UNIT 4 LINEAR PROGRAMMING

Structure

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4.0 INTRODUCTION

We first make the idea clear through an illustration.

Suppose a furniture company makes chairs and tables only. Each chair gives a profit of ₹ 20 whereas each table gives a profit of ₹30. Both products are processed by three machines M_1, M_2 and M_3. Each chair requires 3 hrs 5 hrs and 2 hrs on M_1, M_2 and M_3. respectively, whereas the corresponding figures for each table are 3, 2, 6. The machine M_1 can work for 36 hrs per week, whereas M_2 and M_3 can work for 50 hrs and 60 hrs, respectively. How many chairs and tables should be manufactured per week to maximise the profit?

We begin by assuming that x chairs and y tables, be manufactured per week. The profit of the company will be ₹ $(20x + 30y)$ per week. Since the objective of the company is to maximise its profit, we have to find out the maximum possible value of $P = 20x + 30y$. We call this as the objective function.

To manufacture x chairs and y tables, the company will require $(3x + 3y)$ hrs on machine M_1. But the total time available on machine M_1 is 36 hrs. Therefore, we have a constraint $3x + 3y \leq 36$.

Similarly, we have the constraint $5x + 2y \leq 50$ for the machine M_2 and the constraint $2x + 6y \leq 60$ for the machine M_3.

Also since it is not possible for the company to produce negative number of chairs and tables, we must have $x \geq 0$ and $y \geq 0$. The above problem can now be written in the following format :

Maximise

$$P = 20x + 30y$$

subject to

$$3x + 3y \leq 36$$
$$5x + 2y \leq 50$$
$$2x + 6y \leq 60$$

and $x \geq 0,\ y \geq 0$

We now plot the region bounded by constraints in Fig. 1 The shaded region is called the **feasible region** or the **solution space** as the coordinators of any point lying in this region always satisfy the constraints. Students are encouraged to verify this by taking points (2,2),(2,4) (4,2) which lies in the feasible region.

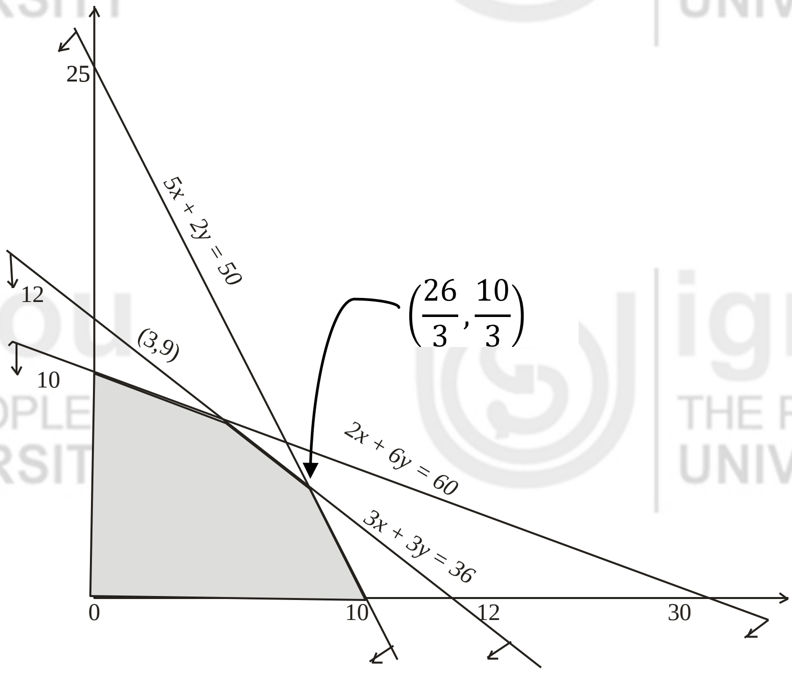

Figure 1

We redraw the feasible region without shading to clarify another concept (see Figure 2).

For any particular value of P, we can draw in the objective function as a straight line with slope $\left(\frac{-2}{3}\right)$ This is because $P = 20x + 30y$ is a straight line, which can be written as

$$y = \left(\frac{-2}{3}\right)x + \frac{P}{30}$$

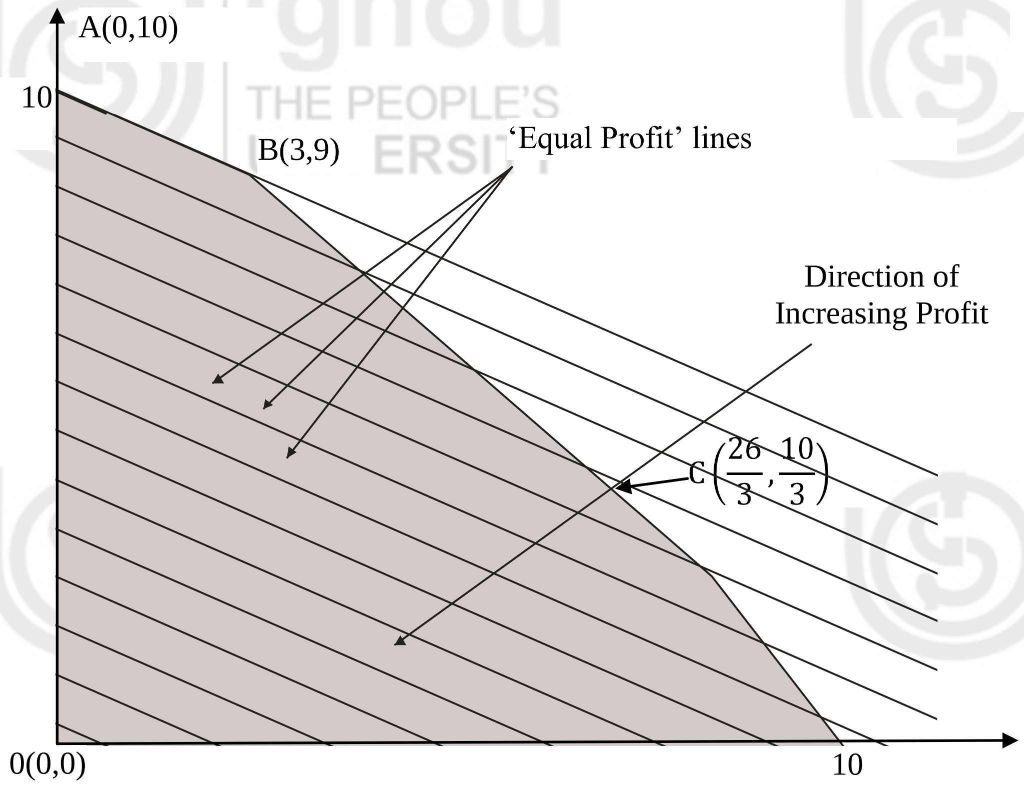

Figure 2

and this generates a family of parallel lines with slope $\left(\frac{-2}{3}\right)$ but with different intercepts on the axes. On any particular line, the different combinations of x and y (chairs and tables) all yield the same profit (P).

However, as P increases so does intercept $\frac{P}{30}$ (the $y-$ intercept), so the lines with higher y-intercepts yield higher profits.

The problem is therefore to maximise the y-intercept, while at the same time remaining within constraints (or, feasible region). The part of the profit line which fall within the feasible region have been heavily drawn. Profit lines drawn farthest away from the origin (0, 0) yield the highest profits. Therefore, the highest profit yielded within the feasible region is at point B(3, 9). Therefore, the maximum profit is given by ₹ (20× 3 + 30× 9) = ₹ 330.

We are now ready for the definition of linear programming – the technique of solving the problem such as above.

What is Linear Programming

Linear because the equations and relationships introduced are linear. Note that all the constraints and the objective functions are linear. *Programming* is used in the sense of method, rather than in the computing sense.

In fact, linear programming is a technique for specifying how to use limited resources or capacities of a business to obtain a particular objective, such as least cost, highest margin or least time, when those resources have alternative uses.

4.1 OBJECTIVES

After studying this unit, you should be able to:

- define the terms-objective function, constraints, feasible region, feasible solution, optimal solution and linear programming;
- draw feasible region and use it to obtain optimal solution;
- tell when there are more than one optimal solutions; and
- know when the problem has no optimal solution.

4.2 LINEAR PROGRAMMING

We begin by listing some definitions.

Definitions

Objective Functions: If $a_1, a_2, . . . , a_n$ are constants and x_1 $x_2, ..., x_n$ are variables, then the linear function $Z = a_1x_1 + a_2x_2 + ... + a_nx_n$ which is to be maximised or minimised is called objective function*.

Constraints: These are the restrictions to be satisfied by the variables x_1, x_2 ..., x_n. These are usually expressed as inequations and equations. ***Non-negative Restrictions:*** The values of the variables x_1 x_2, ..., x_n involved in the linear programming problem (LPP) are greater than or equal to zero (This is so because most of the variable represent some economic or physical variable.)

Feasible Region: The common region determined by all the constraints of an LPP is called the feasible region of the LPP.

Feasible Solution: Every point that lies in the feasible region is called a feasible solution. Note that each point in the feasible region satisfies all the constraints for the LPP.

Optimal Solution: A feasible solution that maximises or minimizes the objective function is called an optimal solution of the LPP.

* In this unit we shall work with just two variables.

Definition

Convex Region

A region ***R*** in the coordinate plane is said to be convex if whenever we take two points A and *B* in the region **R**, the segment joining *A* and *B* lies completely in ***R*.**

The student may observe that a feasible region for a linear programming is a convex region.

One of the properties of the convex region is that maximum and minimum values of a function defined on convex regions occur at the corner points only. Since all feasible regions are convex regions, maximum and minimum values of optimal functions occur at the corner points of the feasible region.

The region of Figure 3 is convex but that of Figure 4is not convex.

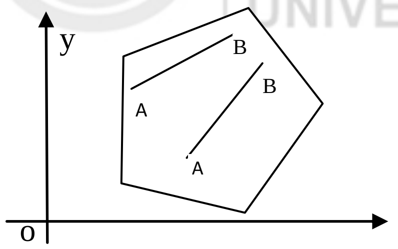

Figure 3 : Convex Region

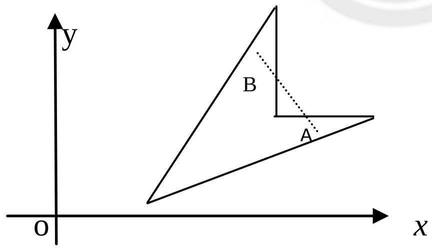

Figure 4 : Not Convex

4.3 TECHNIQUES OF SOLVING LINEAR PROGRAMMING PROBLEM

There are two techniques of solving an L.P.P. (Linear Programming Problem) by graphical method. These are

(i) Corner point method, and

(ii) Iso profit or Iso-cost method.

The following procedure lists the coner point method.

Corner Point Method*

Step 1 Plot the feasible region.

Step 2 Find the coordinates of the cornor points of the feasible region.

Step 3 Calculate the value of the objective function at each of the corner points of the feasible region.

Step 4 Pick up the maximum (or minimum) value of the objective function from amongst the points in step 3.

