# UNIT 4 MATHEMATICAL INDUCTION

## Structure



## 4.0 INTRODUCTION

We begin with the following question. What is the sum of first *n* odd natural numbers ?

If *n* equals 1, the sum equals 1, as 1 is the only summand. The answer we seek is a formula that will enable us to determine this sum for each value *n* without having to add the summands.

Table 4.1 lists the sum $S_n$ of the first *n* odd natural numbers, as *n* takes values from 1 to 10.

**Table 4.1**

| *n* | *Series* | *Sum* ($S_n$) |
|---|---|---|
| 1 | 1 | $1=1^2$ |
| 2 | 1+3 | $4=2^2$ |
| 3 | 1+3+5 | $9=3^2$ |
| 4 | 1+3+5+7 | $16=4^2$ |
| 5 | 1+3+5+7+9 | $25=5^2$ |
| 6 | 1+3+5+7+9+11 | $36=6^2$ |
| 7 | 1+3+5+7+9+11+13 | $49=7^2$ |
| 8 | 1 +3 +……..+15 | $64=8^2$ |
| 9 | 1 +3 +……..+17 | $81=9^2$ |
| 10 | 1 +3 +……..+19 | $100=10^2$ |

### Jumping to a Conclusion

Judging from the pattern formed by first 10 sums, we might conjecture that

$S_n = 1 + 3 + 5 + ... + (2n – 1) = n^2$.

Recognizing a pattern and then simply jumping to the conclusion that the pattern must be true for all values of *n* **is not a logically valid method** of proof in mathematics. There are many instances when the pattern appears to be developing for small values of $n$ and then at some point the pattern fails. Let us look at one example. It was widely believed that $P_n = n^2 + n + 41$ is prime for all natural-numbers. Indeed $p_n$, is prime for all values of $n$ lying between 1 and 39 as shown in Table 4.2.

But the moment we take $n = 40$, we get

$$P_{40} = 40^2 + 40 + 41$$
$$= 1600+40+41 = 1681 = 41^2$$

which is clearly not a prime.

**Table 4.2**

| $n$ | $P_n$ | $n$ | $P_n$ | $n$ | $P_n$ |
|---|---|---|---|---|---|
| 1 | 43 | 11 | 173 | 26 | 743 |
| | | 12 | 197 | 27 | 797 |
| 2 | 47 | 13 | 223 | 28 | 853 |
| 3 | 53 | 14 | 251 | 29 | 911 |
| | | 15 | 281 | 30 | 971 |
| 4 | 61 | 16 | 313 | 31 | 1033 |
| 5 | 71 | 17 | 347 | 32 | 1097 |
| | | 18 | 383 | 33 | 1163 |
| 6 | 83 | 19 | 421 | 34 | 1231 |
| 7 | 97 | 20 | 461 | 35 | 1301 |
| | | | | 36 | 1373 |
| 8 | 113 | 21 | 503 | 37 | 1447 |
| 9 | 131 | 22 | 547 | 38 | 1523 |
| | | 23 | 593 | | |
| 10 | 151 | 24 | 641 | 39 | 1601 |
| | | 25 | 691 | | |

Just because a rule, pattern or formula seems to work for several values of $n$, we cannot simply conclude that it is valid for all values of $n$ without going through a *legitimate proof.*

**How to Legalize a Pattern?**

One way to legalize the pattern is to use the principle of **Mathematical induction.** To see what it is, let us return to our question in the beginning of the chapter. What is the sum of first $n$ odd natural numbers?

We have already seen that the formula

$S_n = 1 + 3 + 5 + ... + (2n - 1) = n^2$ is valid for $n = 1, 2, 3, ..., 10$ (1)

Do we need to compute $S_n$ by adding the first $n$ odd natural numbers ?
A moment's reflection will show that it is not necessary.

Having obtained the value of $S_n$ for some integer $n$, we can obtain the value of

$$S_{n+1} = S_n + 2n + 1$$

if $S_n = n^2$ for some n, then $S_{n+1} = S_n + 2n + 1 = n^2 + 2n + 1 = (n+1)^2$.

That is, if $S_n = n^2$ for some natural number $n$, then the formula holds for the next natural number $n + 1$.

Since the formula $S_n = n^2$ holds for $n = 10$, therefore it must hold $n = 11$. Since, it holds for $n = 11$, therefore, it must hold for $n = 12$. Since, it holds for $n = 12$, it holds for $n = 13$, and so on. The principle underlying the foregoing argument is nothing but the principle of mathematical induction. We state this formally in section 4.3.

## 4.1 OBJECTIVES

After studying this unit, you should be able to:

- use the principle of mathematical induction to establish truth of several formulae and inequalities for each natural number $n$.

