UNIT 1 INTRODUCTION TO METRIC SPACES

1.1 INTRODUCTION

In this unit, we introduce you to the notion of metric spaces. As we already pointed out in the course introduction and block introduction, the metric spaces arose from extending the notions of continuity and convergence on the real line to more abstract spaces.

A metric space is just a set which is equipped with a function called metric which measures the distance between the elements of various pairs from the set. We shall first give the definition of a metric and a metric space and consider various examples in Section 1.2. Then we shall study various properties of these spaces. In Section 1.3 and Section 1.4, we shall consider open and closed sets.

The structure on a metric space allows us to extend the notion of continuity to functions in the context of these spaces. We shall define this notion in Section 1.5 and discuss several examples of continuous functions. Later you will see that the notion of continuity is one of the most important notions for further study of Analysis. Here we talk about two important results about continuous functions which are called Urysohn’s lemma and glueing lemma. Then we explain the notion of uniform continuity through some examples. You will see that the definition of continuity and uniform continuity for metric spaces are similar to those for Euclidean spaces $\mathbb{R}^n$. But extending these notions to metric spaces, provide not only a new perspective but also a deeper insight into their structure and properties.

Objectives

After studying this unit, you should be able to

- state the properties that define a metric and apply them;
- give examples of different metrics on $\mathbb{R}^n$;
- explain a discrete metric space and other metric spaces such as function spaces;
- check whether
  i) a subset of a metric space is open;
  ii) a subset of a metric space is closed;
iii) a function defined on a metric space is continuous;
iv) a function defined on a metric space is uniformly continuous;

- explain Urysohn’s lemma and glueing lemma.

### 1.2 DEFINITIONS AND EXAMPLES

We begin our study of metric spaces by extending those aspects of real and complex number systems that help us in studying their analytical structure and functions on them. What do you think are the most important concepts in Analysis, then? You would agree that the limit and continuity are the ones which come to your mind. Whatever you studied later had made use of these building blocks. Now, instead of a set of real numbers, if we start with any arbitrary set of objects, how can we introduce limit and continuity on it? Obviously, it would have to have some extra properties. Now, if you go back and recall the definition of a limit, we say the limit of \( f(x) \) as \( x \) approaches \( a \) is \( l \), if \( f(x) \) gets arbitrarily close to \( l \) as \( x \) gets sufficiently close to \( a \). To be precise, \( \forall \epsilon > 0, \exists \delta > 0, \) s.t.

\[
0 < |x - a| < \delta \Rightarrow |f(x) - l| < \epsilon
\]

Here \( |x - a| \) and \( |f(x) - l| \) denote the distance between \( x \) and \( a \) and between \( f(x) \) and \( l \), respectively.

So, it seems that if we can introduce the notion of ‘distance’ on our arbitrary set of objects, we would be able to talk about limits in this new set up also.

**Definition 1:** Let \( X \) be a non-empty set. A **metric** \( d \) on \( X \) is a function \( d : X \times X \rightarrow \mathbb{R} \), such that, for \( x, y, z \in X \),

1) \( d(x, y) \geq 0 \) and \( d(x, y) = 0 \) if and only if \( x = y \),
2) \( d(x, y) = d(y, x) \),
3) \( d(x, y) \leq d(x, z) + d(z, y), x, y, z \in X \).

For \( x, y \in X \), \( d(x, y) \) is called the distance between \( x \) and \( y \).

The three properties 1, 2 and 3 listed in Definition 1 above mean that

- the distance between \( x \) and \( y \) is non-negative and it is equal to zero when and only when \( x \) and \( y \) coincide,
- the distance between \( x \) and \( y \) is the same as the distance between \( y \) and \( x \). This is the property of symmetry,
- the distance between \( x \) and \( y \) is not more than the distances between \( x \) and \( z \) and that between \( z \) and \( y \), added together. This is called the triangle inequality.

Take a minute and think. Aren’t these precisely the properties that we expect from ‘distance’, based on our previous knowledge? And where does our previous knowledge come from? From \( \mathbb{R} \) or \( \mathbb{R}^2 \) of course, and so, our first example of a metric space comes from \( \mathbb{R} \).
Example 1: Let us consider $\mathbb{R}$, the set of real numbers. Define a function $d$ on $\mathbb{R} \times \mathbb{R}$ by $d(x, y) = |x - y|$. Then $d$ has the three properties listed in Definition 1, and thus, is a metric on $\mathbb{R}$. We refer to $d$ as the usual metric or the standard metric or Euclidean metric and if we refer to $\mathbb{R}$ as a metric space, we assume that the metric is given by the usual metric $d$ unless otherwise specified. Note that the triangle inequality follows from the inequality $|x + y| \leq |x| + |y|$.