* The method explained in Example 1 is the iso-profit method. You are advised to use corner method unless you are specifically asked to do the problem by the iso-profit or iso-cost method.

Example 1 Find the maximum value of $5x + 2y$ subject to the constraints

$$-2x - 3y \le -6$$
$$x - 2y \le 2$$
$$6x + 4y \le 24$$
$$-3x + 2y \le 3$$
$$x \ge 0,\ y \ge 0.$$

Solution

Let us denote $5x + 2y$ by P.

Note that $-2x - 3y \le -6$ can be written as $2x + 3y \ge 6$.

We can write the given LPP in the following format :

Maximise

$$P = 5x + 2y$$

subject to

$$2x + 3y \ge 6 \qquad \text{(I)}$$
$$x - 2y \le 2 \qquad \text{(II)}$$
$$6x + 4y \le 24 \qquad \text{(III)}$$
$$-3x + 2y \le 3 \qquad \text{(IV)}$$

and $\quad x \ge 0,\ y \ge 0$

We plot the feasible region bounded by the given constraints in Figure 5

The lines I and II intersect in A (18/7, 2/7)

The lines II and III intersect in B (7/2, 3/4)

The lines III and IV intersect in C (3/2, 15/4)

The lines IV and I intersect in D (3/13, 24/13)

Let us evaluate P at A, B, C and D

P(A) = 5 (18/7) + (2/7) = 94/7 P(B) = 5 (7/2) + 2(3/4) =19

P(C) = 5 (3/2) + 2(15/4) = 15 P(D) = 5 (3/13) + 2(24/13)= 63/13

Thus, maximum value of P is 19 and its occurs at $x = 7/2$, $y = 3/4$.

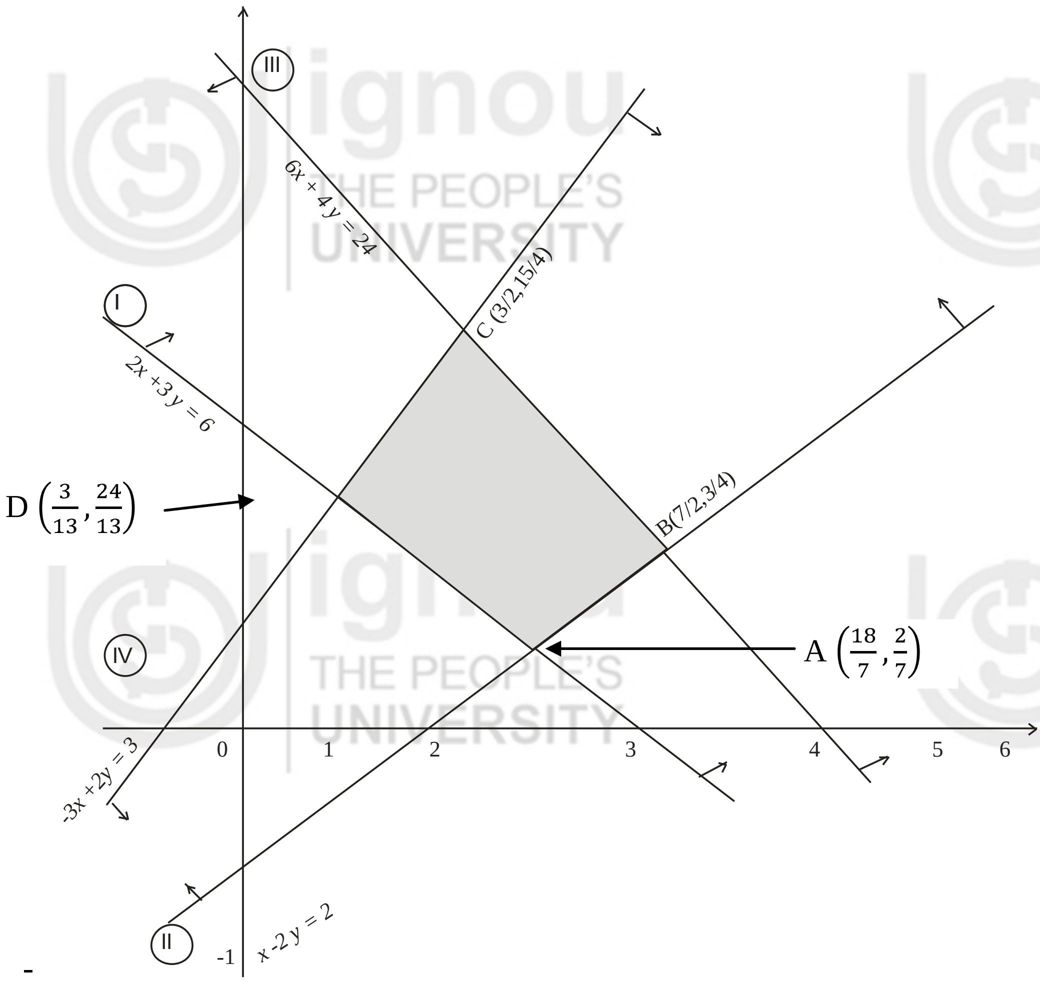

Figure 5

Example 2 Find the maximum value of $2x + y$ subject to the constraints

$$x + 3y \geq 6$$
$$x - 3y \leq 3$$
$$3x + 4y \leq 24$$
$$-3x + 2y \leq 6$$
$$5x + y \geq 5$$
$$x, y \geq 0$$

Solution

We write the given question in the following format :

Maximise

$$P = 2x + y$$

subject to

$$x + 3y \geq 6 \qquad \text{(I)}$$
$$x - 3y \leq 3 \qquad \text{(II)}$$
$$3x + 4y \leq 24 \qquad \text{(III)}$$
$$-3x + 2y \leq 6 \qquad \text{(IV)}$$
$$5x + y \geq 5 \qquad \text{(V)}$$
$$\text{and} \quad x \geq 0,\ y \geq 0 \qquad \text{(VI)}$$

The feasible region is sketched in Figure 6

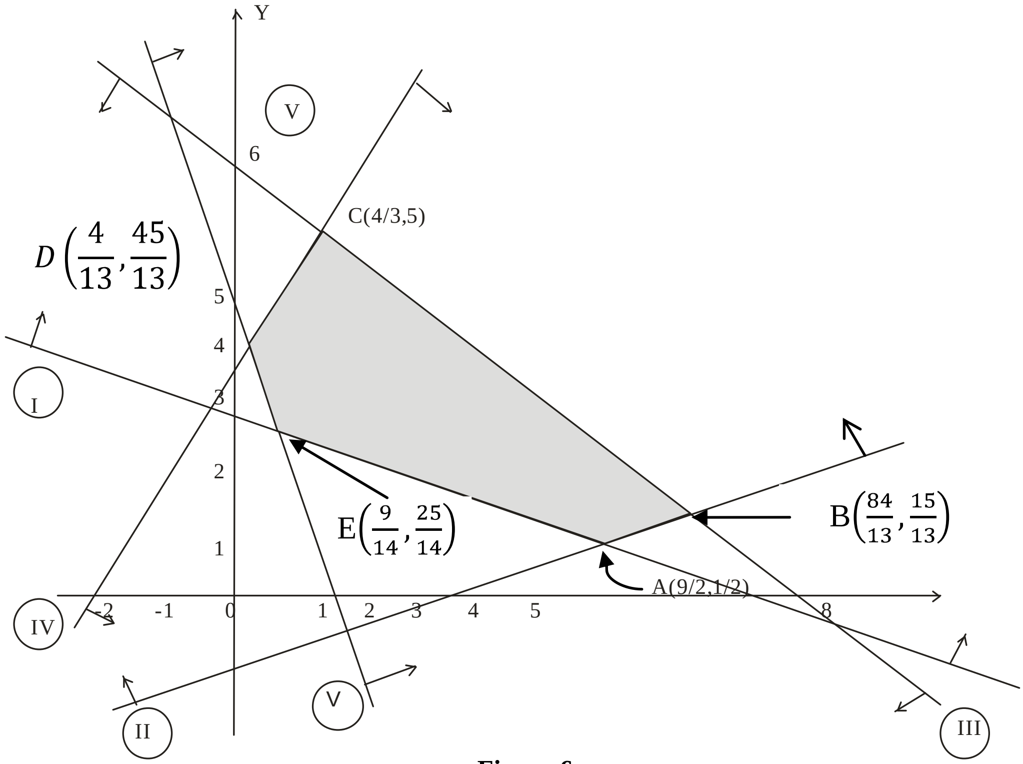

Figure 6

The lines I and II intersect in A(9/2, 1/2)

The lines II and III intersect in B(84/13, 15/13)

The lines III and IV intersect in *C(4/3, 5)*

The lines IV and V intersect in D(4/13, 45/13)

The lines V and I intersect in E(9/14, 25/14).

Let us evaluate *P* at the points A, B, *C, D* and E as follows:

$P(A) = 2(9/2) + 1/2 = 19/2$

$P(B) = 2(84/13) + 15/13 = 183/13$

$P(C) = 2(4/3) + 5 = 23/3$

$P(D) = 2(4/13) + 45/13 = 53/13$

$P(E) = 2(9/14) + 25/14 = 43/14$

The maximum value of P is 183/13 which occurs at $x = 84/13$ and $y = 15/13$.

Example 3: An aeroplane can carry a maximum of 200 passengers. A profit of ₹ 400 is made on each first class ticket and a profit of ₹ 300 is made on each economy class ticket. The airline reserves at least 20 seats for first class. However, at least 4 times as many passengers prefer to travel by economy class than by first class. Determine how many of each type of tickets must be sold in order to maximise the profit for the airline? What is the maximum profit?

Solution

Let number of first class tickets sold be x and the number of economy class tickets sold be y.

As the airline makes a profit of ₹400 on first calss and ₹300 on economy class, the profit of the airlines is $400x + 300y$

As the aeroplane can carry maximum of 200 passengers, we must have $x + y \leq 200$. As the airline reserves at least 20 tickets for the first class, $x \geq 20$.

Next, as the number of passengers preferring economy class is at least four times the number of passengers preferring the first class, we must have $y \geq 4x$.
Also, $x \geq 0, y \geq 0$.

Therefore the LPP is

Maximixe

	$P = 400x + 300y$	[objective function]
subject to	$x + y \leq 200,$	[capacity function]
	$y \geq 20$	[first class function]
	$y \geq 4x$	[preference constraint]
	$x \geq 0, y \geq 0$	[non-negativity]

We plot the feasible region in Figure 7

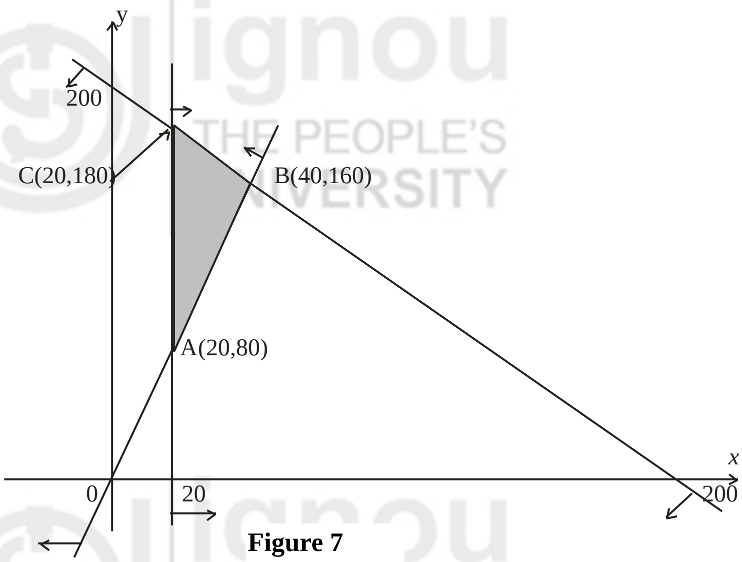

Figure 7

We now calculate the profit at the corner points of the feasible region.