## 4.2 THE PRINCIPLE OF MATHEMATICAL INDUCTION

Let $P_n$ be a statement involving the natural number $n$. If

1. $P_1$ is true, and
2. the truth of $P_k$ implies the truth of $P_{k+1}$, for every interger $k$, then $P_n$ must be true all natural numbers $n$.

In other words, to prove that a statement $P_n$ holds for all natural numbers, we must go through *two* steps; First, we must prove that $P_1$ is ture. Second, we must prove that $P_{k+1}$ is true whenever $P_k$ is true.

**CAUTION** Just proving $P_{k+1}$ whenever $P_k$ is true will not work.

There is an interesting analogue. Suppose we have "sequence" of dominoes standing in a row, as in Fig. 4.1 Suppose (1) the first domino falls, and (2) whenever any domino falls, then the one next to it (to the right in Fig. 4.2) falls as well. Our conclusion is that each domina will fall (see Fig 4.3). This reasoning closely parallels the ideal of induction.

To apply the principles of mathematical induction, we always need to be able to find $P_{k+1}$ for a given $P_k$ . It is important to acquire some skill in writing $P_{k+1}$ whenever $P_k$ is given.

We now take up some illustrations in which we write some particular terms when we know $P_n$. We also take up some illustrations in which we write $P_{k+1}$ when we know $P_k$.

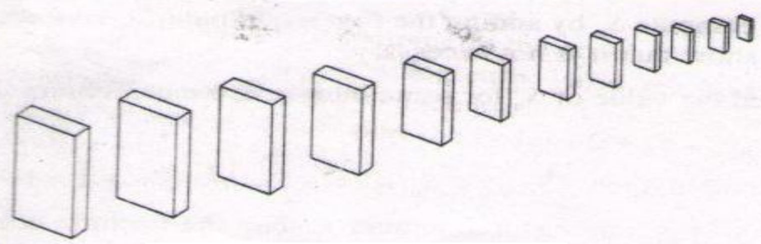

**Figure 4.1**

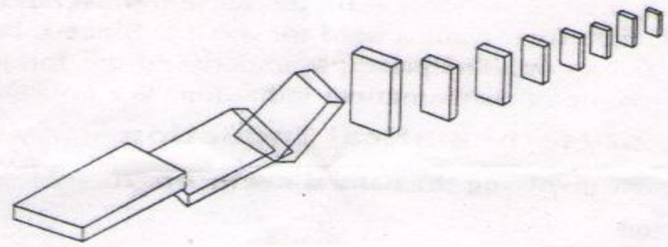

**Figure 4.2**

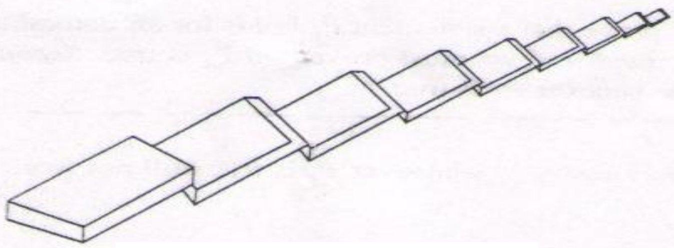

**Figure 4.3**

**Illustration 1** : If $P_n$ is the statement "$n(n + 1)$ is even"., then what is $P_4$ ? What is $P_{10}$ ?

**Solution :**

$P_4$ is the statement "4(4+1) is even", i.e., "20 is even". $P_{10}$ is the statement "10(10+1) is even" i.e., "110 is even".

**Ilustration 2 :** If $P_n$ is the statement "$n(n + 1)(n + 2)$ is divisible by 12", write $P_3$, $P_4$ and $P_5$. Which one of $P_3$, $P_4$ and $P_5$ are true statement ?

**Solution**

$P_3$ is "3 (3+1) (3+2) is divisible by 12" i.e., "60 is divisible by 12"

$P_4$ is "4 (4+1) (4+2) is divisible by 12" i.e., "120 is divisible by 12"

$P_5$ is "5 (5+1) (5+2) is divisible by 12" i.e., "210 is divisible by 12"

Each of $P_3$ and $P_4$ is true. But $P_5$ is false.

**Example 1**

(i) If $P_n$ is the statement "$n^3 + n$ is divisible by 3", is the statement $P_3$ true ? Is the statement $P_4$ true ?

(ii) If $P_n$ is the statement "$2^{3n} - 1$ is an integral multiple of 7", prove that $P_1$, $P_2$ and $P_3$ are true ?

(iii) If $P_1$ is the statement "$3^n > n$" are true $P_1$, $P_2$ and $P_4$ true statements ?

(iv) If $P_n$ is the statement "$2^n > n$" what is $P_{n+1}$ ?

(v) If $P_n$ is the statement "$3^n > n$" prove that $P_{n+1}$ is true whenever $P_n$ is true.