**Definition 2:** A pair $(X, d)$ where $X$ is a non-empty set and $d$ is a metric on $X$ is called a metric space.

Elements of $X$ are referred to as points of the metric space $(X, d)$. When there is no ambiguity about the metric on $X$, we may also denote the metric space simply by $X$.

Our next example identifies $\mathbb{C}$ also as a metric space. You need to recall the definition of the absolute value of a complex number.

Example 2: Consider the function $d : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$ such that $d(z_1, z_2) = |z_1 - z_2|$.

Now, since the absolute value function on $\mathbb{C}$ has the following properties,

i) $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$,

ii) $|-z| = |z|$.

It is easy to verify that $d$ satisfies conditions (1) and (2) in Definition 1. To show the triangular inequality $d(z_1, z_2) \leq d(z_1, z_3) + d(z_3, z_2)$, we use the inequality $|z + w| \leq |z| + |w|$ with $z = z_1 - z_3, w = z_3 - z_2$. You can verify this inequality. This shows that $d$ is a metric.

**Remark 1:** The above example shows that every set $X$ has at least one metric on it.

Example 3: Let $X$ be a non-empty set. We define

$$d : X \times X \rightarrow \mathbb{R}$$

as,

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y. \end{cases}$$

This means that the distance of a point from any other point is exactly 1. Are you convinced that this a metric? Try this by yourself and figure out why it is called a trivial metric (see E1). This metric is called the discrete metric.

**Remark 1:** The above example shows that every set $X$ has at least one metric on it.

Example 4: Let $X$ be $\mathbb{R}^2$. Then consider the function $d : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$
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where \( p_1 = (x_1, y_1) \) and \( p_2 = (x_2, y_2) \) represent any two elements in \( \mathbb{R}^2 \). You recall that the expression on the Right Hand Side of Eqn(1) represents the Euclidean distance between two points in \( \mathbb{R}^2 \). Therefore we can easily verify the condition (1), (2) and (3) in Definition 1. Note that the condition (3) follows from the property of triangles which states that the length of one side of a triangle is always less than or equal to the sum of the lengths of the other two sides (see Fig. 1). Thus the name triangle inequality is justified. This shows that the function \( d \) defined by Eqn.(1) is a metric on \( \mathbb{R}^2 \). The metric \( d \) is called the Euclidean metric or the usual metric on \( \mathbb{R}^2 \). Fig. 2 gives the geometrical picture of the distance measured by this metric.

If we connect \( \mathbb{R}^2 \) to \( \mathbb{C} \) by the map,

\[
(x, y) \rightarrow z = x + iy
\]

then from Example 2, we have

\[
d(z_1, z_2) = |z_1 - z_2| = |(x_1 - x_2) + i(y_1 - y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}
\]

where \( z_1 \) and \( z_2 \) are in \( \mathbb{C} \) represented by \( z_1 = x_1 + iy_1 \) and \( z_2 = x_2 + iy_2 \). This shows that the metrics given in Examples 2 and 4 are the same.

**Example 5:** Let us consider \( \mathbb{R}^2 \). Consider the function \( d_1 : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} \) defined by

\[
d_1(x, y) = |x_1 - x_2| + |y_1 - y_2|, \ x = (x_1, y_1), y = (x_2, y_2)
\]

We claim that \( d_1 \) is another metric on \( \mathbb{R}^2 \). Do you agree with us? Convince yourself by doing E2. The following figure gives the geometrical picture of the distance measured by this metric.
The metric defined by Eqn.(2) is called **Taxicab metric** on $\mathbb{R}^2$ because it models the distance a taxi-cab would have to traverse to get from one point to another if the cab was only allowed to move in cardinal directions (north, south, east, west) and not diagonally. You can see that the distance in this metric is at least as large as that in Euclidean metric.

Yet another metric on $\mathbb{R}^2$ is explained in the next example.

**Example 6**: Let us define another function $d_\infty : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$d_\infty(x, y) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

Then $d_\infty$ satisfies all the properties of metric. We leave this as an exercise for you to verify (see E2). Fig. 2 gives the geometrical picture of the distance.

If you closely look at the examples above, you will see that similar metrics can be defined for the higher dimensional spaces $\mathbb{R}^n$. In the next example we consider this.