P(A) = P(20,80) = (400)(20) + (300)(80) = 8000 + 24000 = 32000

P(B) = P(20,160) = (400)(40) + (300)(160) = 64000

P(C) = P(20,180) = (400)(20) + (300)(180) = 62000

Hence, profit of the airlines maximum at B i.e., when 40 tickets of first class and 160 tickets of ecoomy class are sold. Also, maximum profit is ₹ 64,000.

Example 4: Suriti wants to invest at most ₹ 12000 in Savings Certificate and National Savings Bonds. She has to invest at least ₹ 2000 in Savings Certificate and at least ₹ 4000 in National Savings Bonds. If the rate of interest in Saving Certificate is 8% per annum and the rate of interest on National Saving Bond is 10% per annum, how much money should she invest to earn maximum yearly income ? Find also the maximum yearly income ?

Solution

Suppose Suriti invests ₹ x in saving certificate and ₹ y in National Savings Bonds.

As she has just ₹ 12000 to invest, we must have $x + y \leq 12000$.

Also, as she has to invest at least ₹ 2000 in savings certificate $x \geq 2000$.

Next, as she must invest at least Rs. 4000 in National Savings Certificate $y \geq 4000$. Yearly income from saving certificate = ₹ $\frac{8x}{100}$ = $0.08x$ and from

National Savings Bonds = ₹ $\frac{10y}{100}$ = Rs. $0.1y$

$\therefore$ Her total income is ₹ P where

$P = 0.08x + 0.1y$

Thus, the linear programming problem is

Maximise

$$P = \frac{8x}{100} + \frac{10y}{100}$$

subject to

$x + y \leq 12000$	[Total Money Constraint]
$x \geq 2000$	[Savings Certificate Constraint]
$y \geq 4000$	[National Savings Bonds Constraint]
$x \geq 0, y \geq 0$	[Non-negativity Constraint]

However, note that the constraints $x \geq 0$, $y \geq 0$, are redundant in view of $x \geq 2000$ and $y \geq 4000$.

We draw the feasible region in Figure 8

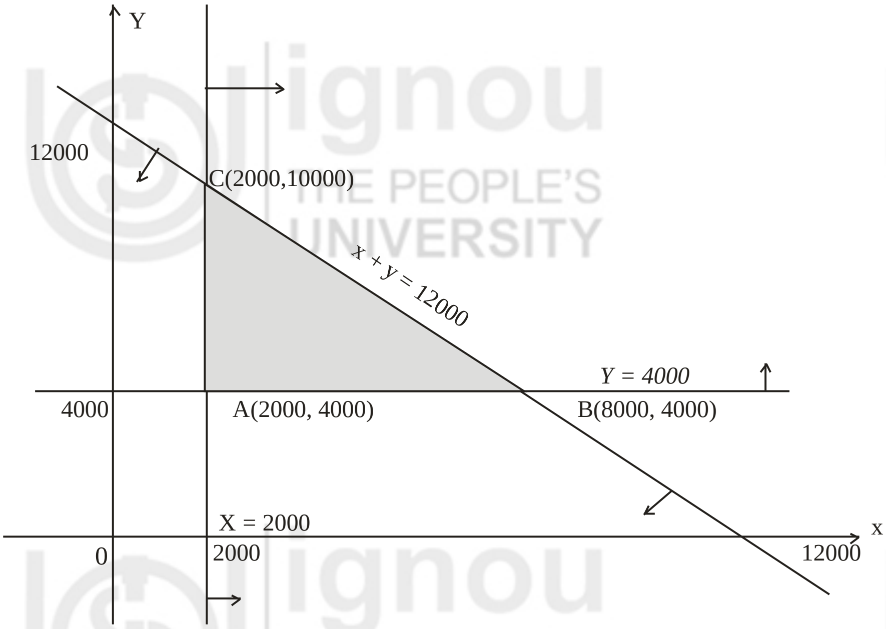

Figure 8

We now calculate the profit at the corner points of the feasible region.
We have

P(A) = P(2000,4000) = (0.08) (2000) + (0.1)(4000)
= 160 +400 = 560

P(B) = P(8000,4000) = (0.08) (2000) + (0.1)(4000)
= 640 +400 = 1040

P(C) = P(2000,10000) = (0.08) (2000) + (0.1)(10000)
= 160 +1000 =1160.

Thus, she must invest ₹ 2000 in Savings certificate and ₹ 10000 in National Savings Bonds in order to earn maximum income.

Example 5 If a young man rides his motor cycle at 25 km per hour, he has to spend ₹ 2 per km on petrol; if he rides it at a faster speed of 40 km per hour, the petrol cost increases to ₹ 5 per km. He wishes to spend at most ₹ 100 on petrol and wishes to find what is maximum distance he can travel within one hour. Express this as a linear programming problem and then solve it.

Solution

Let x km be the distance travlled at the rate of 25 km/h and y km be the distance travelled at the rate of 40 km/h. Then the total distance covered by the young man is D = $(x + y)$ km.

The money spent in travelling x km (at the rate of 25 km/h) is $2x$ and the money spent in travelling y km (at the rate of 40 km/h) is $5y$. Thus, total money spent during the journey is ₹ $(2x + 5y)$. Since the young man wishes to spend at most Rs. 100 on the journey, we must have $2x + 5y \leq 100$.

Also the time taken to travel x km is $\frac{x}{25}h$ and the time taken to travel y km is $\frac{y}{40}h$. Since the young man has just one hour, we must have

$$\frac{x}{25} + \frac{x}{40} \leq 1.$$

Also, note that $x \geq 0, \; y \geq 0$.

The mathematical formulation of the linear programming problem is

Maximise

$$D = x + y$$

subject to

$$2x + 5y \leq 100$$

$$\frac{x}{25} + \frac{x}{40} \leq 1$$

and $x \geq 0, \; y \geq 0$.

The feasible region is sketched in Figure 9.

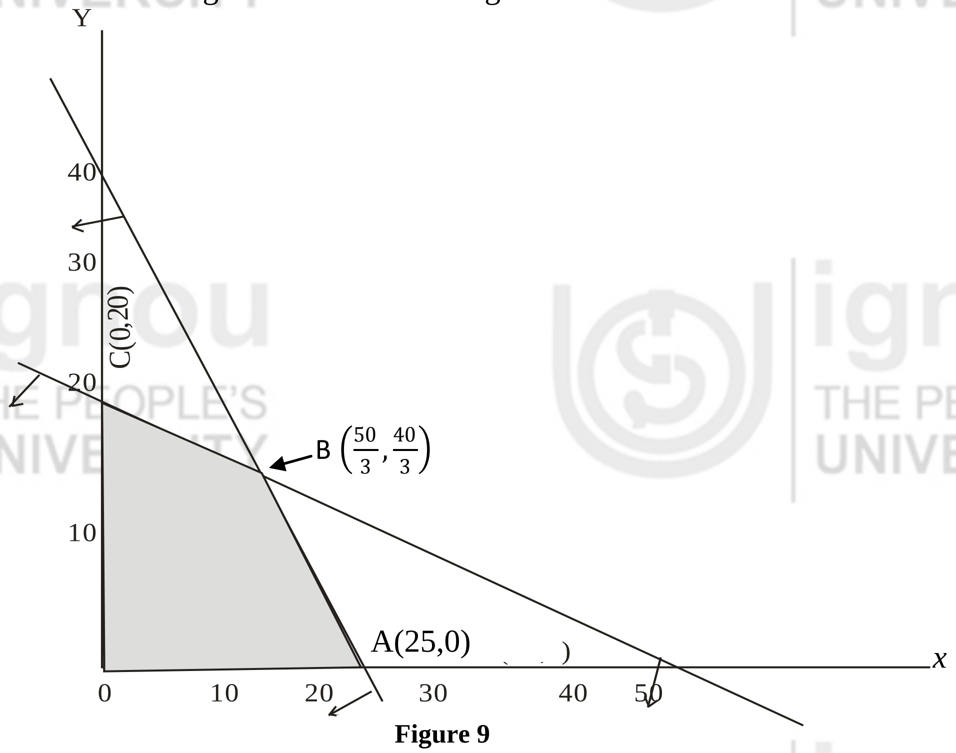

Figure 9

The corner points of feasible region are O(0,0), A(25,0), B$\left(\frac{50}{3}, \frac{40}{3}\right)$ and C(0,20)

Let us evaluate D at these points

D(O) = 0 + 0 = 0

D(A) = 25 + 0 = 25

$$D(B) = \frac{50}{3} + \frac{40}{3} = 30$$

D(C) = 0 + 20 = 20

Thus, the maximum values of D is 30 which occurs at x = 50/3, y = 40/3

When do we have Multiple Solutions ?

Whenever the objective function isoprofit (iso-cost) line is parallel to one of the constraints we have multiple optimal solutions to the linear programming problem. We illustrate this in the following example.

Example 6 The *xyz* company manufactures two products A and B. They are processed on the same machine. A takes 10 mintues per item and B takes 2 minutes per items on the machine. The machine can run for a maximum of 35 hours in a week. Product A requires 1 kg and product B 0.5 kg of the raw material per item, the supply of which is 600 kg per week. Note more than 800 items of product B are required per week. If the product A costs ₹ 5 per items and can be sold for ₹ 10 and Product B costs ₹ 6 per items and can be sold for ₹ 8 per item. Determine how many items per week be produced for A and B in order to maximize the profit.

Solution

We summarise the information given in the question as follows :

	Product A (x)	**Product B(y)**	**Constraint**
Machine	10 (min. per item)	2 (min. per item)	$\leq 35 \times 60 = 2100$
Material	1 (kg per item)	0.5 (kg per item)	≤ 600
Number		for the Proudct B	≤ 800
Profit	₹ (10 –5) = ₹ 5 per item	₹ (8–6) = ₹ 2 per item	Maximise P

Suppose x items of **A** and y items of **B** are be manufactured. The given problem can be written as follows.

Maximise

$$P = 5x + 2y$$

subject to

$$10x + 2y \leq 2100 \qquad \text{(Machine constraint)}$$

$$x + 0.5y \leq 600 \qquad \text{(Material constraint)}$$

$$y \leq 800 \qquad \text{(Restriction on B)}$$

$$x \geq 0, y \geq 0 \qquad \text{(Non-negativity)}$$

The feasible region has been shaded in Figure 10.