(vi) Let $P_n$ is the statement "$n^2 > 100$" prove that $P_{k+1}$ is true whenever $P_k$ is true.

(vii) If $P_n$ is the statement "$2^n > 3n$" and if $P_k$ is true, prove that $P_{k+1}$ is true.

(viii) If $P_n$ is the statement "$2^{3n} - 1$ is a multiple of 7", prove that truth of $P_k$ implies the true of $P_{k+1}$.

(ix) If $P_n$ is the statement "$10^{n+1} > (n + 1)^5$", prove that $P_{k+1}$ is true whenever $P_k$ is true.

(x) Give an example of a statement $P_n$, such that $P_3$ is true but $P_4$ is not true.

(xi) Give an example of statement $P_n$ such that it is not true for any $n$.

(xii) Give an example of a statement $P_n$ in which $P_1$, $P_2$, $P_3$ are not true but $P_4$ is true.

(xiii) Give an example of a statement $P_n$ which is true for each $n$.

**Solution :**

(i) $P_3$ is the statement "$3^3 + 3$ is divisible by 3" i.e., "30 is divisible by 3". which is clearly true.

$P_4$ is the statement "$4^3 + 4$ is divisible by 4" i.e., "68 is divisible by 3"

This is clearly not true.

(ii) $P_1$ is the statement "$2^3 - 1$ is an integral multiple of 7", i.e., "7 is an integral multiple of 7". This is a true statement.

$P_2$ is the statement "$2^6 - 1$ is an integral multiple of 7", i.e., 63 is an integral multiple of 7". This also is a true statement.

$P_3$ is the statement "$2^9 - 1$ is an integral multiple of 7", i.e., "511 is an integral multiple of 7". This again is a true statement.

(iii) $P_1$ is $3^1 > 1$", which is clearly true.

$P_2$ is "$3^2 > 2$". This also is a true statement.

$P_4$ is "$3^4 > 4$". This again is a true statement.

(iv) $P_{n+1}$ is the statement "$2^{n+1} > n+1$".

(v) We are given that $3^n > n$.

we are interested to show that $3^{n+1} > n+1$

we have $\dfrac{n+1}{n} = 1 + \dfrac{1}{n} \leq 1 + 1 < 3$

$\Rightarrow n + 1 < 3n < 3.3^n = 3^{n+1}$.

This show that if $P_n$ is true, then $P_{n+1}$ is true.

(vi) We are given that $k^2 > 100$.

we wish to show that $(k+1)^2 > 100$

we have

$(k + 1)^2 = k^2 + 2k + 1 > k^2 > 100$ $[\because 2k+1 > 0]$

$\Rightarrow (k+1)^2 > 100$.

This shows that $P_{k+1}$ is true whenever $P_k$ is true.

(vii) Since $P_k$ is true, we get $2^k > 3k$.

we wish to show that $2^{k+1} > 3(k+1)$

we have

$2^{k+1} = 2.2^k = 2^k + 2^k > 3k + 3k$ [by assumption]

$> 3k + 3$

$\Rightarrow 2^{k+1} > 3(k+1)$

this proves that $P_{k+1}$ is true.

(viii) Since $P_k$ is true we have $2^{3k} - 1$ is a multiple of 7, i.e., there exists an integer $m$ such that $2^{3k} - 1 = 7m$

We wish to show that $2^{3(k+1)} - 1$ is a multiple of 7.

We have

$2^{3(k+1)} - 1 = 2^{3k} . 2^3 - 1 = (7m+1). (8) - 1$

$= 56m + 8 - 1 = 56m + 7 = 7(8m + 1)$

This shows that $2^{3(k+1)} - 1$ is a multiple of 7, i.e. $P_{k+1}$ is true.

(ix) Since $P_k$ is true, we have $10^{k+1} > (k+1)^5$

We wish to show that $10^{k+2} > (k+2)^5$

We have

$$\frac{(k+2)^5}{(k+1)^5} = \left(1 + \frac{1}{k+1}\right)^5$$

$$\text{As } k \geq 1, k+1 \geq 2 \Rightarrow \frac{1}{k+1} \leq \frac{1}{2}$$

$$\text{therefore, } \quad \left(1 + \frac{1}{k+1}\right)^5 \leq \left(1 + \frac{1}{2}\right)^5 = \left(\frac{3}{5}\right)^5 = \frac{243}{32} < 10$$

Thus, $(k+2)^5 < 10\,(k+1)^5 < 10.10^{k+1} = 10^{k+2}$.

Therefore, $P_{k+1}$ is true.

(x) Let $P_n$ be that statement "$n \leq 3$", then $P_3$ is true but $P_4$ is not true.

(xi) Let $P_n$ be the statement "$n\,(n+1)$ is odd". Then $P_n$ is false for every $n$.

(xii) Let $P_n$ $n \geq 4$.