**Example 7**: Let $\mathbb{R}^n$ denote the set of elements $x = (x_1, \ldots, x_n)$ where $x_i \in \mathbb{R}$ for $i = 1, \ldots, n$.

Now define $d, d_1, d_\infty$ on $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, as follows:
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For \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \)

\[
d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},
\]

(4)

\[
d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|
\]

\[
d_\infty(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}
\]

Then \( d, d_1 \) and \( d_\infty \) define metrics on \( \mathbb{R}^n \).

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The metric \( d \) is called the usual metric (or the Euclidean metric) and \( (\mathbb{R}^n, d) \) is called the Euclidean space. The metric \( d_1 \) is called the Taxicab metric on \( \mathbb{R}^n \).

**Remark 2:** To verify the triangle inequality for the metric \( d \) in (4) we need the following two inequalities which are useful in many other contexts also.

\[
\left| \sum_{i=1}^{n} x_i y_i \right| \leq \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2} \left( \sum_{j=1}^{n} y_j^2 \right)^{1/2}
\]

(5)

\[
\left( \sum_{i=1}^{n} (x_i + y_i)^2 \right)^{1/2} \leq \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2} + \left( \sum_{j=1}^{n} y_j^2 \right)^{1/2}
\]

(6)

where \( x_i, y_i, i = 1, \ldots, n, \) are real numbers.

The inequality in (5) is called Cauchy-Schwarz inequality and in the inequality in (6) is called Minkowski's inequality.

In the following example we consider a metric on the set of bounded real sequences.

**Example 8:** Let \( X \) be the set of all bounded real sequences \( x = (x_1, x_2, x_3 \ldots) \). Define a metric \( d_\infty \) on \( X \) by

\[
d_\infty((x_1, x_2, x_3, \ldots), (y_1, y_2, y_3, \ldots)) = \sup\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|, \ldots\}
\]

This space is denoted by \( \ell^{\infty}(\mathbb{R}) \). The verification that \( d_\infty \) is a metric is left as an exercise (see E3).

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Next we show an important inequality.

**Proposition 1:** In any metric space \( (X, d) \), we have

\[
|d(x, z) - d(y, z)| \leq d(x, y), x, y, z \in X.
\]

(7)

**Proof:** Let \( x, y, z \in X \). Since \( d(x, z) \leq d(x, y) + d(y, z) \) it follows that

\[
d(x, z) - d(y, z) \leq d(x, y)
\]

Interchanging \( x \) and \( y \) in this inequality, we see that

\[
d(y, z) - d(x, z) \leq d(y, x) = d(x, y)
\]
Thus we get
\[ d(x, z) - d(y, z) \geq -d(x, y) \]

Therefore
\[ -d(x, y) \leq d(x, z) - d(y, z) \leq d(x, y), \]

from which it follows that
\[ |d(x, z) - d(y, z)| \leq d(x, y). \]

We want you to try some exercises.

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**E1** Show that the function \( d \) in Example 3 is a metric.

**E2** Show that the functions \( d, d_1 \) and \( d_\infty \) given in Example 5, 6 and 7 satisfy all the three properties of a metric.

**E3** Show that the function \( d_\infty \) given in Example 8 is a metric.

**E4** Which of the following functions \( d : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are metrics on \( \mathbb{R} \)?
   i) \( d(x, y) = 5|x - y| \)
   ii) \( d(x, y) = x^2 + y^2 \)
   iii) \( d(x, y) = (x - y)^2 \)

**E5** Check whether the function \( d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) given by
\[ d(p_1, p_2) = |x_1 - x_2| |y_1 - y_2| \]
where \( p_1 = (x_1, y_1) \) and \( p_2 = (x_2, y_2) \) is a metric or not.

**E6** Let \((X, d)\) be a metric space. Show that the following functions give metrics on \( X \).
   i) \( D(x, y) = \frac{d(x, y)}{1 + d(x, y)} \)
   ii) \( \rho(x, y) = \min\{1, d(x, y)\} \)

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So far we have been considering sets whose elements are finite or infinite sequences of real or complex numbers. But quite often we need to consider sets whose elements are either functions or matrices. In the next few examples we consider such sets and discuss certain metrics on these sets.