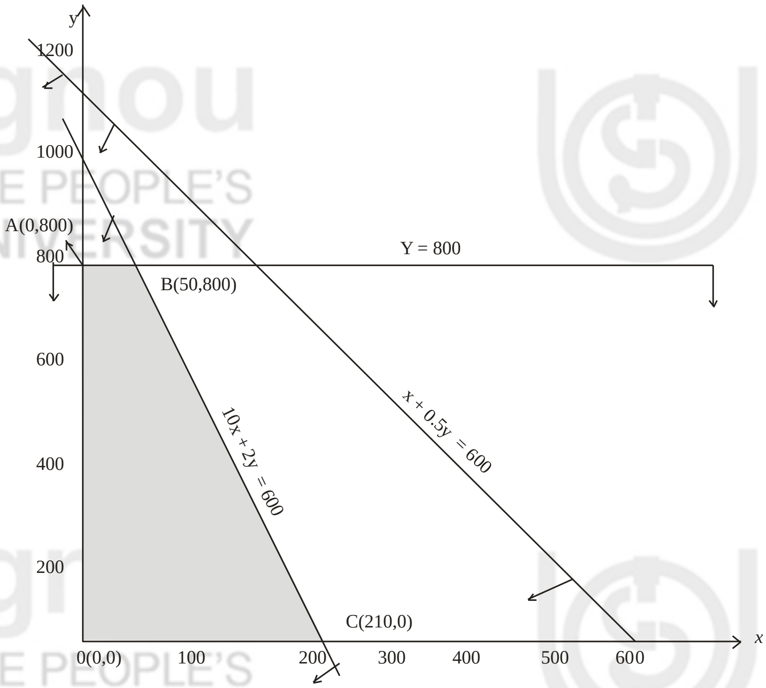

Figure 10

We now calculate the value of *P* at the corner points of the feasible region

P(A) = P(0,800) = 1600

P(B) = P(50,800) = 1850

P(C) = P(210,0) = 1050

P(O) = P(0,0) = 0

Maximum profit is ₹ 1850 for x = 50 and y = 800

Redundant Constraints

In the above example, the constraitn $x + 0.5, y \leq 600$ does not affect the feasible region. Such a constraints is called as redundant constraint. Redundant constraints are unnecessary in the formulation and solution of the problem, because they do not affect the feasible region.

Example 7 The manager of an oil refinery wants to decide on the optimal mix of two possible blending processes 1 and 2, of which the inputs and outputs per product runs as follows :

Process	Crude *A*	Crude *B*	Gasoline *X*	Gasoline *Y*
1	5	3	5	8
2	4	5	4	4

The maximum amounts available of crudes **A** and **B** are 200 untis and 150 units, respectively. At least 100 units of gasoline *X* and 80 untis of *Y* are required. The profit per production run from processes 1 and 2 ₹ 300 and Rs. 400 respectively. Formulate the above as linear programming and solve it by graphical method.

Solution

Let process 1 be run x times and process 2 be y times. The mathematical formulation of the above linear programming problem is given by

Maximise

$$P = 300\,x + 400\,y$$

subject to

$5x + 4y \leq 200$ (constraint on Crude A)

$3x + 5y \leq 200$ (constraint on Crude B)

$5x + 4y \geq 100$ (constraint on gasoline X)

$8x + 4y \geq 80$ (constraint on gasoline Y)

$x \geq 0, y \geq 0$ (non-negativity)

The feasible region has been shaded in Figure 11.

We have

$P(\mathrm{A}) = P(0,30) = 12000$

$$P(B) = P\left(\frac{400}{13}, \frac{150}{13}\right) = \frac{180000}{13} = 13486\ \frac{2}{13}$$

$P(\mathrm{C}) = P(40,0) = 12{,}000$

$P(\mathrm{D}) = P(25,0) = 7500$

$P(\mathrm{E}) = P(0,200) = 8000$

Thus, the profit is maximum when $x = \frac{400}{13}, y = \frac{150}{13}$. Hence, 30 runs of process 1 and 11 runs of process 2 will give maximum profit.

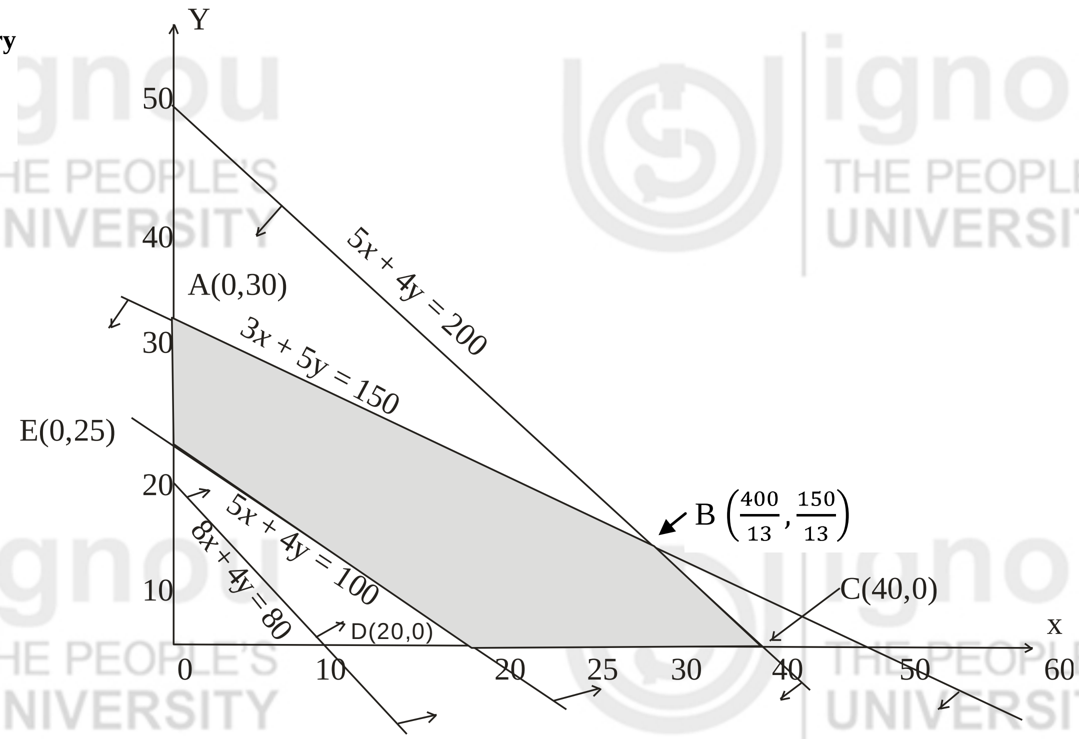

Figure 11

Remark : The constraint $8x + 4y \geq 80$ does not affect the feasible region, that is, the constraint $8x + 4y \geq 80$ is a redundant constraint.

No Feasible Solution

In case the solution space or the feasible region is empty, that is, there is no point which satisfies all the constraints, we say that the linear programming problem has no feasible solution.

Illustration : We illustrate this in the following linear programming problem.

Maximise

$$P = 2x + 5y$$

subject to

$$x + 2y \leq 10$$
$$x \geq 12$$

and $x \geq 0,\ y \geq 0$

we draw the feasible region in Fig 12.

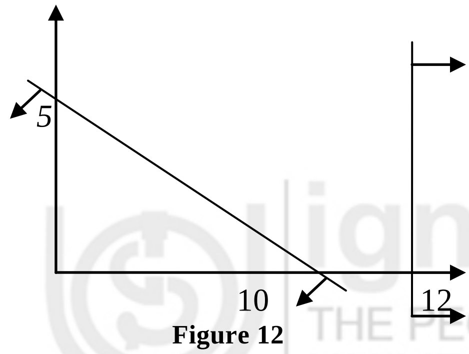

Figure 12

The direction of arrows indicate that the feasible region is empty. Hence, the given linear programming problem has no feasible solution.

Unbounded solution

Sometimes the feasible region is unbounded. In such cases, the optimal solution may not exist, because the value of the objective function goes on increasing in the unbounded region.

Illustration Let us look at the following illustration.

Maximise

$$P = 7x + 5y$$

subject to

$$2x + 5y \geq 10$$
$$x \geq 4$$
$$y \geq 3$$
$$x \geq 0,\ y \geq 0$$

has no bounded solution.

We draw the feasible region in Fig. 13

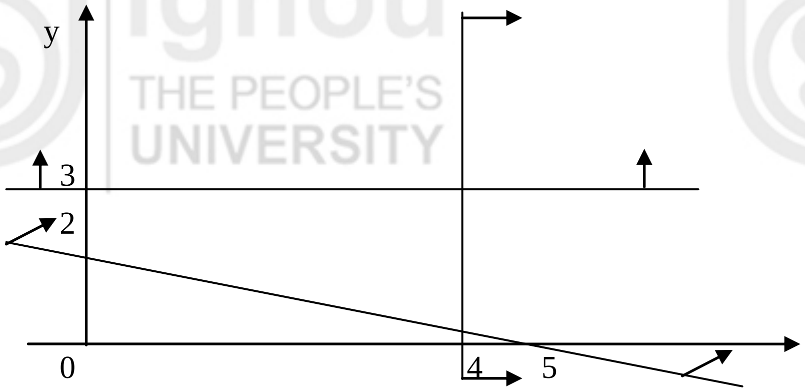

Figure 13

The constraint $2x + 5y \geq 10$ is a redundant constraint.

The feasible region is unbounded. Note that the linear programming problem has no bounded solution.

Chcek Your Progress – 1

1. Best Gift Packs company manufactures two types of gift packs, type A and type B. Type A requires 5 minutes each for cutting and 10 minutes assembling it. Type B requires 8 minutes each for cutting and 8 minutes each for assembling. There are at most 200 minutes available for cutting and at most 4 hours available for assembling. The profit is ₹ 50 each for type A and ₹ 25 each for type B. How many gift packs of each type should the company manufacture in order to maximise the profit ?

2. A manufacturer makes two types of furniture, chairs and tables. Both the products are processed on three machines A_1, A_2 and A_3. Machine A_1 requires 3 hrs for a chair and 3 hrs for a table, machine A_2 requires 5 hrs for a chair and 2 hrs for a table and machine A_3 requires 2 hrs a chair and 6 hrs for a table. Maximum time available on machine A_1, A_2, A_3 is 36 hrs, 50 hrs and 60 hrs respectively. Profits are ₹ 20 per chair and ₹ 30 per table. Formulate the above as a linear programming problem to maximise the profit and solve it.