(xiii) Let $P_n$ be the statement "$n \geq 1$". The $P_n$ is true for each $n$.

**Example 2:** Use the principle of mathematical induction to prove that

$$2 + 4 + 6 + \ldots\ldots\ldots\ldots\ldots\ldots + 2n = n(n+1)$$

for each natural number $n$.

**Solution :**

Mathematical induction consists of two distinct parts. First, we must show that the formula holds for $n = 1$.

Let $P_n$ denote the statement

$$2 + 4 + 6 + \ldots\ldots\ldots\ldots\ldots\ldots + 2n = n(n+1)$$

**Step 1**. When $n = 1$, $P_n$ becomes $2 = 1(1+1)$

which is clearly true.

The second part of mathematical induction has two steps. The first step is to assume that the formula is valid for *some* integer $k$. The second step is to use this assumption to prove that formula is valid for the next natural number $k+1$.

**Step 2**. Assume that $P_k$ is true for some $k \in \mathbf{N}$, that is, assume that

$$2 + 4 + 6 + \ldots\ldots\ldots\ldots\ldots\ldots + 2k = k\,(k+1)$$

is true. We must show that $P_{k+1}$ is true, where $P_{k+1}$

$$2 + 4 + 6 + \ldots\ldots\ldots\ldots\ldots\ldots + 2k + 2\,(k+1) = (k+1)(k+2) \qquad (1)$$

**Not to Forget**

While writing LHS of $P_{k+1,}$ you must remember that not only should you write the last term of the series, but also a term prior to the last term. If you, now supress the last term of the LHS of $P_{k+1}$ what remain of the LHS of $P_k$.

LHS of (1) = 2 + 4 + 6 + ……………… + 2$k$ + 2 ($k$ + 1)

= $k$ ($k$ + 1) + 2($k$+1) [induction assumption]

= ($k$ + 1) ($k$ + 2) [taking $k$ +1 common]

= RHS of (1)

This shows that the result holds for $n = k+1$; therefore, the truth of $P_k$ implies the truth of $P_{k+1}$. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number $n$.

**CAUTION** You will lose at least one mark if you do not write this last paragraph.

**Example 3** Use the principle of mathematical induction to show that

$$1+ 4 + 7 + \ldots\ldots + (3n-2) = \frac{1}{2}n(3n-1)$$

for every natural number $n$.

**Solution :** Let $P_n$ denote the statement.

$$1+4+7+\ldots\ldots(3n-2) = \frac{1}{2}n\,(3n-1)$$

When $n = 1$, $P_n$ becomes $1 = \frac{1}{2}(1)[3(1) - 1]$ or $1 = 1$

which is clearly true.

This shows that the result holds $n = 1$.

Assume that $P_k$ is true for some $k \in \mathbf{N}$. That is, assume that

$$1 + 4 + 7 + \ldots\ldots(3k-2) = \frac{1}{2}k\,(3k-1)$$

We shall now show that the truth of $P_k$ implies the true of $P_{k+1}$ where $P_{k+1}$ is

$$1 + 4 + 7 + \ldots\ldots + (3k-2) + \{3(k+1) - 2\} = \frac{1}{2}(k+1)(3(k+1)-1)$$

or $1 + 4 + 7 + \ldots\ldots + (3k-2) + (3k+1) = \frac{1}{2}(k+1)(3k+2)$ (1)

LHS of (1)

$= 1 + 4 + 7 + \ldots\ldots + (3k-2) + (3k+1)$

$= \frac{1}{2}k(3k-1) + (3k+1)$ [ induction assumpation]

$= \frac{1}{2}\ [3k^2 - k + 6k + 2]$

$= \frac{1}{2}\ (3k^2 + 5k + 2)$

$= \frac{1}{2}\ [3k^2 + 3k + 2k + 2]$

$= \frac{1}{2}\ [3k(k+1) + 2(k+1)]$

$= \frac{1}{2}\ (k+1)(3k+2) =$ RHS of (1)

This shows that the result holds for $n = k + 1$; therefore, the truth of $P_k$ implies the truth of $P_{k+1}$. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number $n$.

**Example 4 :** Use the principle of mathematical induction to prove that

$1^3 + 2^3 + \ldots\ldots\ldots\ldots + n^3 = \frac{1}{4}n^2(n+1)^2$ for every natural number $n$.

