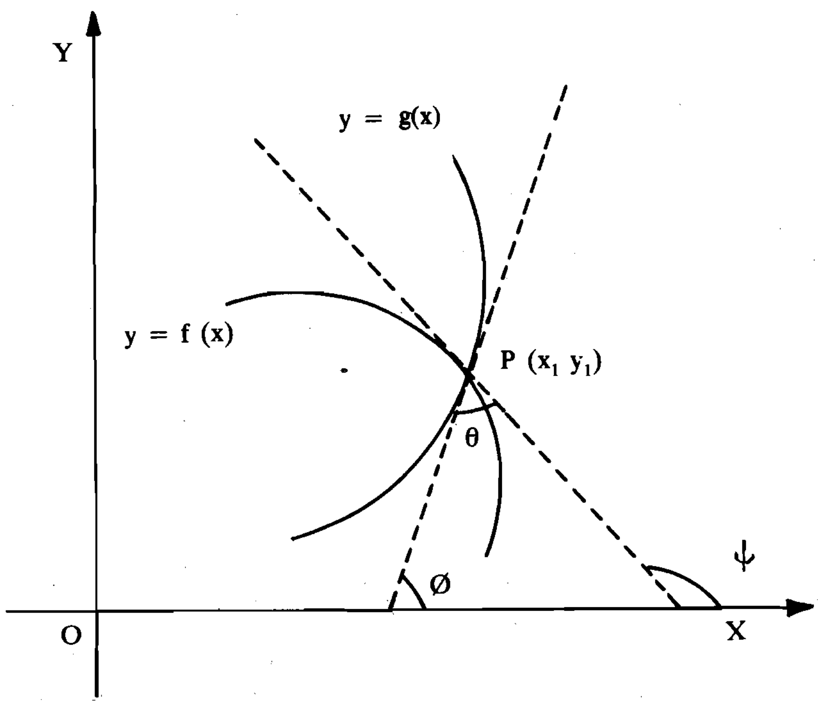


Fig. 4

We now prove a theorem which gives us the angle of intersection at a point when the equations of the two curves are known.

Theorem 1 If two curves $y = f(x)$ and $y = g(x)$ intersect at a point $P(x_1, y_1)$, then the angle θ of intersection of these curves at $P(x_1, y_1)$ is given by

$$\tan \theta = \left| \frac{f'(x_1) - g'(x_1)}{1 + f'(x_1)g'(x_1)} \right|$$

Proof From Fig. 4, $\tan \theta = \tan (\Psi - \Phi)$

$$= \frac{\tan \Psi - \tan \Phi}{1 + \tan \Psi \tan \Phi}$$

$$= \frac{f'(x_1) - g'(x_1)}{1 + f'(x_1)g'(x_1)}$$

Fig. 4 shows $\Psi - \Phi$ to be an acute angle. But if the curves f and g were as in Fig. 5, then angle $\theta = \pi - (\Psi - \Phi)$, since we take the acute angle as the angle of intersection.

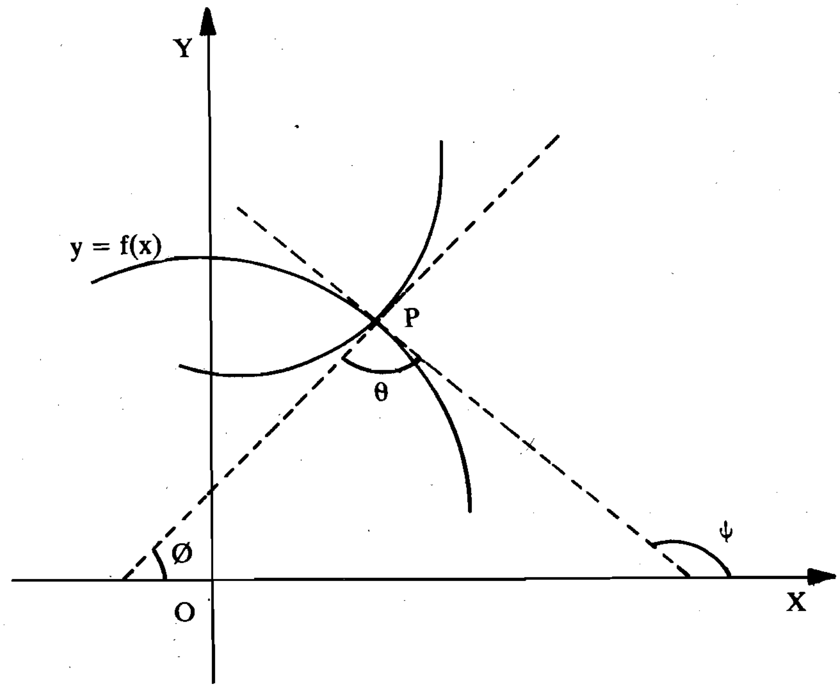


Fig. 5

In this case, $\tan\theta = \tan[\pi - (\Psi - \Phi)] = -\tan(\Psi - \Phi)$

But we are not in a position to decide whether we should take $\tan\theta$ as $\tan(\Psi - \Phi)$ or as $-\tan(\Psi - \Phi)$, unless we have drawn the curves. Since it would be tedious to first draw the curves and then decide, we think of an alternate scheme. We observe that since θ lies between 0 and $\pi/2$, $\tan\theta$ is non-negative. Thus, we take $\tan\theta$ to be $|\tan(\Psi - \Phi)|$.

Hence, $\tan\theta = \left|\dfrac{f'(x_1) - g'(x_1)}{1 + f'(x_1)\,g'(x_1)}\right|$

Having proved this theorem, we can easily deduce the following corollaries.

Corollary 1 Two curves $y = f(x)$ and $y = g(x)$ touch each other at (x_1, y_1), that is, have common tangents at (x_1, y_1), iff $\theta = 0$, that is, iff

$f'(x_1) = g'(x_1)$.

Corollary 2 Two curves cut each other at right angles, or orthogonally, at (x_1, y_1) iff

$f'(x_1)\,g'(x_1) = -1$.

If you study Example 5 carefully, you will have no difficulty in solving the exercises later.

Example 5 Let us find the angle of intersection of the parabola $y^2 = 2x$ and the circle $x^2 + y^2 = 8$.

First we find the points of intersection of these curves, if there are any. The coordinates of these points will satisfy both the equation to the parabola and the equation to the circle.

So substituting $y^2 = 2x$ in $x^2 + y^2 = 8$, we get

$x^2 + 2x = 8$, or $x = -4$ or 2.

It is clear from $y^2 = 2x$ that the abscissa $x\ (= y^2/2)$ of any common point must be non-negative. So we reject the value -4 of x. When $x = 2$, $y = \pm 2$. Hence the common points are P(2, 2) and Q(2, $-$ 2). Since both curves are symmetric about the x-axis (see Fig. 6) and since P and Q are reflections of each other w.r.t. the x-axis, it is sufficient to find the angle at one point, say P; the angle at Q being equal to the angle at P.