3. A manufacturer wishes to produce two types of steel trunks. He has two machines A and B. For completing, the first type of trunk, he requires 3 hrs on machine A and 2 hrs on machine B whereas the second type of trunk requires 3 hrs on machine A and 3 hrs on machine B. Machines A and B can work at the most for 18 hrs and 14 hrs per day respectively. He earns a profit of ₹30 and ₹ 40 per trunk of first type and second type respectively. How many trunks of each type must he make each day to make maximum profit? What is his maximum profit?
4. A new businessman wants to make plastic buckets. There are two types of available plastic bucket making machines. One type of machine makes 120 buckets a day, occupies 20 square metres and is operated by 5 men. The corresponding data for second type of machine is 80 buckets, 24 square metres and 3 men. The available resources with the businessman are 200 sq. metres and 40 men. How many machines of each type the manufacturer should buy, so as to maximise the number of buckets?
5. A producer has 20 and 10 units of labour and capital respectively which he can use to produce two kinds of goods X and Y. To produce one unit of goods X, 2 units of capital and 1 unit of labour is required. To produce one unit of goods Y, 3 units of labour and 1 unit of capital is required. If X and Y are priced at ₹ 80 and ₹ 100 per unit respectively, how should the producer use his resources to maximize the total revenue? Solve the problem graphically.
6. A firm has available two kinds of fruit juices – pineapple and orange juice. These are mixed and the two types of mixtures are obtained which are sold as soft drinks A and B. One tin of A needs 4 kgs of pineapple juice and 1 kg of orange juice. One tin of B needs 2 kgs of pineapple juice and 3 kgs of orange juice. The firm has available only 46 kgs of pineapple juice and 24 kgs of orange juice. Each tin of A and B sold at a profit of ₹ 4 and ₹ 2 respectively. How many tins of A and B should the firm produce to maximise profit?

4.4 COST MINIMISATION

We illustrate the concept by the following example.

Example 7 *:* A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. In view of the need to ensure, certain nutrient constituents, it is necessary to buy two products (call them A and B) in addition. The contents of the various products, per unit, in nutrient constituents *(e.g.*, vitamins, proteins, etc.) is given in the following table:

Nutrients	Nutrient content in product A	B	Minimum amount of nutrient
M_1	36	6	108
M_2	3	12	36
M_3	20	10	100

The last column of the above table gives the minimum amounts of nutrient constituients M_1, M_2, M_3 which must given to the pigs. If products A and B cost ₹ 20 and ₹ 40 per unit respectively, how much each of these two products should be bought, so that the total cost is minimised?

Solution

Let x units of A and y units B be purchased. Our goal is to minimise the total cost

$$C = 20x + 40y$$

By using x units of A and y units of B, we shall get $36x + 6y$ units of M_1 Since we need at least 108 units of M_1, we must have

$$36x + 6y \geq 108$$

Similarly for M_2 we have $3x + 12y \geq 36$ and for M_3 we have $20x + 10y \geq 100$. Also, we cannot use negative numbers of x and y. Thus, our problem is

Minimise

$$C = 20x + 40y$$

subject to

$$36x + 6y \geq 108$$

$$3x + 12y \geq 36$$

$$20x + 10y \geq 100$$

$$x \geq 0, \; y \geq 0$$

We now draw the constraints on the same graph to obtain the feasible region. Since $x \geq 0$, $y \geq 0$, we shall restrict ourself only to the first quadrant. See Figure 14. We obtain point B by solving $36x + 6y = 108$ and $20x + 10y = 100$ and point C by solving $3x + 12y = 36$ and $20x + 10y = 100$. The feasible region has been shaded.

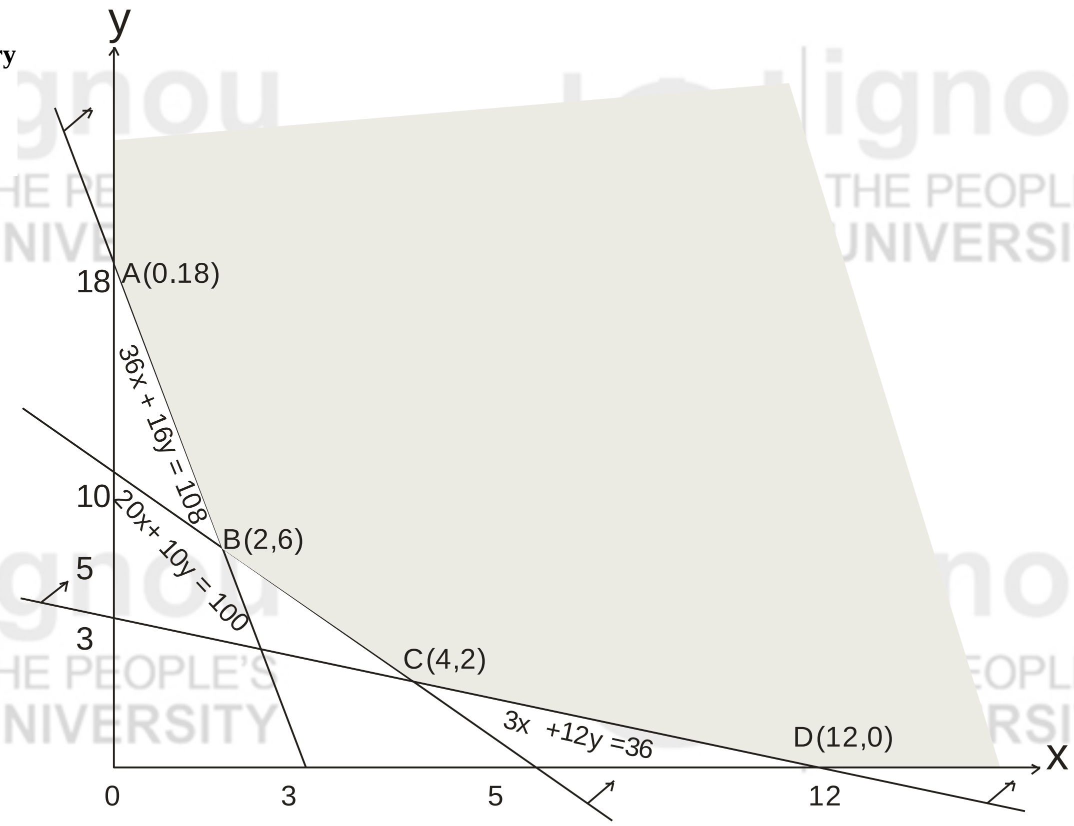

Figure14

We next draw a family of straight lines

$$y = -\frac{1}{2}x - \frac{C}{40}$$

with varying values of **C**. See Figure 15

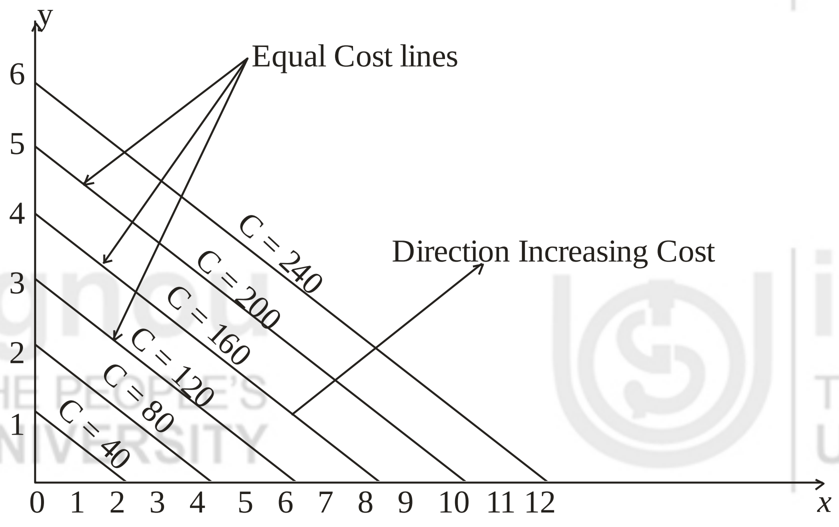

Figure 15

It is clear from here that in order to have least possible cost, we should take a cost line which intersects the feasible region and is as near to the origin as possible. Therefore we should consider only the corner points of the feasible region as possible candidates for least cost.

$C(A) = C(0, 18) = 720$

$C(B) = C(2, 6) = 20\times 2 + 40\times 6 = 280$

$C(C)=C(4,2)=20\times4+40\times2= 160$

$C(D) = C(12, 0) = 240$

Minimum cost is obtained at **C**, that is, for $x = 4$, $y= 2$. Minimum possible cost is ₹160.

Remark: The procedure for finding least cost is the same as that for the maximization of profit.

Example 8 A diet for a sick person must contain at least 1400 units of vitamins, 50 units of minerals and 1400 of calories. Two foods A and B are available at a cost of ₹ 4 and ₹ 3 per unit, respectively. If one unit of A contains 200 units of vitamins, one unit of mineral and 40 calories and one unit of food B contains 100 units of vitamins, two units of minerals and 40 calories. Find what combination of food be used to have least cost?

Solution

Let x units of food A and y units of food B be used to give the *sick* person the least quantities of vitamins, minerals and calories.

The total cost of the food is $C = 4x + 3y$. We wish to minimise C.

By consuming x units of A and y units of B, the sick person will get $200x + 100y$ units of vitamins. Since the least quantity of vitamin required is 1400, we must have $200x + 100y \geq 1400$

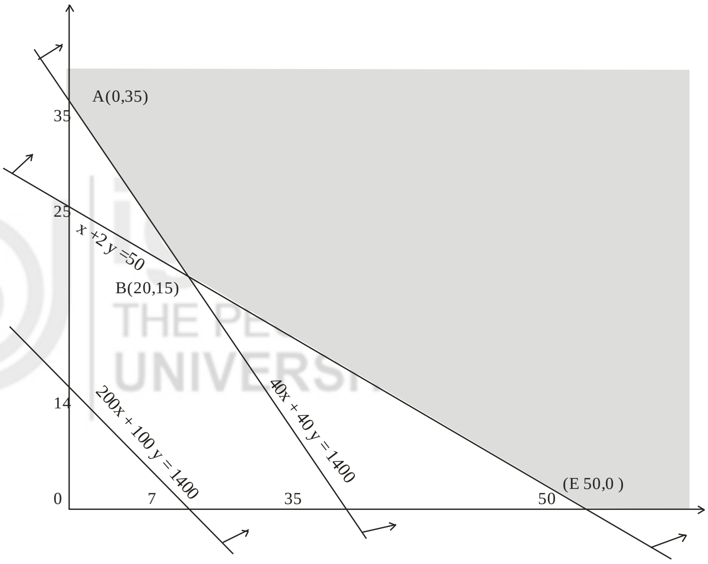

Figure16

Similarly, we must have

Minerals : $x + 2y \geq 50$

Calories : $40x + 40y \geq 1400$

Also, since x and y cannot be negative, we must have $x \geq 0$, $y \geq 0$
Thus, the linear programming problem is

Minimise

$$C = 4x + 3y$$

subject to

$$200x + 100y \geq 1400$$
$$x + 2y \geq 50$$
$$40x + 40y \geq 1400$$

and $\qquad x \geq 0, y \geq 0$

We draw the feasible region of the above linear programming problem in fig. 16. Note that constraint $200x + 100y \geq 1400$ is a redundant constraint. The other two intersect in (20,15).

The corner points of the feasible regions are A(0, 35), B(20,15) and E(50,0). We find the value of C at each of these corner points.

C(A) = 4(0) + 3(35) = 105
C(B) = 4(20) + 3(15) = 125
C(E) = 4(50) + 3(0) = 200

Thus, least cost is occurs at $x = 0$, $y = 35$.