**Solution:** Let $P_n$ denote the statement

$$1^3 + 2^3 + \ldots\ldots\ldots\ldots + n^3 = \frac{1}{4}\ n^2\ (n+1)^2.$$

When $n = 1$, $P_n$ becomes

$$1^3 + 2^3 + \ldots\ldots\ldots\ldots + n^3 = \frac{1}{4}(1^2)\ (1+1)^2 \text{ or } 1 = 1$$

This shows that the result holds for $n = 1$. Assume that $P_k$ is true for some $k \in \mathbf{N}$. That, is assume that

$$1^3 + 2^3 + \ldots\ldots\ldots\ldots + k^3 = \frac{1}{4}k^2\ (k+1)^2$$

We shall now show that the truth of $P_k$ implies the truth of $P_{k+1}$ where $P_{k+1}$ is

$$1^3 + 2^3 + \ldots\ldots\ldots\ldots + k^3 = \frac{1}{4}\ (k+1)^2\ (k+1)^2 \qquad (1)$$

LHS of (1)

$= 1^3 + 2^3 + \ldots\ldots\ldots\ldots k^3 = + (k+1)^3$

$= \frac{1}{4}\ k^2\ (k+1)^2\ (k+1)^3$

$= \frac{1}{4}\ k^2\ (k+1)^2\ (k+1)^3$ [induction assumption]

$= \frac{1}{4}\ (k+1)^2\ [k^2 + 4(k+1)] = \frac{1}{4}\ (k+1)^2(k+2)^2$

= RHS of (1)

This shows that the result holds for $n = k+1$; therefore, the truth of $P_k$ implies the truth of $P_{k+1}$. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number $n$.

**Example 5 :** Use the principle of mathematical induction to prove that

$$\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \ldots\ldots\ldots\ldots\ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for every natural number *n*.

**Solution:** Let $P_n$ denote the statement

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

When $n = 1, Pn$ becomes $\frac{1}{1.2} = \frac{1}{1+1}$ or $\frac{1}{2} = \frac{1}{2}$

This shows that the result holds for $n = 1$. Assume that $P_k$ is true for some $k \in \mathbf{N}$. That, is assume that

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{k(k+1)} = \frac{k}{k+1} \qquad (1)$$

We shall now show that the true of $P_k$ implies the truth of $P_{k+1}$ where $P_{k+1}$ is

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{k(k+1)} = \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

$$\text{LHS of (1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)} \qquad \text{[induction assumption]}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

= RHS of (1)

This shows that the result holds for $n = k+1$; therefore, the truth of $P_k$ implies the truth of $P_{k+1}$. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number *n*.

**Example 6 :** Use the principle of mathematical induction to show that

$$2 + 2^2 + \ldots\ldots\ldots\ldots + 2^n = 2^{n+1} - 2$$

for every natural number *n*.

**Solution :** Let $P_n$ denote the statement

$$2 + 2^2 + \ldots\ldots\ldots\ldots + 2^n = 2^{n+1} - 2$$

When $n = 1$, $P_n$ becomes

$$2 = 2^{1+1} - 2 \text{ or } 2 = 4 - 2$$

This shows that the result holds for $n = 1$.

Assume that $P_k$ is true for some $k \in \mathbf{N}$.

That is, assume that

$$2 + 2^2 + \ldots\ldots\ldots + 2^k = 2^{k+1} - 2$$

We shall now show that truth of $P_k$ implies the truth of $P_{k+1}$ is

$$2 + 2^2 + \ldots\ldots\ldots + 2^k + 2^{k+1} = 2^{k+1} - 2 \quad (1)$$

LHS of (1) $= 2 + 2^2 + \ldots\ldots\ldots + 2^k + 2^{k+1}$

$= (2^{k+1} - 2) + 2^{k+1}$ [induction assumption]

$= 2^{k+1}(1 + 1) - 2$

$= 2^{k+1}2 - 2 = 2^{k+2} - 2$

$=$ RHS of (1)

This shows that the result holds for $n = k+1$; therefore, the truth of $P_k$ implies the truth of $P_{k+1}$. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number $n$.

**Example 7:** Show that $2^{3n} - 1$ is divisible by 7 for every natural number $n$.

**Solution :** Let $P_n$ denote the statement $7|(2^{3n} - 1)$

For $n = 1$, $P_n$ becomes $7|(2^3 - 1)$

Since $2^3 - 1 = 8 - 1 = 7$, we have $7|7$. This shows that the result is ture for $n = 1$.
Assume that $P_k$ is true for some $k \in \mathbf{N}$.

That is, assume that $7 \mid (2^{3k} - 1)$

That is, assume that $2^{3k} - 1 = 7m$ for some $m \in \mathbf{N}$.

We shall now that that the truth of $P_k$ implies the truth of $P_{k+1}$, where $P_{k+1}$ is

$7(2^{3(k+1)} - 1)$

Now

$2^{3(k+1)} - 1 = 2^{3k+3} - 1 = 2^{3k}.\,2^3 - 1$

$= (7m + 1)(8) - 1$ [$\because 2^{3k} - 1 = 7m$]

$= 56m + 8 - 1 = 56m + 7 = 7(8m+1)$

$\Rightarrow \quad 7|[2^{3(k+1)} - 1]$

This shows that the result holds for $n = k+1$; therefore, the truth of $P_k$ implies the truth of $P_{k+1}$. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number $n$.