**Example 9:** Let \( T = [a, b] \) and let \( B(T) \) denote the set of all real-valued bounded functions defined on \( T \). For \( f, g \in B(T) \), let
\[ d_\infty(f, g) = \sup_{x \in T} \{|f(x) - g(x)|\} \]

Note that \( \{ |f(x) - g(x)| : x \in T \} \) is a non-empty set of non-negative real numbers which is bounded above. Therefore the set has its supremum in \( \mathbb{R} \). Thus \( d_\infty \) is a function from \( B(T) \times B(T) \to \mathbb{R} \). We can verify that \( d_\infty \) satisfies all the conditions 1, 2 and 3 of Definition 1. Therefore \( d_\infty \) defines a metric on \( B(T) \).
B(T) with the metric $d_\infty$ is called the **metric space of bounded real functions on** $T$.

**Note:** A metric space in which the elements are functions is called a **function space**. Thus $B(T)$ is a function space.

In the next example we consider another function space.

**Example 10:** For $T = [a, b]$, let $C(T)$ denote the set of all real-valued functions on $T$ which are continuous. From your undergraduate Real Analysis course you know that a continuous function defined on a closed and bounded interval is bounded. Therefore each functions in $C(T)$ is bounded; that means $C(T)$ is a subset of $B(T)$.

From the algebra of continuous functions you know if $f, g \in C(T)$, then $f + g \in C(T)$ and $(-f) \in C(T)$. We can consider the function $d_\infty$ as in Example 10 for $C(T)$ as well. Accordingly, for $f, g \in C(T)$, we have

$$d_\infty(f, g) = \sup_{t \in [a, b]} \{|f(t) - g(t)|\}$$

Hence $(C(T), d_\infty)$ is a metric space. This metric space is usually denoted by $C(T)$ only. It is called the **metric space of continuous functions**.

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**Remark 3:** In the two examples above the only difficulty is to verify the triangle inequality. For that you only need to show the following inequality for $f, g, h \in C(T)$ or $B(T)$.

$$\sup_{t \in [a, b]} |f(t) - g(t)| \leq \sup_{t \in [a, b]} |f(t) - h(t)| + \sup_{t \in [a, b]} |h(t) - g(t)|$$

The following figure illustrates the geometrical meaning of the distance defined by $d_\infty$ for $B(T)$ or $C(T)$.

![Fig. 5](image)

**Example 11:** Let $X = C[0, 1]$. For $f, g \in C[0, 1]$ define $d : X \times X \to \mathbb{R}$ by

$$d_1(f, g) = \int_0^1 |f(t) - g(t)|\,dt \quad (8)$$

where the integral is the Riemann integral. We first of all note that the integral on the R.H.S (8) exists since $|(f - g)|$ is a continuous function. We claim that $d_1$ is a metric. All conditions except the condition (1) follows from the properties
of Riemann integral. The condition (1) follows from a result which we state below.

**Proposition 2:** Let \( f : [0, 1] \to \mathbb{R} \) be continuous with \( f(t) \geq 0 \) for \( t \in [0, 1] \). Then \( \int_0^1 f(t) \, dt = 0 \) if and only if \( f(t) = 0 \) for all \( t \in [0, 1] \).

You might be already familiar with this result from your undergraduate Real Analysis course.

The figure below illustrates the geometric meaning of \( d_1(f, g) \).

![Graph of f and g](image)

\( d_1(f, g) \) is the shaded area

**Fig. 6**

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**Remark 4:** Note that if we assume that \( f \) is non-negative, Riemann integrable on \([0, 1]\) and such that \( \int_0^1 f(t) \, dt = 0 \), not necessarily continuous then we cannot conclude that \( f = 0 \) on \([0, 1]\). For instance consider the function \( g(t) = 0 \) if \( t \neq \frac{1}{2} \) and \( g \left( \frac{1}{2} \right) = 0 \). Then \( \int_0^1 g(t) \, dt = 0 \), whereas \( g \) is non-zero. So, the extra condition of continuity is put to avoid this situation.

**Remark 5:** In the example above, we have considered sets of functions on the closed interval, \([0, 1]\). There is no special status attached to \([0, 1]\). We could take any bounded closed interval in its place. We could also try to define a similar metric for larger spaces of functions (see E6).

**Example 12:** Let \( M(n, \mathbb{R}) \) denote the set of all \( n \times n \) real matrices. We identify a matrix \( A = (a_{ij})_{i,j=1}^n \) in \( M(n, \mathbb{R}) \) with the element

\[
(a_{11}, a_{12}, \ldots, a_{1n}, \ldots, a_{n1}, \ldots, a_{nn}) \in \mathbb{R}^{n^2}
\]

Then the Euclidean metric \( d \) on \( \mathbb{R}^{n^2} \) defines a metric on \( M(n, \mathbb{R}) \) which makes \( M(n, \mathbb{R}) \) a metric space.