Example 9 Every gram of wheat provides 0.1 g of proteins and 0.25 g of carbohydrates. The corresponding values for rice are 0.05 g and 0.5 g, respectively. Wheat costs ₹2 per kg and rice ₹ 8. The minimum daily requirements of protein and carbohydrates for an average child are 50 g and 200 g, respectively. In what quantities should wheat and rice be mixed in the daily diet to provide the minimum daily requirements of protein and carbohydrates at minimum cost? (The protein and carbohydrate values given here are fictitious and may be quite different from the actual values.)

Solution

Let x kg of wheat and y kg of rice be given to the child to give him at least the minimum requirements of protein and carbohydrates. Then cost of the food is $(2x + 8y) = C$ (say).

Since one gram wheat contains 0.1 g of proteins, x kg of wheat will contain $(1000x)(0.1) = 100x$ grams of protein.

Similarly, y kg of rice contain $(100y)(0.05) = 50y$ grams of protein. Thus, x kg of wheat and y kg of rice will contain $(100x + 50y)$ gms of protein.

As the minimum requirement of protein is 50 g, we must have

$$100x + 50y \geq 50.$$

Similarly, for carbohydrate, we must have

$$(1000x)(0.25) + (1000y)(0.5) \geq 200$$

$$\text{or} \quad 250x + 500y \geq 200$$

Also, x and y must be non-negative, that is, $x \geq 0$, $y \geq 0$.

Thus, the linear programming problem is

Minimise

$$C = 2x + 8y$$

subject to

$$100x + 50y \geq 50$$

$$250x + 500y \geq 200$$

$$\text{and} \quad x \geq 0, \; y \geq 0.$$

We draw the feaible region of its problem in Figure 17

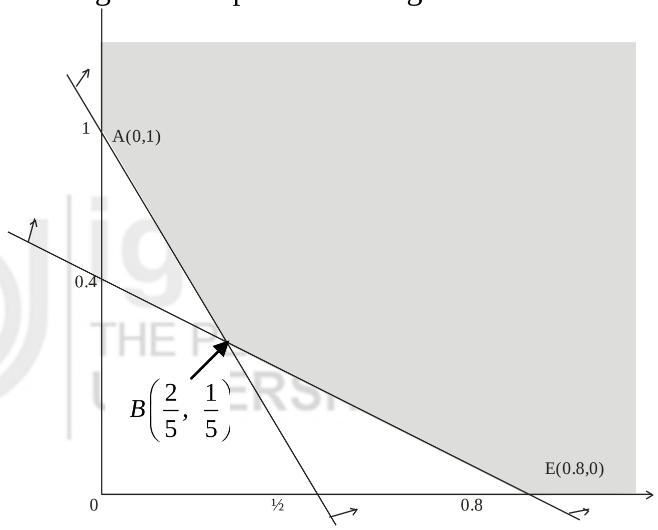

Figure17

The corner points of the feasible region are $A(0,1)$, $B(2/5,1/5)$ and $E(0.8,0)$.
Let us evaluate C at the conrer points of the feasible region.

$C(A) = 2(0) + 8(1) = 8$

$$C(B) = 2\left(\frac{2}{5}\right) = 8\left(\frac{1}{5}\right) = \frac{12}{5} = 2.4$$

$C(E) = 2(0.8) + 8(0) = 1.6$

Thus, the cost is least when $x = 0.8$, $y = 0$. The least cost is Rs. 1.60.

Example 10 An animal feed manufacturer produces a compound mixture from two materials A and B. A costs ₹2 per kilograms and B, ₹ 4 per kilogram. Material A is supplied in packs of 25 kilograms and material B in packs of 50 kilograms. A batch of at least 1,00,000 kilograms of the mixture is to be produced with the specification that at least 40,000 kilograms of material A, should be used in the manufacture, which ensures the minimum guaranteed content of the ingredient.

How many packs of A and *B* should be purchased in order to minimise cost?

Solution

Let x packs of A and y packs of B be purchased in order to meet the requirement. Our goal is to minimise the total cost. The total cost is

$$C = 2 \times 25x + 4 \times 50y = 50x + 200y$$

The constraints are

$$25x + 50y \geq 100000$$
$$50x \geq 40000$$

Therefore, the linear programming problem is

Minimise

$$C = 50x + 200y$$

subject to

$$25x + 50y \geq 100000$$
$$50x \geq 40000$$
$$x \geq 0, \; y \geq 0$$

We can rewrite the problem as

Minimise

$$C = 50x + 200y$$

subject to

$$x + 2y \geq 4000$$
$$x \geq 800$$
$$x \geq 0, \; y \geq 0$$

The feasible region of the above linear programming problem is given in Figure 17

Let us evaluate C at the corner points of the feasible region.

$$C(A) = 50 \times 800 + 50 \times 1600$$
$$= 40000 + 80000 = 120000$$
$$C(B) = 50 \times 4000 + 50 \times 0 = 200000$$

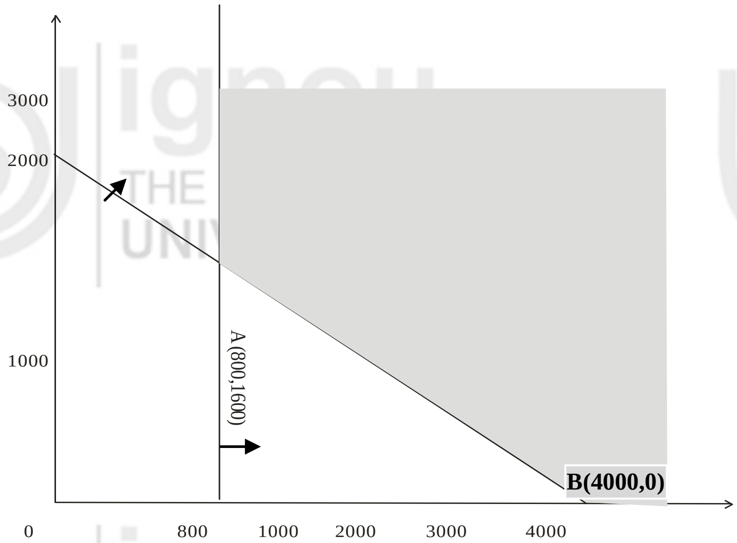

Figure 18

Therefore, cost is minimum when $x = 800, y = 1600$. The minimum possible cost is Rs. 1,20,000.

Check Your Progress – 2

1. Two tailors, A and B, earn ₹ 150 and ₹ 200 per day respectively. A can stitch 6 shirts and 4 pants while B can stitch 10 shirts and 4 pants per day. How many days shall each work if it is desired to produce (at least) 60 shirts and 32 pants at a minimum labour cost ? Also calculate the least cost.
2. A dietician mixes together two kinds of food in such a way that the mixture contains at least 6 units of vitamin A, 7 units of vitamin B, 11 units of vitamin C and 9 units of vitamin D. The vitamin contents of 1 unit food X and 1 unit of food Y are given below:

	Vitamin A	Vitamin B	Vitamin C	Vitamin D
Food X	1	1	1	2
Food Y	2	1	3	1

One unit of food X costs ₹ 5, whereas one unit of good Y cost ₹ 8. Find the least cost the mixture which will produce the dersired diet. Is there any redundant constraint ?

3. A diet for a sick person must contain at least 4000 units of vitamins, 50 units of minerals and 1400 of calories. Two foods, A and B, are available at a cost of ₹ 4 and ₹ 3 per unit respectively. If one unit of A contains 200 units of vitamin, 1 unit of mineral and 40 calories and one unit of food B contains 100 units of vitamin, 2 units of minerals and 40 calories, find what combination of foods should be used to have the least cost ? Also calculate the least cost.

4. A tailor needs at least 40 large buttons and 60 small buttons. In the market, buttons are available in boxes or cards. A box contains 6 large and two small buttons and a card contains 2 large and 4 small buttons. If the cost of a box is 30 paise and card is 20 paise. Find how many boxes and cards should he buy as to minimise the expenditure ?

4.5 ANSWERS TO CHECK YOUR PROGRESS

1. Type A : x, Type B : y then LPP is

Maximize

$P = 50x + 25y$

subject to

$5x + 8y \leq 200$ [Cutting constraint]

$10x + 8y \leq 240$ [Assembly constraint]

$x \geq 0$, $y \geq 0$ [Non- negativity]

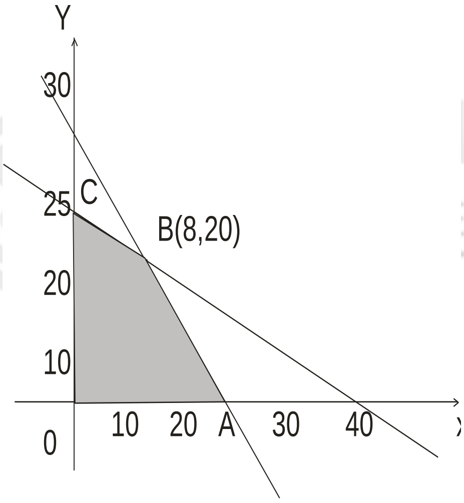

Figure 19

P(A) = 1200

P (B) = 900

P (C) = 625

P(0) = 0

Thus, profit is maximum when $x = 24$, y = 0

Maximum profit = Rs. 1200

2. Chairs : x, Tables : y, then LPP is

Maximize

$$P = 20x + 30y$$

subject to

$3x + 3y \leq 36$ [Machine A_1 constraint]

$5x + 2y \leq 50$ [Machine A_2 constraint]

$2x + 6y \leq 60$ [Machine A_3 constraint]

$x \geq 0, y \leq 0$ [Non-negativity]

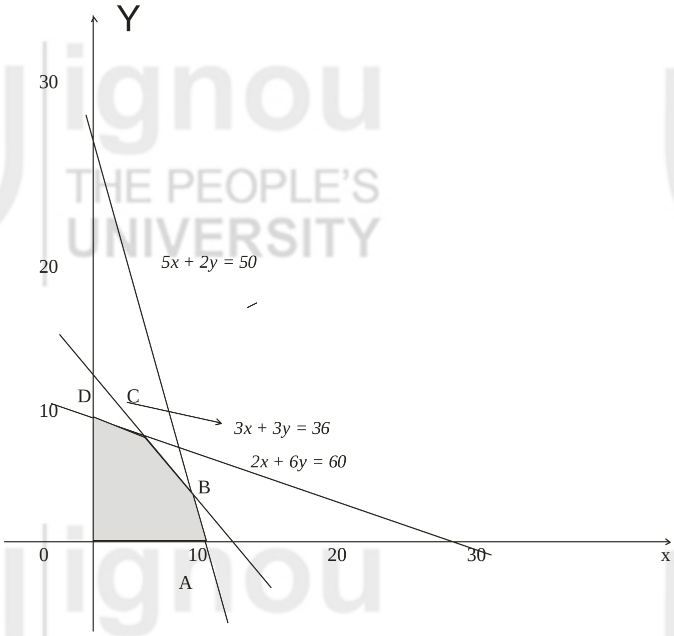

Figure 20

P (A) = 20 (10) + 30(0) = 200

$$P\text{ (B)} = 20\left(\frac{26}{3}\right) + 30\left(\frac{10}{3}\right) = \frac{820}{3}$$

P (C) = 20 (3) + 30(9) = 330

P (D) = 20 (10) + 30(0) = 200

P(0) = 20(0) + 30(0) = 0

Thus, profit is maximum

when $x = 3$, $y = 9$.