Differentiating the two equations w.r. to x, we get

$$2y\frac{dy}{dx} = 2 \text{ and } 2x + 2y\frac{dy}{dx} = 0$$

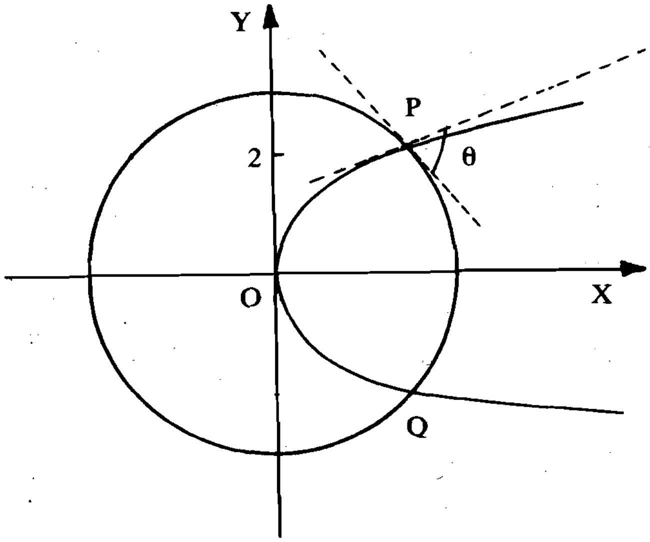


Fig. 6

Hence the values of $f'(x)$ and $g'(x)$, that is, the slopes of the tangents to the two curves at (x,y) are $1/y$ and $-x/y$. Hence the slopes of the tangents at (2, 2) to the two curves are $1/2$ and -1. Hence if θ is the required angle, then

$$\tan\theta = \left|\frac{1/2-(-1)}{1+1/2(-1)}\right| = 3$$

Hence, $\theta = \tan^{-1}3 \simeq 71.56°$

You can try these exercises now.

E

E 7) Find the angle of intersection of the parabolas $y^2 = 4x$ and $x^2 = 4y$.

E

E 8) Show that

a) the ellipse $x^2 + 4y^2 = 8$ and the hyperbola $x^2 - 2y^2 = 4$ cut each other orthogonally (at right angles) at four points.

b) the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other (have a common tangent) at two points.

You know that given a pair of axes in a plane, the position of a point in the plane can be determined if we know its distances from the x-and y-axes. There is one more way in which we can determine its position. Suppose we are given an initial line OX in a plane (see Fig. 7(a)). Then a point P can be located if we know

i) r, its distance from O, and

ii) θ, the angle made by OP with OX.

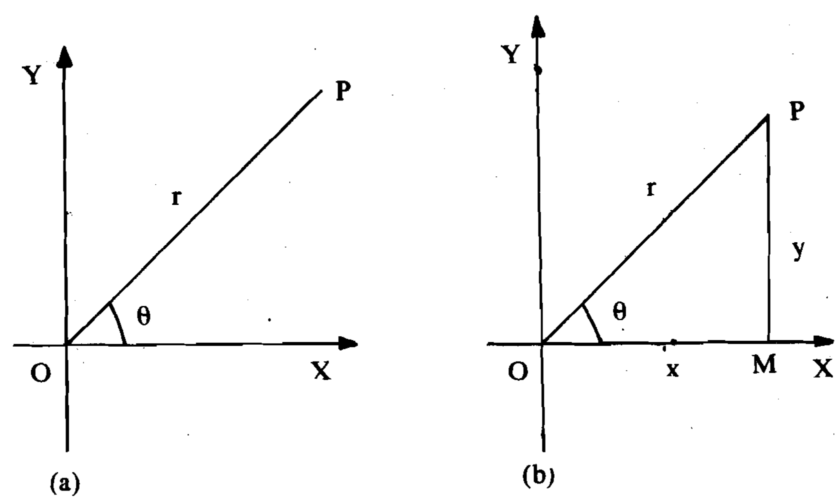


Fig. 7

r, and θ are called the **polar coordinates** of P. r is always taken to be non-negative, and θ takes values between 0 and 2π. From Fig. 7(b) it is clear that if x and y are the cartesian tan θ = y/x. The equation of a curve is sometimes expressed in polar coordinates by an equation r = f(θ). For example, the equation of a circle with centre O and radius r is r = a. Now let us turn once again to the problem of finding the angle of intersection of two curves.

The method that we have been following till now, cannot be used if the equation of the curve is given in the polar form. In this case we follow a somewhat indirect method.

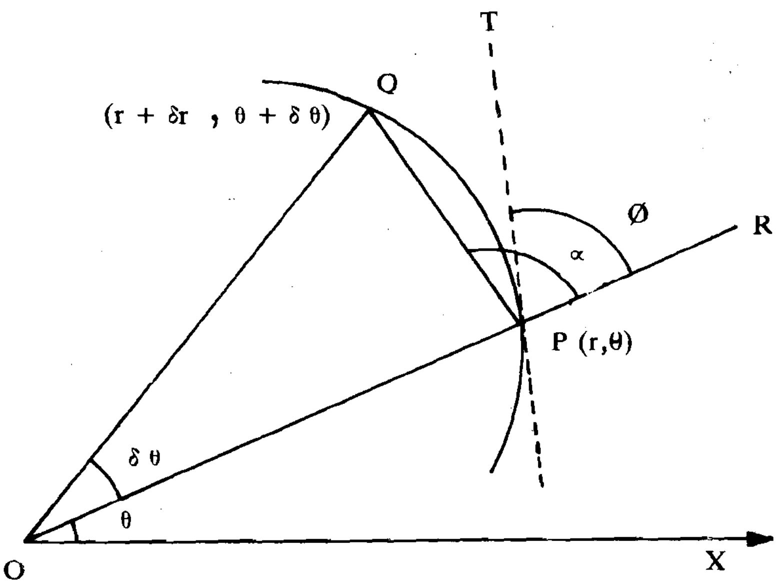


Fig. 8

Take a look at Fig. 8. It shows a curve whose equation is given in the polar form as r = f(θ). P(r,θ) and Q(r + δr, $\theta + \delta\theta$) are two points on this curve. PT is the tangent at P and OPR is the line through the origin and the point P. We shall now try to find θ, the angle between PT and OR.

We note here, that the tangent PT is the limiting position of the secant PQ. If we denote the angle between PQ and OR by α, then we can similarly say that ϕ is the limit of α as Q→P along the curve.