**Example 8** Show that $n(n+1)(2n+1)$ is a multiple of 6 for every natural number $n$.

**Solution :** Let $P_n$ denote the statement $n(n + 1)(2n + 1)$ is a multiple of 6.
When $n = 1$, $P_n$ becomes $1(1 + 1)((2)(1) + 1) = (1)(2)(3) = 6$ is a multiple of 6.
This shows that the result is true for $n = 1$.

Assume that $P_k$ is true for some $k \in \mathbf{N}$. That is assume that $k(k + 1)(2k + 1)$ is a mutliple of 6.

Let $\qquad k(k + 1)(2k + 1) = 6m$ for some $m \in \mathbf{N}$.

We now show that the truth of $P_k$ implies the truth of $P_{k+1}$, where $P_{k+1}$ is $(k + 1)(k + 2)[2(k + 1) + 1] = (k + 1)(k + 2)(2k + 3)$ is a multiple of 6.

We have

$(k + 1)(k + 2)(2k + 3)$

$$= (k + 1)(k + 2)[(2k + 1) + 2]$$

$$= (k + 1)[k(2k + 1) + 2(2k+1) + 4)]$$

$$= (k + 1)[k(2k + 1) + 6(k + 1)]$$

$$= k(k + 1)(2k + 1) + 6(k + 1)^2$$

$$= 6m + 6(k + 1)^2 = 6[m + (k + 1)^2]$$

Thus $(k + 1)(k + 2)(2k + 3)$ is multiple of 6.

This shows that the result holds for $n = k+1$; therefore, the truth of $P_k$ implies the truth of $P_{k+1}$. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number $n$.

**Example 9** : Show that 11 divides $10^{2n-1} + 1$ for every natural number $n$.

**Solution :** Let $P_n$ denote the statement

$$11 | (10^{2n-1} + 1).$$

When $n = 1$, $P_n$ becomes $11 | (10^{2-1} + 1)$.
As $10^{2-1} + 1 = 10 + 1 = 11$ we have $11 | 11$.
This shows tht the result is true for $n = 1$

Assume that $P_k$ is true for some $k \in \mathbf{N}$. That is, assume that

$$11(10^{2k-1} + 1).$$

That is, assume that $10^{2k-1} + 1 = 11$ m for some $m \in \mathbf{N}$.

We shall now that the truth of $P_k$ implies $P_{k+1}$, where $P_{k+1}$ is

$$11(10^{2(k+1)-1} + 1)$$

$$\begin{aligned}
\text{Now, } 10^{2(k+1)-1} + 1 &= 10^{2k+2-1} + 1 \\
&= 10^{2k-1+2} + 1 = 10^{2k-1} \,.\, 10^2 \\
&= (11m - 1)\, 10^2 + 1 \qquad [\because 10^{2k-1} + 1 = 11m] \\
&= 1100\, m - 100 + 1 = 1100\, m - 99 = 11(100m - 9)
\end{aligned}$$

$$\Rightarrow \qquad 11 \mid (10^{2(k+1)-1} + 1).$$

This shows that the result holds for $n = k+1$; therefore, the truth of $P_k$ implies the truth of $P_{k+1}$. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number $n$.

**Example 10** Show that 133 divides $11^{n+2} + 12^{2n+1}$ for every natural number $n$.

**Solution** Let $P_n$ denote the statement

$$133|(11^{n+2} + 12^{2n+1})$$

When $n = 1$, $P_n$ becomes $133|(11^{1+1} + 12^{2+1})$ .

As $11^{1+2} + 12^{2+1} = 11^3 + 12^3 = 1331 + 1728 = 3059$

$= (133)(29)$, we have $133|(11^{1+2} + 12^{2+1})$

This shows that the result is true for $n = 1$. Asumme that $P_k$ is true for some $k \in \mathbf{N}$. That is assume that

$$133|(11^{k+2} + 12^{2k+1})$$

That is, assume that $11^{k+2} + 12^{2k+1} = 133m$ for some $m \in \mathbf{N}$. We shall now show that the truth of $P_k$ implies the truth of $P_{k+1}$, where $P_{k+1}$ is

$$133 \mid (11^{k+1+2} + 12^{(2k+2+1)})$$

Now, $11^{k+1+2} + 12^{(2k+2+1)}$

$= 11^{k+2}\, 11^2 + 12^{2k+1}\, 12^2$

$= 11^{k+2}\, 11 + (133m - 11^{2k+1})\, 12^2$ [by induction assumption]

$= 11^{k+2}\, 11 + (133m)(144) - (11^{k+2})(144)$

$= 133(144m) - 133(11^{k+2})$

$= 133(144m - 11^{k+2})$

Thus, $133 \mid (11^{k+1+2} + 12^{2(k+1)+1})$

This shows that the result holds for $n = k + 1$; therefore, the truth of $P_k$ implies the truth of $P_{k+1}$. The two steps required for a proof by mathematical induction have been completed, so $P_n$ is true for each $n \in \mathbf{N}$.