Maximum Profit = Rs. 330

3. First type trunks : x Second type trunks : y

The LPP is

Maximise $P = 30x + 40y$,

subject to

$3x + 3y \le 18$ [Machine A constraint]

$2x + 3y \le 14$ [Machine B constraint]

$x \ge 0, y \ge 0$ [non-negativity]

P(A) = 180

P(B) =200

P (C) = 560/3

P(O) = 0

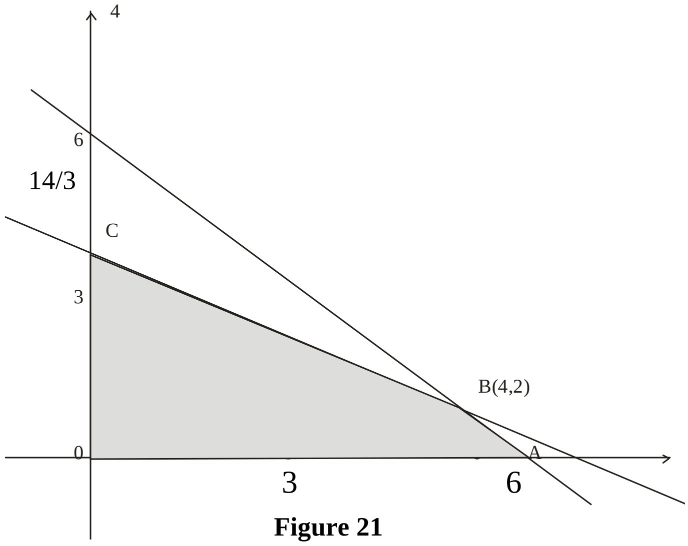

Figure 21

Thus, profit is maximum when $x = 4$, $y = 2$ and maximum profit is Rs. 200.

4. We write the information in the question in tabular form as follows :

	First type of machine (x)	First type of machine (x)	Constraint
Area	20 (sq. m per machine)	24 (sq. m per machine)	≤ 200
Labour	5 (per machine)	3 (per machine)	≤ 40
Buckets	120 (per machine)	80 (per machine)	Maximise N

Let x machines of the first x pe and y machines of the second type be purchased.

We have to

Maxmise

$$N = 120\,x + 80y$$

subject to

$20\,x + 24\,y \leq 200$ (Area constraint)

$5x + 3y \leq 40$ (Labour constraint)

$x \geq 0, y \geq 0$ (Non-negativity)

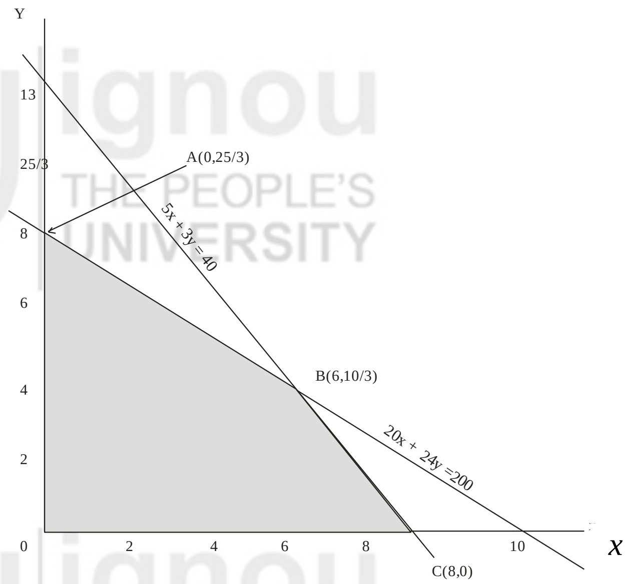

Figure 22

The feasible region for the above linear programming problem has been shaded in the figure.

We find the value of N at the cornor points of the feasible region. We have

$$N\,(A) = N\left(0, \frac{25}{3}\right) = \left(\frac{2{,}000}{3}\right) = 666\frac{2}{3}$$

$$\text{N (B)} = \text{N}\left(6, \frac{10}{3}\right) = \frac{2960}{3} = 986\frac{2}{3}$$

$$\text{N (C)} = \text{N}(8,0) = 960$$

$$\text{N (O)} = \text{N}(0,0) = 0$$

Thus, the value of N is maximum when $x = 6, y = 10/3$. As y cannot be in fraction, we take $x = 6, y = 3$.

5. We summarise the information given in the question in tabular form as follows:

	Good X	Good Y	Constraint
Capital	2	1	10
Labour	1	3	20
Revenue	80	100	Maximize R

Let x units of X and y units of Y be produced. The above problem can be written as

Maxmize

$$\text{R} = 80\,x + 100y$$

subject to

$2x + y \leq 10$ (Capital constraint)

$x + 3y \leq 20$ (Labour constraint)

$x \geq 0, y \geq 0$ (Non-negativity)

We shade the feasible region in following figure.

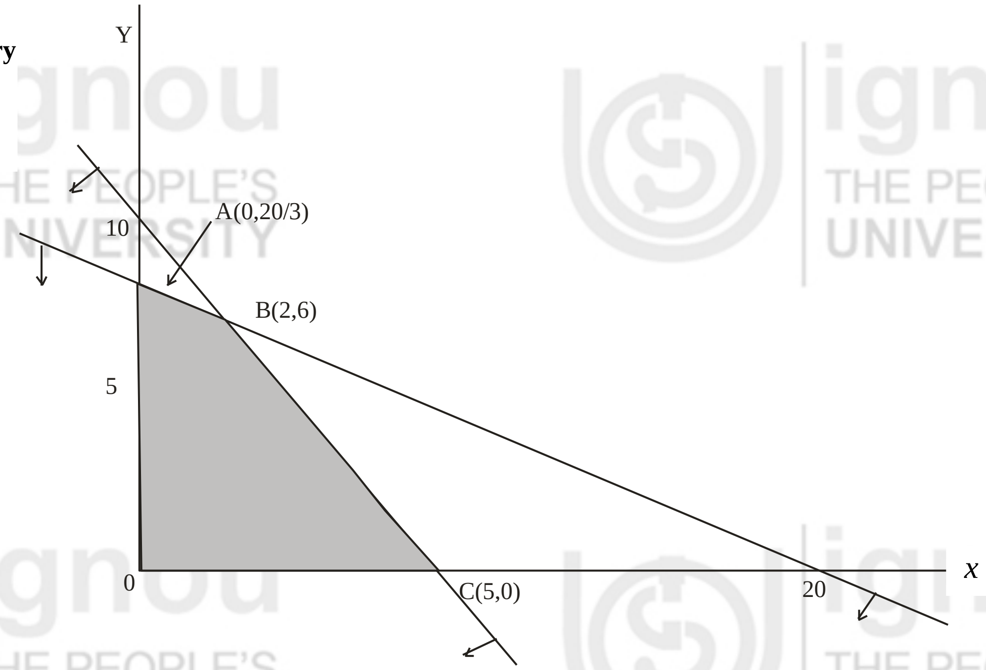

Figure 23

We now check the revenue at corner points of the feasible region.

R (A) = 80(0) + 100(20/3)= 2000/3 = $666\frac{2}{3}$

R(B) = 80(2) + 100(6) = 760

R(C) = 80(5) + 100(0) = 400

R(0) = 80(0) + 100(0) = 0

This shows that the revenue is maximum when x = 2 and y = 6

i.e. when 2 units of x and 6 units y are produced and the maximum revenue is Rs. 760.

6. We summarise the information given in the above question in the following table.

	Drink A	Drink B	Constraint
Pineapple juice	4(kg per tin)	2(kg per tin)	≤ 46
Orange juice	1(kg per tin)	3(kg per tin)	≤ 24
Profit	4	2	Maximize P

Let x tins of drink A and y tins of drink B be filled up. The above problem can be written as

Maximise

$$P = 4x + 2y$$

Subject to

$4x + 2y \leq 46$ (pineabpple juice constraint)

$x + 3y \leq 24$ (organge juice constraint)

$x \geq 0, y \geq 0$ (non-negativity)

The feasible region has been shaded. See the following figure.

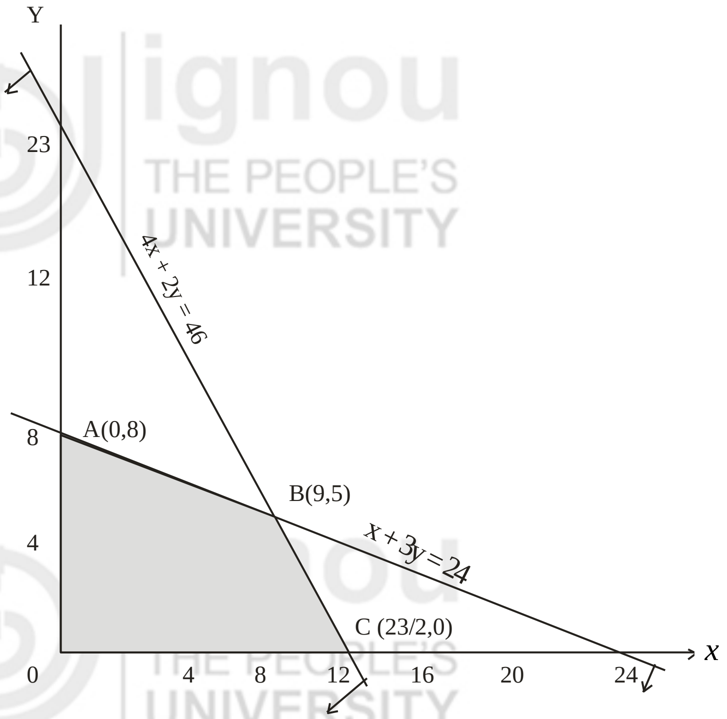

Figure 24

We have

P(A) = 4 (0) + 2(8) = 16

P(B) = 4(9) + 2(5) = 46

$$P(C) = 4\left(\frac{23}{2}\right) + 2(0) = 46$$

P(0) = 0

We have maximum profit at *B* and *C*. In this case, we say that we have multiple solutions. In fact, each point on the segment *BC* gives a profit of ₹ 46. This is because the segment *BC* is one of the isoprofit lines.