Now from ΔOPQ we have

Remember the sine rule for a Δ ABC?

$$\frac{\sin A}{A} = \frac{\sin B}{B} = \frac{\sin C}{C}.$$

$$\frac{OQ}{OP} = \frac{\sin \angle OPQ}{\sin \angle OQP}$$

$$\text{or} \quad \frac{r + \delta r}{r} = \frac{\sin(\pi - \alpha)}{\sin(\alpha - \delta\theta)}$$

$$\text{or} \quad 1 + \frac{\delta r}{r} = \frac{\sin(\pi - \alpha)}{\sin(\alpha - \delta\theta)}$$

$$\text{or} \quad \frac{\delta r}{r} = \frac{\sin\alpha - \sin(\alpha - \delta\theta)}{\sin(\alpha - \delta\theta)} \quad (\text{since } \sin(\pi - \alpha) = \sin\alpha)$$

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

$$\text{or} \quad \frac{1}{r}\frac{\delta r}{\delta\theta} = \frac{2\cos(\alpha - \delta\theta/2).\sin(\delta\theta/2)}{\sin(\alpha - \delta\theta).\delta\theta}$$

$$= \frac{2\cos(\alpha - \delta\theta/2)}{\sin(\alpha - \delta\theta)} \quad \frac{\sin(\delta\theta/2)}{\delta\theta/2}$$

As Q→P, $\alpha \to \phi$. $\delta\theta \to 0$ and δr→0. Hence as Q→P, we get

Recall $\lim\limits_{\delta\theta \to 0} \frac{\sin(\delta\theta/2)}{\delta\theta/2} = 1$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\phi}{\sin\phi} = \cot\phi$$

$$\text{so that } \tan\phi = r.\frac{d\theta}{dr}$$

This formula helps us to find the angle between OP and the tangent at the point P on the

We shall use this result to find the angle between two curves C_1 and C_2 which intersect at P (say). If the angles between OP and the tangents to C_1 and C_2 at P are ϕ_1 and ϕ_2, respectively, the angle of intersection of C_1 and C_2 will be $|\phi_1 - \phi_2|$ (see Fig. 9).

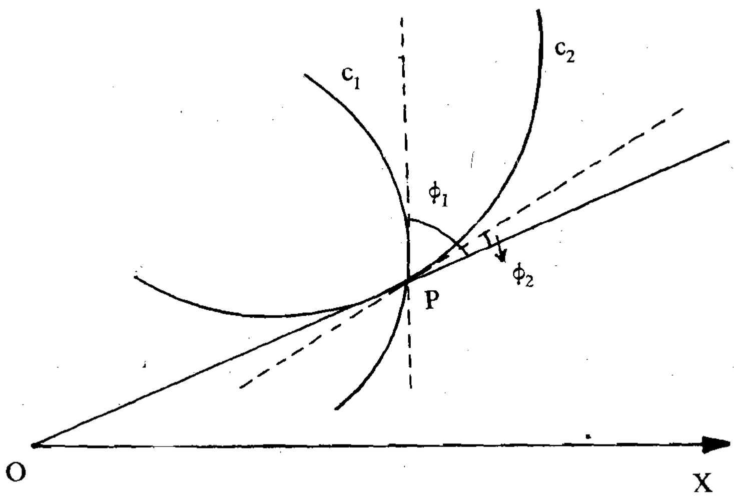


Fig. 9

This can be easily calculated as we know tan ϕ_1 and tan ϕ_2.

$$\text{Thus, } \tan|\phi_1 - \phi_2| = \left|\frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2}\right|$$

Further, if the curves intersect orthogonally, $\tan\phi_1 . \tan\phi_2 = -1$. The following examples will help clarify this discussion.

Example 6 Suppose we want to find the angle of intersection of the curves $r = a \sin 2\theta$ and $r = a \cos 2\theta$ at the point $P(a/\sqrt{2}, \pi/8)$. The coordinates of P satisfy both the equations $r = a \sin 2\theta$ and $r = a \cos 2\theta$.

If ϕ_1 is the angle between OP and the tangent to $r = a \sin 2\theta$, then

$$\tan\phi_1 = r\frac{d\theta}{dr} = \frac{a \sin 2\theta}{dr/d\theta} = \frac{a \sin 2\theta}{2a \cos 2\theta} = \frac{1}{2} \tan 2\theta = \frac{1}{2}$$

Similarly, if ϕ_2 is the angle between OP and the tangent to $r = a \cos 2\theta$, then

$$\tan\phi_2 = r\frac{d\theta}{dr} = -\frac{1}{2} \cot 2\theta = -\frac{1}{2}$$

$$\text{Thus, } \tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2} = \frac{1/2 + 1/2}{1 - 1/4} = \frac{4}{3}$$

Thus, $\phi_1 - \phi_2 = \tan^{-1}(4/3) \simeq 53.13°$, which is the required angle.

Now try to do a few exercises on your own.

E E 9) Find the angle between the line joining a point $P(r,\theta)$ on the curve to the origin and the tangent for each of the following curves.

a) $r^2 = a^2 \cos 2\theta$ b) $1/r = 1 + e \cos\theta$

c) $r^m = a^m \cos m\theta$ b) $r^m = a^m(\cos m\theta - \sin m\theta)$

E 10) Check whether the following two curves intersect orthogonally.

a) $r = ae^{\theta}$ and $re^{\theta} = b$

b) $r = a(1 + \sin\theta)$ and $r = a(1 - \sin\theta)$

8.4 SINGULAR POINTS

In this section we shall study a category of points on curves, called singular points. But to properly classify singular points we have to find the nature of tangents at these points. So let us first study on easy method of finding the tangents to a curve at the origin. This knowledge will then help us to find the tangents at any point of the curve easily.

8.4.1 Tangents at the Origin

We shall now give you a simple method of finding the tangent to a curve at the origin, when the equation of the curve is given by a polynomial equation. That is, the equation is of the form $f(x,y) = 0$ where f is a polynomial in x and y. You will agree that the constant term in this polynomial is zero since the curve passes through the origin. For such a curve the equation of the tangent at the origin can be found out by equating to zero the lowest degree terms in x and y (we shall not prove this here).

Thus, if $x^3 + 3xy + 2x + y = 0$ is the equation of a curve, the equation of the tangent at the origin is $2x + y = 0$.

Similarly, if the equation of a curve is $x^4 + x^2 - y^2 = 0$, then the equation of the tangent to this curve at (0,0) is $x^2 - y^2 = 0$, or $x^2 = y^2$, or $x = \pm y$.

Hence we get two equations $x = y$ and $x = -y$. This means that the curve $x^4+x^2-y^2 = 0$ has two tangents at the origin. We shall consider such eventualities in the next sub-section.