**Example 11** : Show that $14 \mid (3^{4n-2} + 5^{2n-1})$ for all natural number $n$.

**Solution :** Let $P_n$ denote the statement $14 \mid (3^{4n-2} + 5^{2n-1})$.

For $n = 1$, we have

$3^{4n-2} + 5^{2n-1} = 3^2 + 5 = 14$ which is divisible by 14.

Assume that $P_n$ is true for some natural number $n$, say $k$. That is, assume that $14 \mid (3^{4k-2} + 5^{2k-1})$ is true for some natural number $k$. Suppose $3^{4k-2} + 5^{2k-1} = 14m$ for some natural number m. We now show that the truth of $P_k$ implies the truth of $P_{k+1}$, that is, we show that $14 \mid [(3^{4(k+1)-2} + 5^{2(k+1)-1})]$.

We have

$$= 3^{4(k+1)-2} + 5^{2(k+1)-1} = 3^{4k-2} \cdot 3^4 + 5^{2k-1} 5^2$$

$$= (14\, m\, 5^{2k-1}) \cdot 3^4 + 5^{2k-1} \cdot 5^2 \qquad [3^{4k-2} = 14m - 5^{2k-1}]$$

$$= (14\, m - (81) + 5^{2k-1}\ (-81+25)$$

$$= (14m)\,(81) - (5^{2k-1})(56) = 14[81m - 4.5^{2k-1}]$$

$$\Rightarrow \quad 14 \mid [3^{4(k+1)-2} + 5^{2(k+1)-1}]$$

This shows that the result holds for $n = k+1$; therefore, the truth of $P_k$ implies the truth of $P_{k+1}$. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number $n$.

**Check Your Progress – 1**

Use the principle of mathematical induction to prove the following formulae.

1. $1 + 3 + 5 + \ldots\ldots\ldots + (2n-1) = n^2 \qquad \forall\, n \in \mathbf{N}$
2. $1^2 + 2^2 + \cdots \ldots\ldots\ldots\ldots + n^2 = \frac{1}{6} n(n+1)(2n+1) \qquad \forall\, n \in \mathbf{N}$
3. $\frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \ldots\ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1} \qquad \forall\, n \in \mathbf{N}$
4. $1(2^2) + 2\,(3^2) + \ldots\ldots\ldots\ldots + n\,(n+1)^2 \qquad \forall\, n \in \mathbf{N}$
5. $8 \mid (3^n - 1) \qquad \forall\, n \in \mathbf{N}$
6. $24 \mid (5^{2n} - 1) \qquad \forall\, n \in \mathbf{N}$
7. $1 + \frac{1}{2} + \ldots\ldots\ldots + \frac{1}{2^n} < 2. \qquad \forall\, n \in \mathbf{N}$
8. $1 + 2 + \ldots\ldots\ldots + n < (2n+1)2 \qquad \forall\, n \in \mathbf{N}$

## 4.3 ANSWERS TO CHECK YOUR PROGRESS

**Check Your Progress – 1**

1. Let $P_n$ denote the statement

$1 + 3 + 5 + \ldots\ldots\ldots\ldots + (2n - 1) = n^2$

For $n = 1$, $P_n$ becomes $1 = 1^2$ or $1=1$ which is clearly true.

Assume that $P_k$ is ture for $k \in \mathbf{N}$

That is, assume that

$1 + 3 + 5 + \ldots\ldots\ldots\ldots + (2k - 1) = k^2$

For $n = k + 1$, we have

$P_{k+1} : 1 + 3 + \ldots\ldots\ldots\ldots + (2k - 1) + (2k + 1) = (k + 1)^2$

Now,

$1 + 3 + \ldots\ldots\ldots + (2k - 1) + (2k+1)$

$= k^2 + (2k + 1)$ [by induction assumption]

$= (k + 1)^2$

2. Clearly result is true for $n =1$ . Assume that result holds for $n = k$, that is,

$1^2 + 2^2 + \ldots\ldots\ldots\ldots + k^2 = \frac{1}{6} k (k + 1) (2k + 1)$

For $n = k+1$,

LHS $= 1^2 + 2^2 + \ldots\ldots\ldots\ldots + k^2 + (k+1)^2$

$$= \frac{1}{6} k (k + 1) (2k + 1) + (k + 1)^2$$

$$= \frac{1}{6}(k + 1) [k (2k + 1) + 6(k + 1)]$$

$$= \frac{1}{6}(k + 1) [2k^2 + 7k + 6 ]$$

$$= \frac{1}{6}(k + 1)[2k^2 + 3k + 4k + 6 ]$$

$$= \frac{1}{6}(k + 1) [k(2k + 3) + 2(2k + 3)]$$

$$= \frac{1}{6}(k + 1)(k + 2)(2k + 3)$$

$$= \frac{1}{6}(k + 1)(k + 1 + 1)(2k + 1) + 1)$$

The result holds for $n = k + 1$.