Check Your Progress 2

1. Suppose tailor A works for *x* days and tailor *B* work for *y* days.
   The LPP is

   Minimise
   $$C = 150x + 200y$$
   subject to

   $6x + 10y \geq 60$ [Shirts constraints]

   $4x + 4y \geq 32$ [Pants constraint]

   $x \geq 0, y \geq 0$ [Non–negativity]

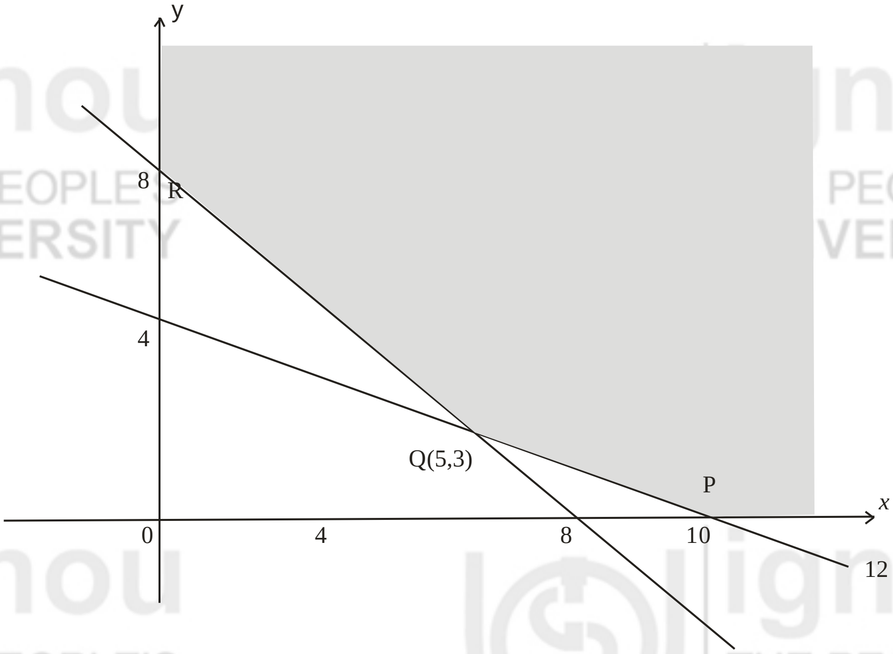

Figure 25

C (P) = 1500

C (Q) = 1350

C(R) = 1600

Thus, cost is least when x =5 and y = 3

2. Food X : x units and Food Y : y units LPP is

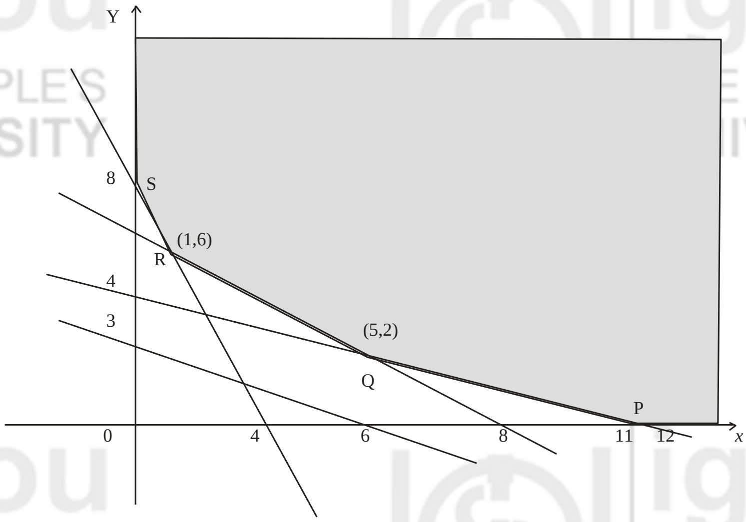

Figure 26

Minimise

$$C = 5x + 8y$$

subject to

$$x + 2y \geq 6$$

$$x + y \geq 7$$

$$x + 3y \geq 11$$

$$2x + y \geq 8$$

$$x \geq 0, y \geq 0$$

Now, C(P) = 55, C (Q) = 41

C(R) = 30, C(S) = 64

Thus, C is least when $x = 5$, y = 2

least Cost is Rs. 41

The constraint $x + 2y \geq 6$ is a redundant constraint.

3. Let x units of food A and y units of food B be used. The LPP is

Minimise

$$C = 4x + 3y$$

subject to

$200x + 100y \geq 400$	(vitamins constraints)
$x + 2y \geq 50$	(minerals constraints)
$40x + 40y \geq 1400$	(calories constraints)
$x \geq 0, y \geq 0$	(non-negativity)

We have

C(P) = 200, C (Q) = 125, C (R) = 110,

C(S) = 120

Thus, C is least when $x = 5$, $y = 30$.

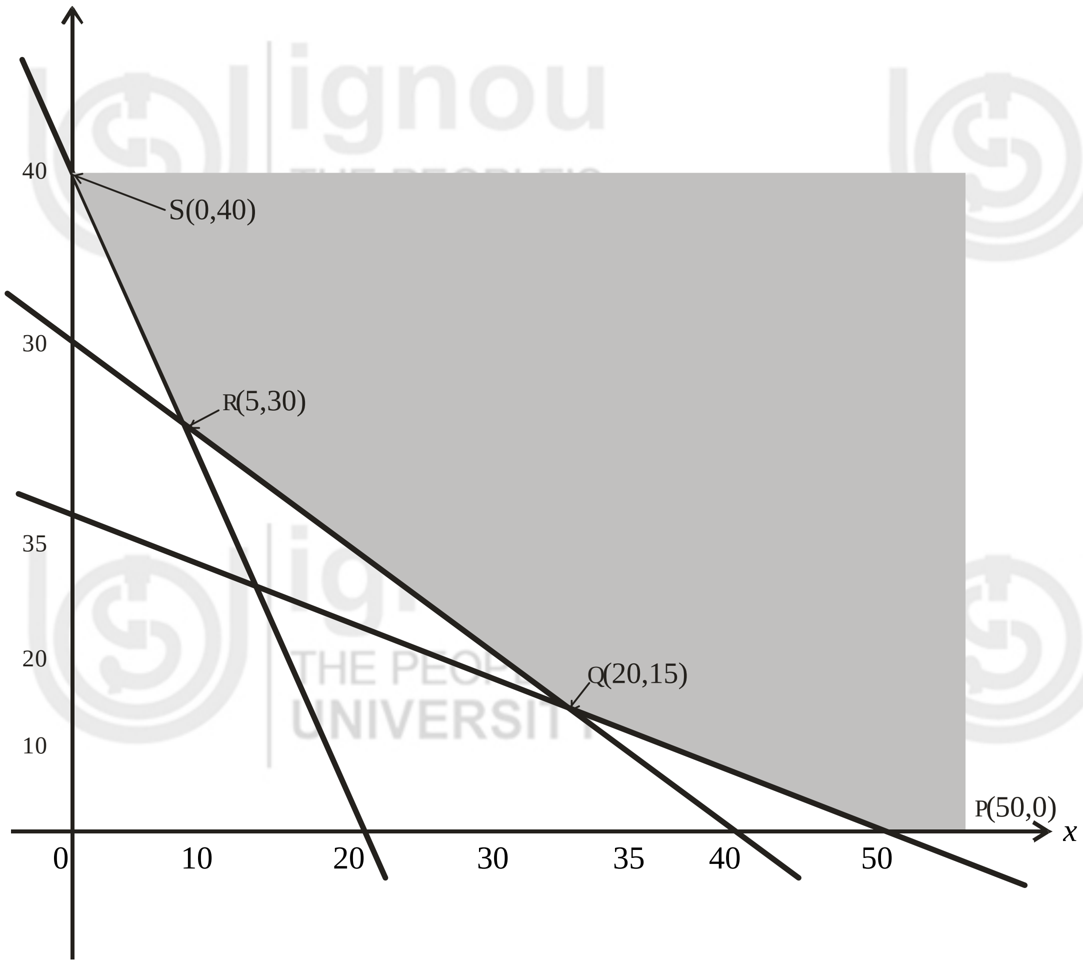

Figure 27

4. Let x boxes and y cards be purchased.

Cost of x boxes is $30x$ paise and cost of y cards is $20y$ paise.

$\therefore$ Cost incurred by the tailor (in paise) is $30x + 20y$

Number of large buttons obtained from x boses and y cards is $2x + 4y$.

According to given condition.

$$6x + 2y \geq 40$$

Number of small buttons obtained from x boxes and y cards is $2x + 4y$.

According to the given condition

$$2x + 4y \geq 60$$

Also, $x \geq 0, y \geq 0$

Thus, the linear programming problem is

Minimize

$C = 30\,x + 20y$ [objective function]

subject to

$6x + 2y \geq 40$ [large button constraint]

$2\,x + 4y \geq 60$ [small button constraint]

$x \geq 0, y \geq 0$ [non-negativity]

We draw the feasible in the following figure.

We now calculate the cost at the corner points of the feasible region.

$C(A) = C(0,20) = 30(0) + (20)(20) = 400$

$C(B) = C(5/2, 55/2) = (30)\,(5/2) = (20)\,(55/2) = 75 + 550 = 525$

$C(D) = C\,(30,0) = (30)(30) + (20)(0) = 900$

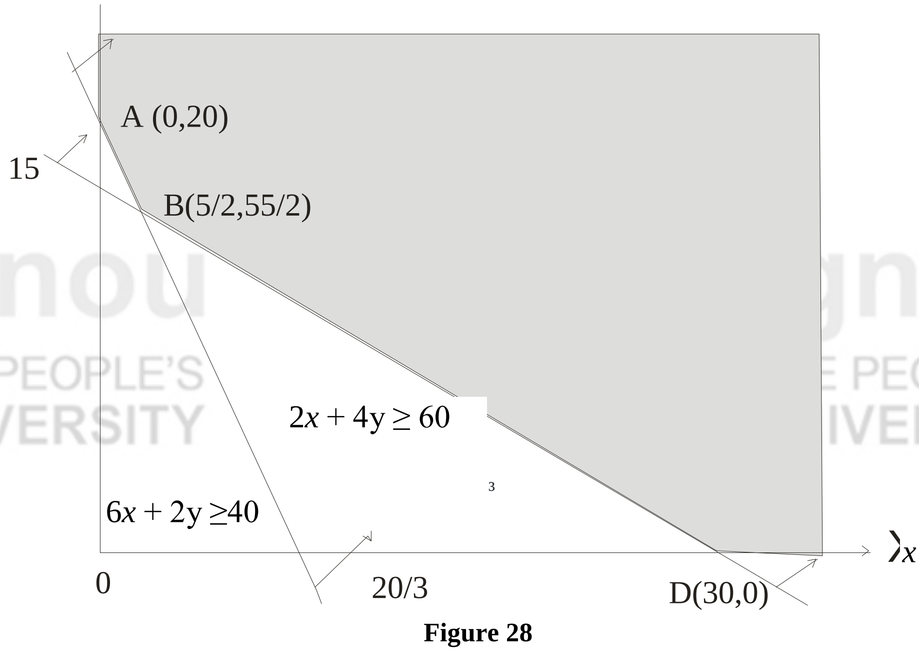

Figure 28

Thus, the least cost occurs when the tailor purchases just 20 cards and the least cost is 400.

4.7 SUMMARY

The unit is about the mathematical discipline of linear programming. In **section 4.0**, a number of relevant concepts including that of objective function, feasible region/solution space are introduced. Then the nomenclature 'linear programming' is explained. In **section 4.2** the above concepts alongwith some other relevant concepts are (formally) defined. **Section 4.3** explains the two graphical methods for solving linear programming problems (L.P.P.). viz. (i) corner point method (ii) iso-profit and iso-cost method. The methods are explained through a number of examples. **Section 4.4** discusses methods of cost minimisation in context of linear programming problems.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 4.5**.