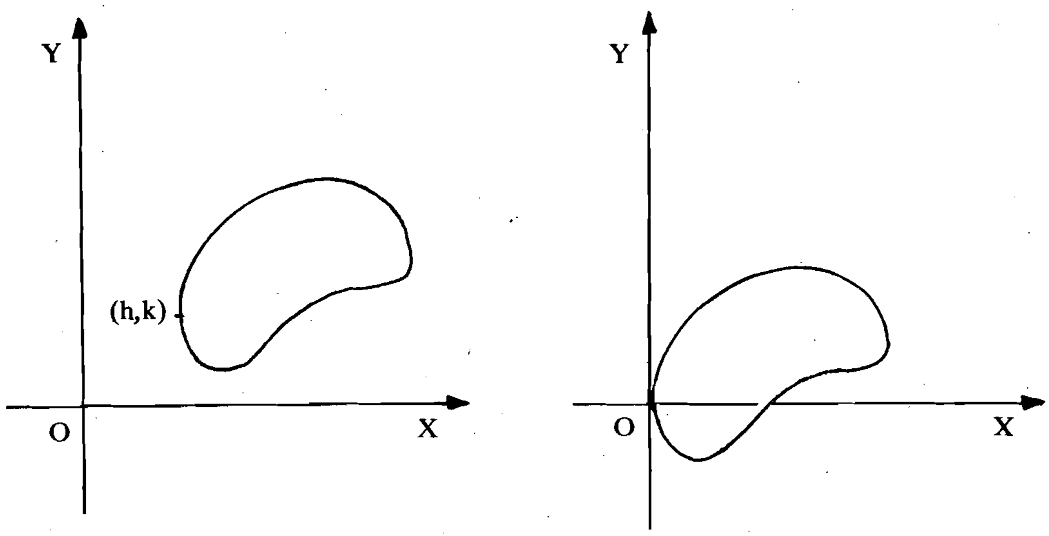
When the origin is shifted to (h, k) the coordinates of a point P(x, y) in the new coordinate system are given by

$x' = x - h$
$y' = y - k$

Now, consider a curve given by $g(x,y) = 0$, where $g(x, y)$ is some polynomial in x and y. Suppose we want to find the equation of the tangent to this curve at some point, say (h, k), on it. What we do is, we shift the origin to (h, k). Then with respect to this new origin, the equation of the curve will be

$g(x' + h, y + k) = 0$. We can also express it by $G(x , y) = 0$, where $G(x', y') = g(x + h, y + k)$.

Note that this change of origin does not change the shape of the curve. Now the tangent to the curve $G(x,y) = 0$ at the origin will be the tangent to the curve $g(x,y) = 0$ at the point (h, k) (see Fig. 10(a) and (b)).



(a) Fig. 10 (b)

The method for finding the equation of the tangent at any point of a curve will be clear to you when you read our next example.

Example 7 Consider the curve defined by the equation $ay^2 = x(x + a)^2$. Let us find the equation of the tangent to this curve at the point (− a, 0).

For this, we first shift the origin to (− a, 0). The equation of the curve then becomes

$ay^2 = (x - a)(x - a + a)^2$
or $ay^2 = x^2(x - a)$.

We can also write this as

$a(x^2 + y^2) = x^3$

Now, the equation of the tangent to this curve at the origin will be given by

$$a\,(x^2 + y^2) = 0$$
$$\Rightarrow x^2 + y^2 = 0$$
$$\Rightarrow x^2 = -\,y^2$$

This is impossible, since the square of any real number has to be non-negative. But we can write this as $x = \pm\, iy$, where $i = \sqrt{-1}$ is an imaginary number.

Thus the equations of the tangents to the given curve at the point $(-\,a, 0)$ are $x + a = \pm\, iy$ (shifting back the origin).

In such cases we say that the curve has imaginary tangents at the point $(-\,a, 0)$.

Now that you have seen how to find the tangents to curves given by polynomial equations, let us try and categorise the points on a given curve with the help of the tangents at those points.

Applying the procedure used in the Example 7, you should be able to solve this exercise.

E 11) Find the equations of the tangents at the origin to each of the following curves

a) $16y^2 = x^2(16 - x^2)$

b) $(y - x^2)^2 = x^4 + 3x^3$.

c) $x^3 + 6x^2y - 8y^2 = 0$.

8.4.2 Classifying Singular Points

An equation of the type $y = f(x)$ determines a unique value of y for a given value of x. This means, every straight line parallel to the y-axis meets the curve $y = f(x)$ in a unique point. However the equation of a curve is often given as $f(x, y) = 0$. If $f(x, y)$ is not a linear expression in y, then it may not be possible to write $f(x, y) = 0$ in the form $y = F(x)$ uniquely. For example, if $f(x,y) = y^3 - x^2$, then $f(x, y) = 0$ gives

$y^3 = x^2$.

This gives us two relations $y = \sqrt{a^2 - x^2}$ and $y = -\sqrt{a^2 - x^2}$ of the type $y = F(x)$.

The curve has 2 branches, as you can see from Fig. 11.

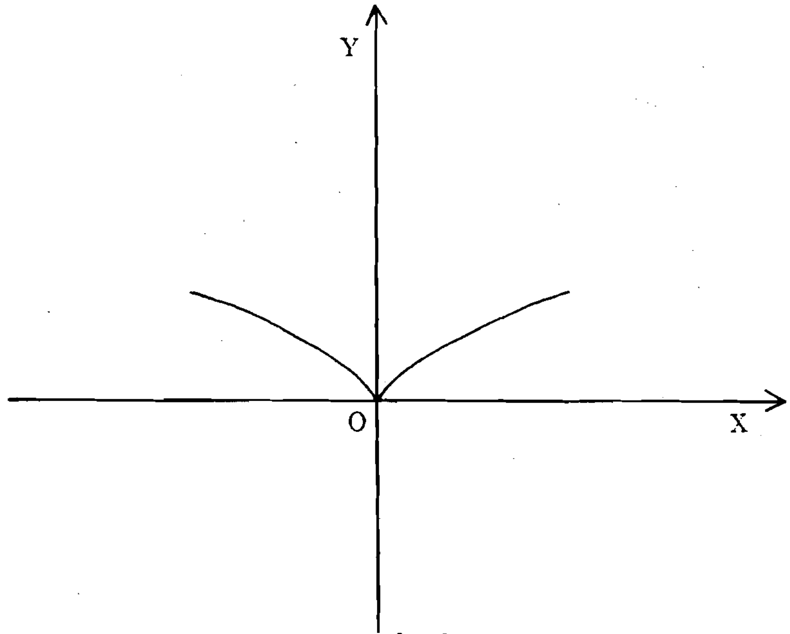


Fig. 11 . $y^3 = x^2$

The origin is common to the two branches. Put, differently we can say that two branches of the circle $x^2 + y^2 = a^2$ pass through points A and B. We have a generic name, singular points, for points like O. A precise definition is as follows.