3. Result holds for $n$ =1.

Assume that

$$\frac{1}{(1)(3)}+\frac{1}{(3)(5)}+\ \ldots\ldots..+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

For $n = k + 1$,

$$\text{LHS }=\frac{1}{(1)(3)}+\frac{1}{(3)(5)}+\ \ldots\frac{1}{(2k-1)\ (2k+1)}+\frac{1}{(2k+1)\ (2k+3)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)\ (2k+3)}$$

$$=\frac{1}{(2k+1)(2k+3)}\ [k(2k+3)+1]$$

$$=\frac{1}{(2k+1)(2k+3)}\ [2k^2+2k+k+1]$$

$$=\frac{(k+1)(2k+1)}{(2k+1)(2k+3)}=\frac{(k+1)}{(2k+3)}=\frac{(k+1)}{2(k+1)+1}$$

Result holds for $n = k + 1$.

4. The result holds for $n$ =1.

Assume that

$1(2^2)+\ 2(3^2)+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+k+(k+1)^2=$

$$\frac{1}{12}\ k(k+1)(k+2)(3k+5)$$

For $n = k + 1$.

LHS = $(1)\ (2)^2+\ldots\ldots\ldots\ldots\ldots\ldots..+\ k\ (k+1)^2+(k+1)\ (k+2)^2$

$$=\frac{1}{12}k\ (k+1)\ (k+2)\ (3k+5)\ +\ (k+1)\ (k+1)^2$$

$$=\frac{1}{12}k\ (k+1)\ (k+2)\ [k(3k+5)\ +12(k+2)$$

$$=\frac{1}{12}k\ (k+1)\ (k+2)\ [3k^2+17k+24]$$

$$=\frac{1}{12}k\ (k+1)(k+2)[3k^2+8k+9k+24]$$

$$=\frac{1}{12}k\ (k+1)\ (k+2)\ [k(3k+8)k+3(k+8)]$$

$$=\frac{1}{12}k\ (k+1)\ (k+2)\ (k+3)(3k+8)$$

$$=\frac{1}{12}k\ (k+1)\ (k+1+1)\ (k+1+2)[(3k+1)+5]$$

The result holds for $n = k$+1.

5. The result holds for $n = 1$.

   Assume that $8 | (3^{2k} - 1)$ for same $k \in$ **N.**

   Let $3^{2k} - 1 = 8m$ for some $m \in$ **N.**

   Next, $3^{2(k+1)} - 1 = 3^{2k}.3^2 - 1 = (8m + 1)(9) - 1$

   $= 72m + 9 - 1 = 8(9m + 1)$

   This shows that $8 | (3^{2(k+1)} - 1)$

   The result holds for $n = k+1$.

6. The result holds for $n = 1$

   Assume $24|(5^{2k} - 1)$ for same $k \in$ **N.**

   $\Rightarrow 5^{2k} - 1 = 24m$ for same m $\in$ **N.**

   For $n = k+1$,

   $5^{2k+2} - 1 = 5^{2k}.5^2 - 1$

   $= (24m + 1)(25) - 1$

   $= (24)(25)m + 25 - 1$

   $= 24(25m + 1)$

   Thus, $24 ( 5^{2k+2} - 1)$

   The result holds for $n = k+1$.

7. The result holds for $n = 1$, as $1 + \frac{1}{2} < 2$

   Assume that

   $1 + \frac{1}{2} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{2^k} < 2$ *for same* $k \in$ **N.**

   $\Rightarrow \frac{1}{2}(1 + \frac{1}{2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{2^k}) < 1$

   $\Rightarrow 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{2^{k+1}} < 1 + 1 = 2.$

   The result holds for $n = 1$.

8. The result holds for $n = 1$.

   Assume that the result holds for $n = k$, that is ,

   $1 + 2 + \ldots\ldots\ldots\ldots + k < (2k + 1)^2$

   We have,

   $1 + 2 + \ldots\ldots\ldots\ldots + k + (k+1)$

   $< (2k + 1)^2 + (k + 1)$

   $= 4k^2 + 4k + 1 + k + 1$

   $< 4k^2 + 12k + 9$ $\quad [\because 7k + 7 > 0]$

   $= (2k + 3)^2$

   Thus, the result holds for $n = k + 1$.

## 4.4 SUMMARY

The unit is for the purpose of explaining the Principle of Mathematical Induction, one of the very useful mathematical tools. A large number of examples are given to explain the applications of the principle.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 4.3**.