Definition 1 If k branches of a curve pass through a point P on the curve $f(x, y) = 0$ and $k > 1$, then P is said to be a **singular point** or a **multiple point** of order k.

Singular points of order two are known as **double points.** Thus the points A and B in Fig. 11(c) are double points. Obviously, a curve will have more than one tangent at a singular point (one corresponding to each branch). Depending upon whether tangents at double points are distinct, coincident or imaginary, we shall give special names to such points.

Definition 2 A double point is known as

i) a **node** if the two tangents at that point are real and distinct,

ii) a **cusp** if the two tangents are real and coincident,

iii) a **conjugate** (or **isolated**) point if the two tangents are imaginary.

In Fig. 12 we show an example of each. For the curve $f(x, y) = 0$, the origin is a node. For the curve $g(x,y) = 0$, the points P_1, P_2, P_3 and P_4 are cusps, while the point Q on the curve $h(x, y) = 0$ is a conjugate point.

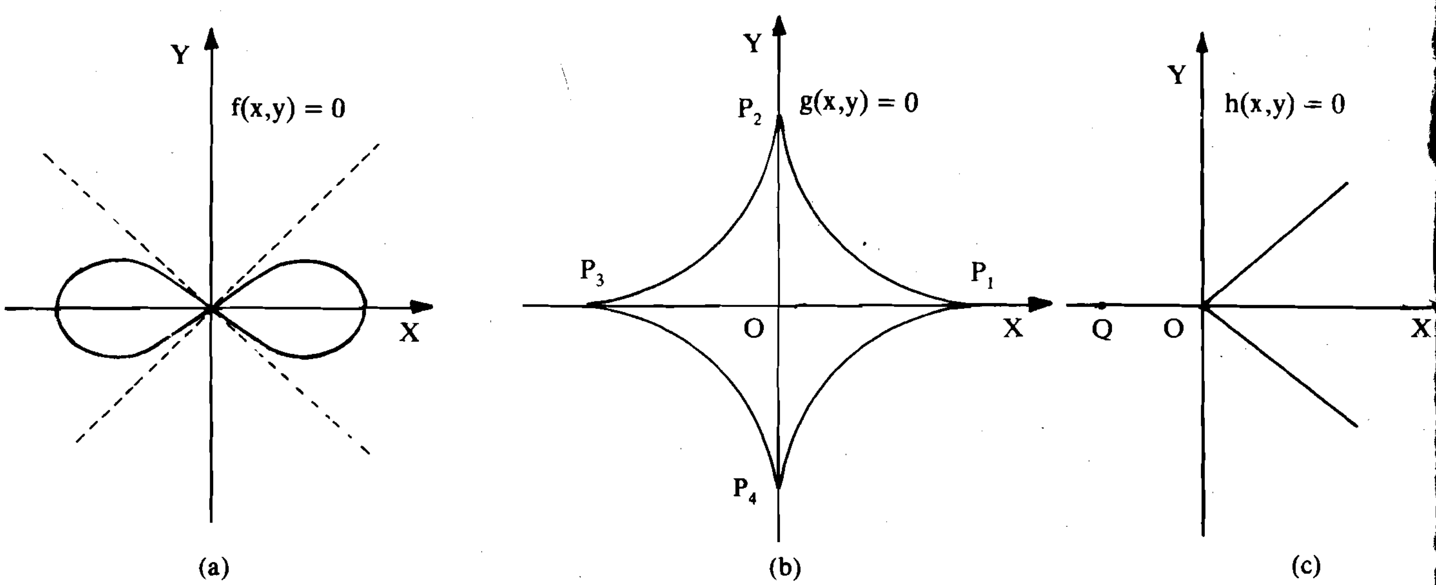


Fig. 12

In Example 7 we have seen one more example of a conjugate point. See if you can solve this exercise now.

E 12) Show that $(-1, -2)$ is a singular point on each of the following curves.

a) $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$

b) $x^3 + xy^2 + 2(x^2 + y^2) + 4xy + 5x + 8y + 8 = 0.$

Also determine the nature (cusp, node etc.) of this singular point in both cases.
(Hint : Shift the origin to $(-1, -2)$ and check the tangents at the new origin.

8.5 ASYMPTOTES

In this section we shall study another feature of curves which will prove very useful in tracing curves as you will see in the next unit. This involves taking limits as $x = \rightarrow \pm \infty$ or $y \rightarrow \pm \infty$.

You have come across such limits in Unit 2. Let us define an asymptote now.

Definition 3 A straight line is said to be an asymptote to a curve, if as a point P moves to infinity along the curve, the perpendicular distance of P from the straight line tends to zero.

Example 8 Consider the rectangular hyperbola $xy = c$ shown in Fig. 13. $xy = c$ implies $y = c/x$ and this implies that as $x \rightarrow \infty$ or $-\infty$, $y \rightarrow 0$. Now $|y|$ is the distance of a point P(x, y) on the hyperbola from the x-axis. So, we can ... that as $x \rightarrow \infty$ or $-\infty$, the distance of a point, P(x, y) on the hyperbola from the x-axis approaches zero. In other words, this means that the x-axis is an asymptote of the hyperbola.

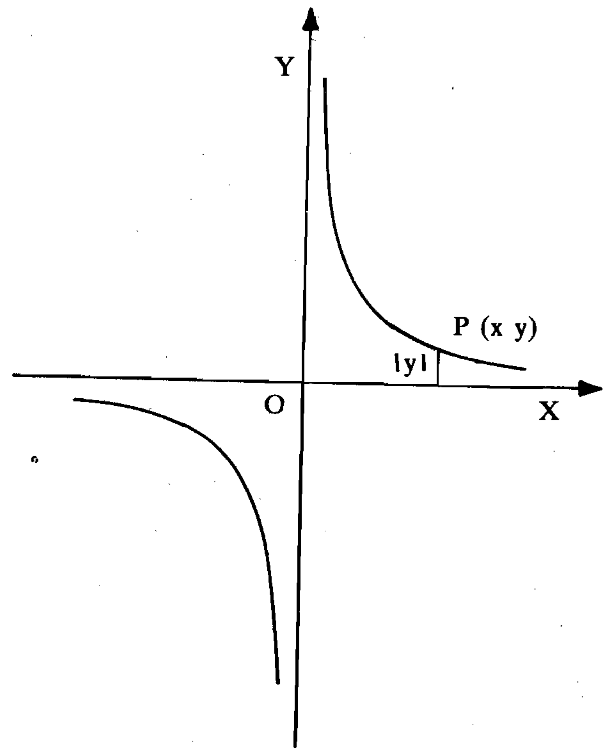


Fig. 13

Writing $xy = c$ as $x = c/y$, and repeating the arguments exactly as above, we can prove that the y-axis is also an asymptote of the hyperbola.

Example 9 Let us prove that the x-axis is an asymptote of the curve $y = \frac{10}{1+x}$ shown in Fig. 14.

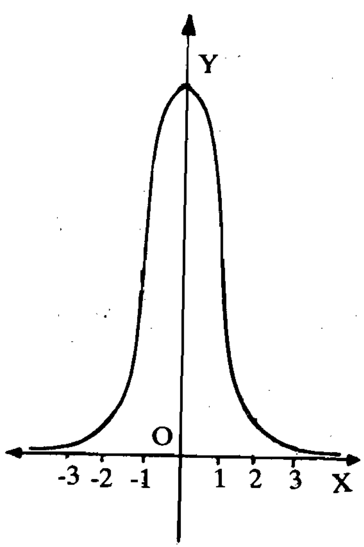


Fig. 14

From the equation of the curve, it is quite clear that $y \to 0$ as $x \to \infty$ or $-\infty$. Again, this means that the distance of the point P(x, y) on the curve from the x-axis tends to zero as $x \to \infty$ or $-\infty$. This proves that the x-axis is an asymptote of the curve.

8.5.1 Asymptotes Parallel to the Axes

Here we shall derive tests to decide whether a given curve has asymptotes parallel to the x and y axes. For this we shall consider a curve given by $f(x, y) = 0$, where $f(x, y)$ is a polynomial in x and y.

Theorem 2 A straight line $y = c$ is an asymptote of a curve $f(x, y) = 0$ iff $y - c$ is a factor of the co-efficient of the highest power of x in f(x, y).

Proof Arrange f(x, y) in descending powers of x so that the equation of the curve is written as

$$g_0(y)x^n + g_1(y)x^{n-1} + \ldots\ldots + g_n(y) = 0$$

$$g_0(y) + g_1(y)\frac{1}{x} + \ldots\ldots + g_n(y)\frac{1}{x^n} = 0$$

P(x, y) tends to infinity means atleast one of x and y must tend to infinity.

The perpendicular distance PM of P(x, y) from the line $y = c$ is $|y - c|$. (Check this by drawing a suitable figure). Now according to Definition 3, $y = c$ is an asymptote iff PM tends to zero as P tends to infinity, that is iff $y \to c$ as P tends to infinity. If the y- coordinate y of P $\to c$ (a finite number) as P tends to infinity, then its x-coordinate x must tend to infinity. Now since P is a point on the curve, its coordinates satisfy the equation $f(x, y) = 0$

So, as P tends to infinity along the curve, we get $\lim\limits_{P \to \infty} f(x, y) = 0$

From this we can say that $y = c$ is an asymptote iff $\lim\limits_{\substack{x \to \infty \\ y \to c}} f(x, y) = 0$

Hence, $y = c$ is an asymptote of (1) iff

$$\lim_{\substack{x \to \infty \\ y \to c}} [g_0(y) + g_1(y)\frac{1}{x} + \ldots.. + g_n(y)\frac{1}{x^n}] = 0$$

$\Longleftrightarrow g_0(c) = 0,$

$\Longleftrightarrow y - c$ is a factor of $g_0(y)$, the co-efficient of the highest power of x in f(x, y).

This theorem can also be interpreted as follows.

Asymptotes parallel to the x-axis are obtained by equating to zero the real linear factors of the co-efficient of the highest power of x in the equation of the curve.

We can also state a theorem, similar to Theorem 2, giving a test to decide whether a given curve has an asymptote parallel to the y-axis or not.

Theorem 3 Asymptotes parallel to the y-axis are obtained by equating to zero the real linear factors $ax + b$ of the co-efficient of the highest power of y in the equation of the curve.

Proof : Similar to that of Theorem 2

Example 10 Let us find the asymptotes parallel to either axis for the curve $y = x + \frac{1}{x}$

Writing the given equation in the form $f(x, y) = 0$, we have $x^2 - xy + 1 = 0$. You can see the graph of this curve in Fig. 15. The co-efficient of the highest power of x is 1. It has no factors of the form $y - c$. Hence there are no asymptotes parallel to the x-axis. The co-efficient of the highest power of y when equated to zero gives $x = 0$. Hence there is one asymptote parallel to the y-axis and moreover, it is the y-axis itself.

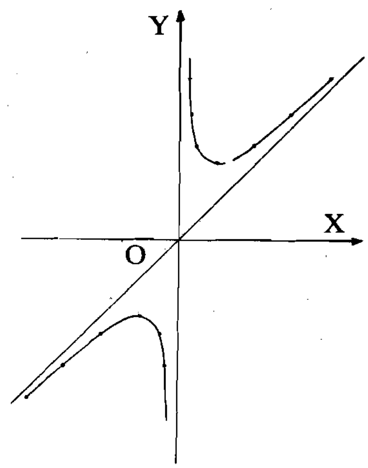


Fig. 15

See if you can do these exercises on your own.

E 13) For each of the following curves, find asymptotes parallel to either axis, if there are any,

a) $x^2y = 2 + y$ b) $xy^2 = 16x^2 + 20y^2$

c) $(x + y)^2 = x^2 + 4$ d) $x^2y^2 = 9(x^2 + y^2)$

e) $y = \frac{1}{x^2 + 1}$ f) $y = \frac{3 - 10x}{x^2 + 10}$

8.5.2 Oblique Asymptotes

You may be wondering whether an asymptote must always be parallel to a coordinate axis. No, there are many curves having asymptotes which are not parallel to either axis. Such asymptotes are generally referred to as oblique asymptotes. We shall now learn how to find oblique asymptotes $y = mx + c$ to rational algebraic curves $f(x, y) = 0$. The problem is to determine m and c so that $y = mx + c$ may be an asymptote to $f(x, y) = 0$.

The perpendicular distance of a point $P(x_1, y_1)$ from the line $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \neq\right|$

Suppose that the line $y = mx + c$ is an oblique asymptote to the curve $f(x, y) = 0$. This means that $m = 0$. The perpendicular distance PM of a point P(x, y) on this curve from $y = mx + c$ is given by $PM = \frac{|y - mx - c|}{\sqrt{(1 + m^2)}}$ Now, since $m \neq 0$, P can be at infinity on the curve only when x (as also y), tends to ∞. Thus, as $x \to \infty$, $PM \to 0$. This means that as $x \to \infty$, $(y - mx - c) \to 0$..

or, $\lim\limits_{x \to \infty} (y - mx - c) = 0$

That is, $c = \lim\limits_{x \to \infty} (y - mx)$. .... (1)

Thus c would be known as soon as m is known. Now,

$$\lim_{x \to \infty} \left(\frac{y}{x} - m\right) = \lim_{x \to \infty} \frac{(y - mx)}{x}$$
$$= \lim_{x \to \infty} (y - mx) \lim_{x \to \infty} \left(\frac{1}{x}\right)$$
$$= c.0 = 0, \quad \text{using (1)}$$

Hence $m = \lim\limits_{x \to \infty} \left(\frac{y}{x}\right)$.

Thus, given any curve $f(x, y) = 0$, we first find $\lim\limits_{x \to \infty} \frac{y}{x} = m$ and then use this m to calculate $c = \lim\limits_{x \to \infty} (y - mx)$.

The following example will clarify this procedure.

Example 11 Let us examine the curve $x^3 - y^3 = 3xy$ for oblique asymptotes.

Suppose that the given curve has an oblique asymptote $y = mx + c$. The equation of the curve can be written as

$x^3 - y^3 - 3xy = 0$

Dividing throughout by x^3 we get

$$1 - \frac{y^3}{x^3} - \frac{3y}{x} \cdot \frac{1}{x} = 0$$

Thus, $\lim\limits_{x \to \infty} [1 - \frac{y^3}{x^3} - \frac{3y}{x} \cdot \frac{1}{x}] = 0$

$$\Rightarrow 1 - \lim_{x \to \infty} (\frac{y^3}{x^3}) - 3 \lim_{x \to \infty} (\frac{y}{x}) \lim_{x \to \infty} (\frac{1}{x}) = 0$$

$$\Rightarrow 1 - \lim_{x \to \infty} (\frac{y^3}{x^3}) = 0, \text{ since } \lim_{x \to \infty} \frac{1}{x} = 0$$

$\Rightarrow m^3 = 1 \Rightarrow m = 1$, the other roots of $m^3 - 1 = 0$ being complex numbers.

Rewriting the equation of the curve as $(x - y)(x^2 + xy + y^2) = 3xy$, we have

$$c = \lim_{x \to \infty} (y - x) = \lim_{x \to \infty} \left[\frac{-3xy}{x^2 + xy + y^2}\right]$$

$$= \lim_{x \to \infty} \left[\frac{-3}{\frac{x^2}{xy} + \frac{xy}{xy} + \frac{y^2}{xy}}\right]$$

$$= \frac{-3}{1 + 1 + 1}, \text{ since } \lim_{x \to \infty} \frac{x}{y} = - \lim_{x \to \infty} \left(\frac{y}{x}\right)^{-1} = 1.$$

$$= -1$$

Hence the required asymptote is $y = x - 1$.

Try to solve these exercises now.

E E 14) Find oblique asymptotes to each of the following curves.

a) $x^3 + y^3 = 3ax^2$ b) $x^4 - y^4 + xy = 0$

8.6 SUMMARY

In this unit we have covered the following points.

1) The equation of the tangent at (x_0, y_0) to the curve $y = f(x)$ is

   $y - y_0 = f'(x_0)(x - x_0)$

2) The curve has a vertical tangent at (x_0, y_0) if $\frac{dx}{dy} = 0$ at this point.

3) The angle θ of intersection of two curves $y = f(x)$, $y = g(x)$ is the acute angle between the tangents at that point to the curves. It is given by the relation

   $\tan\theta = \left|\frac{f'(x) - g'(x)}{1 + f'(x)\,g'(x)}\right|$

4) $y = f(x)$ and $y = g(x)$ cut each other orthogonally at (x_0, y_0) if $f'(x_0)\ g'(x_0) = -1$.

5) The angle ϕ between the tangent and the radius vector of the curve $r = f(\theta)$ at the point θ is given by $\tan\phi = r\frac{d\theta}{dr}$.

6) The tangents at the origin to any curve (which passes through the origin) are obtained by equating to zero the lowest degree terms in the equation of the curve.

7) If k branches of a curve pass through a point P on the curve $f(x, y) = 0$ and $k > 1$, then P is said to be a singular point or a multiple point of order k. Singular points of order two are known as double points. A double point is known as a node, a cusp or a conjugate (isolated) point according as the two tangents at that point are real and distinct, real but coincident, or imaginary.

8) A straight line is said to be an asymptote to an infinite branch of a curve, if, as a point P on the curve moves to infinity along the curve, the perpendicular distance of P from the straight line tends to zero.

9) Asymptotes parallel to the coordinate axes are obtained by equating to zero the real linear factors in the co-efficients of the highest power of x and the highest power of y in the equation of the curve.

10) If $y = mx + c$ is an oblique asymptote of the curve $f(x, y) = 0$, $\lim_{x\to\infty} \frac{y}{x} = m$ and

   $\lim_{x\to\infty} (y - mx) = c$.

8.7 SOLUTIONS AND ANSWERS

E 1) a) $\left.\frac{dy}{dx}\right|_{x=1} = 4$. Equation of the tangent at (1, 4) is

$(y - 4) = 4(x - 1)$

Slope of the normal at $(1, 4) = -1/4$

Equation of the normal at (1, 4) is $(y - 4) = (-1/4)(x - 1)$.

b) Slope of the tangent $= -b/a$

Slope of the normal $= a/b$

at $t = \pi/4$, $x = a/\sqrt{2}$, $y = b/\sqrt{2}$. Equation of the tangent :

$(y - b/\sqrt{2}) = -b/a\ (x - a/\sqrt{2})$

c) Slope of the tangent $= 3/4$

Slope of the normal $= -4/3$

Equation of the tangent : $y - 4 = (3/4)(x + 3)$
Equation of the normal : $y - 4 = (-4/3)(x + 3)$

E 2) a) Tangents are parallel to the x-axis at $x = (1 \pm \sqrt{7})/3$

b) Tangents are parallel to the x-axis at all points where $x = n\pi + \pi/2$ for some integer n. There are no tangents parallel to the y-axis.

E 3) a) Tangent : $ty = x + at^2$
Normal : $y + tx = at(2 + t^2)$

b) Tangent : $(1 + \cos t)y = \sin t(x - at)$. Equivalently,
$\sin(t/2)x - \cos(t/2)y = at \sin(t/2)$
Normal : $\sin(t/2)y + \cos(t/2)x = 2a \sin(t/2) + at \cos(t/2)$

E 4) a) $y - y_0 = -\left(\frac{x_0 + 2}{y_0 + 3}\right)(x - x_0)$

b) $y - y_0 = (-y_0/x_0)(x - x_0)$

E 5) $3y = e^{-2x} \Longrightarrow \left.\frac{dy}{dx}\right|_{x=0} = -\frac{2}{3}$. (0, 1/3) is a point on this curve. The tangent at (0, 1/3) is given by

$$y - \frac{1}{3} = -\frac{2}{3}x$$

or $2x + 3y = 1$.

E 6) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Longrightarrow \left.\frac{dy}{dx}\right|_{(a)\sqrt{2}, (b)} = \frac{b\sqrt{2}}{a}$

$\Longrightarrow$ Slope of the normal $= -a/b\sqrt{2}$

$\Longrightarrow$ Equation of the normal is $y - b = \frac{-a}{b\sqrt{2}}(x - a\sqrt{2})$

E 7) $y^2 = 4x \Longrightarrow x = y^2/4 \Longrightarrow x^2 = y^4/16 = 4y$ at the point of intersection.

$\Longrightarrow y^4 - 64y = 0$
$\Longrightarrow y(y^3 - 64) = 0$
$\Longrightarrow y(y - 4)(y^2 + 4y + 16) = 0$
$\Longrightarrow y = 0, 4$ (other roots are complex)
$\Longrightarrow x = 0$ or 4.

Slope of the tangent to $y^2 = 4x$ at $(4, 4) = 1/2$
Slope of the tangent to $x^2 = 4y$ at $(4, 4) = 2$
$\Longrightarrow$ angle of intersection $= \tan^{-1}(3/4)$
The tangent at (0, 0) to $y^2 = 4x$ is vertical, and the tangent at (0, 0) to $x^2 = 4y$ is horizontal.
Hence the angle of intersection at (0, 0) is $\pi/2$.

E 8) a) The four points are $(4/\sqrt{3}, \pm \sqrt{2/3}$, $(-4/\sqrt{3}, \pm \sqrt{2/3}$

$\frac{dy}{dx}$ for $x^2 + 4y^2 = 8$ is $-x/4y$

$$\therefore \left.\frac{dy}{dx}\right|_{x = \frac{4}{\sqrt{3}}, y = 1\sqrt{\frac{2}{3}}} = \frac{-1}{\sqrt{2}}$$

$\frac{dy}{dx}$ for $x^2 - 2y^2 = 4$ is $x/2y$

$$\left.\frac{dy}{dx}\right|_{x = 4/\sqrt{3}, y = \sqrt{\frac{2}{3}}} = \sqrt{2}$$

$\therefore$ They cut orthogonally.

E 9) a) $2r = -2a^2 \sin 2\theta \frac{d\theta}{dr} \Rightarrow \frac{d\theta}{dr} = \frac{-r}{a^2 \sin 2\theta}$

$\Rightarrow$ angle $= \tan^{-1}(r\frac{d\theta}{dr}) = \tan^{-1}\left(\frac{-r^2}{a^2 \sin 2\theta}\right) = \tan^{-1}(-\cot 2\theta)$

$= \tan^{-1}(\tan(\frac{(2n+1)\pi}{2} + 2\theta)$

$= (2n+1)\pi/2 + 2\theta.$

b) $\tan^{-1}(\frac{1 + e\cos\theta}{e\sin\theta})$ c) $(2n+1)\pi/2 + m\theta$

d) $m\theta - \pi/4$

E 10) a) $r = ae^{\theta} \Rightarrow 1 = ae^{\theta}.\frac{d\theta}{dr} \Rightarrow \frac{d\theta}{dr} = \frac{1}{ae^{\theta}}$

$\Rightarrow \tan\phi_1 = r\frac{d\theta}{dr} = \frac{r}{ae^{\theta}} = 1$

$re^{\theta} = b \Rightarrow r = be^{-\theta} \Rightarrow 1 = -be^{-\theta}\frac{d\theta}{dr}$

$\Rightarrow \frac{d\theta}{dr} = \frac{-1}{be^{-\theta}} \Rightarrow \tan\phi_2 = r\frac{d\theta}{dr} = \frac{-r}{be^{-\theta}} = -1$

$\Rightarrow \tan\phi_1 \tan\phi_2 = -1 \Rightarrow$ the curves cut orthogonally.

b) The curves cut orthogonally.

E 11) a) $16y^2 = 16x^2 - x^4$
The equation of the tangent is
$16y^2 - 16x^2 = 0 \Rightarrow y^2 - x^2 = 0 \Rightarrow y = \pm x.$

b) $y^2 - 2x^2y + x^4 = x^4 + 3x^3.$
The equation of the tangent is $y^2 = 0 \Rightarrow y = 0.$

c) Equation : $y^2 = 0$ or $y = 0.$

E 12) a) Change the origin to $(-1, -2)$.
Then the equation of the curve is
$(x-1)^3 + 2(x-1)^2 + 2(x-1)(y-2) - (y-2)^2 + 5(x-1) - 2(y-2) = 0.$
$\Leftrightarrow x^3 - x^2 + 2xy - y^2 = 0$
The equation of the tangents at the origin is
$x^2 - 2xy + y^2 = 0$
$\Leftrightarrow (x-y)^2 = 0 \Leftrightarrow x = y.$
There are two real and coincident tangents at this point. Hence it is a cusp.

b) After shifting the origin we get the equation :
$(x-1)^3 + (x-1)(y-2)^2 + 2[(x-1)^2 + (y-2)^2] + 4(x-1)(y-2) + 5(x-1) + 8(y-2) + 8 = 0.$
The equation of the tangents at the origin is $y^2 - x^2 = 0$, that is,
$\Leftrightarrow y^2 = x^2$ or $y = \pm x.$
There are two real and distinct tangents at this point. Hence it is a node.

E 13) a) $x^2y = 2 + y \Leftrightarrow x^2y - y - 2 = 0$
Highest power of x is 2. The coefficient of x^2 is y. Hence $y = 0$ is an asymptote.
Highest power of y is 1. The coefficient of y is $x^2 - 1 = (x-1)(x+1)$.
Hence $x = -1$ and $x = 1$ are two asymptotes.

b) No asymptotes parallel to the x-axis.
$x = 20$ is an asymptote.

c) No asymptotes parallel to the y-axis.
$y = 0$ is an asymptote.

d) $y = \pm 3$ are asymptotes.
$x = \pm 3$ are asymptotes.

e) $y = 0$ is an asymptote.

f) $y = 0$ is an asymptote.

E 14) a) $x^3 + y^3 = 3ax^2$

$$\Longrightarrow 1 + (y/x)^3 = 3a/x$$

$$\Longrightarrow 1 + \lim_{x \to \infty} \left(\frac{y}{x}\right)^3 = \lim_{x \to \infty} \frac{3a}{x}$$

$$\Longrightarrow 1 + m^3 = 0 \Longrightarrow m^3 = -1 \Longrightarrow m = -1.$$

$$c = \lim_{x \to \infty} (y - mx) = \lim_{x \to \infty} (y + x)$$

$$= \lim_{x \to \infty} \frac{3ax^2}{x^2 - xy + y^2} = \lim_{x \to \infty} \frac{3a}{1 - y/x + (y/x)^2}$$

$$= \frac{3a}{1 + 1 + 1} = a$$

Hence the equation of the asymptote is $y + x = a$.

b) $m = 1, c = 0$, Equation : $y = x$
$m = -1, c = 0$, Equation : $y + x = 0$.