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Try this exercise now

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**E7** Let \( X = \mathbb{R}[a, b] \) denote the set of all Riemann integrable functions on \([a, b]\). Define a function \( d \) on \( X \times X \to \mathbb{R} \) by

\[
d(f, g) = \int_a^b |f(t) - g(t)| \, dt
\]
Definition 3: Let \((X, d)\) be a metric space and let \(A \subset X\) be a non-empty subset of \(X\). For \(x, y \in A\), define \(\delta(x, y) = d(x, y)\). Then \(\delta\) is a metric on \(A\) and it is called the induced metric on the subset \(A\).

Note: We adopt the convention that if a subset \(A\) of metric space \(X\) is referred to as a metric space, then it is assumed that the metric on \(A\) is the one induced by the metric on \(X\), unless otherwise stated.

Definition 4: (Product Metric): Let \((X_1, d_1)\) and \((X_2, d_2)\) be two metric spaces. Let \(X = X_1 \times X_2\), i.e. the Cartesian product of \(X_1\) and \(X_2\). Then define a function \(d\) on \(X \times X\) by

\[
d((x_1, x_2), (y_1, y_2)) := \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}
\]

where \(x_1, y_1 \in X_1\) and \(x_2, y_2 \in X_2\).

Then \(d\) defines a metric on the product set \(X = X_1 \times X_2\). We refer to this metric as the product metric.

Example 13: Let \(X = S_1 \times [0, 1]\) where \(S_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}\). Then \(X = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 = 1, 0 \leq x_3 \leq 1\}\) is a subset of \(\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3\).

You can note that both \(S_1\) and \([0, 1]\) are metric spaces with Euclidean metric induced from \(\mathbb{R}^2\) and \(\mathbb{R}\) respectively. Then the product metric defined on \(X\) is the metric defined by

\[
d((x_1, x_2, x_3), (y_1, y_2, y_3)) = \max\left\{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, |x_3 - y_3|\right\}
\]

Can you think of other metrics on \(X_1 \times X_2\) coming from the original metrics on \(X_1\) and \(X_2\)? What about the \(D\) and \(\rho\) given by

\[
D(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2) \\
\rho(x, y) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}
\]

for \(x = (x_1, x_2), y = (y_1, y_2)\).

Here is an exercise for you.

E8) Let \((X_1, d_1)\) and \((X_2, d_2)\) be two discrete metric spaces. Then verify that the product metric on \(X_1 \times X_2\) is discrete.

We hope that you have understood the new concept of distance. In what follows, we shall try to see what other new concepts can be developed in this context.

1.3 OPEN SETS

In this section we study certain subsets of metric spaces called open sets. We
first introduce two related terms: ‘the distance of a point from a set’, and ‘the
diameter of a set’.

Let us consider a metric space \((X, d)\). Let \(A\) be a non-empty subset of \(X\). Then
for any \(x \in X\), consider the set \(A_x = \{d(x, y) : y \in A\}\). Then what can we say
about the set \(A_x\)?

From the condition 2 in the definition of a metric, we have \(d(x, y) \geq 0\) for all
\(x, y \in X\). That means \(A_x\) is a non-empty set of non-negative real numbers which
is bounded below by 0. Then this set has a greatest lower bound (infimum).

**Definition 5:** Let \((X, d)\) be a metric space. If \(x \in X\) and \(A \subset X, A \neq \emptyset\), then
the **distance of \(x\) from \(A\)**, \(d(x, A)\) is taken as the greatest lower bound of the
set \(A_x\) that is,

\[d(x, A) = \inf\{d(x, y) : y \in A\}\]

**Remark 6:** If \(x \in A\), then clearly, \(0 \in \{d(x, y) : y \in A\}\), and hence \(d(x, A) = 0\).

Let us see an example.

**Example 14:** Let us consider the function space \(C\left[0, \frac{\pi}{2}\right]\). Let
\(g(t) = 0, 0 \leq t \leq \frac{\pi}{2}\) and \(f_n(t) = \sin nt, 0 \leq t \leq \frac{\pi}{2}\). Let \(A = \{f_n : n \in \mathbb{N}\}\). Then
you can observe that \(d(g, A) = 1\). This follows from the fact that \(-1 \leq f_n(t) \leq 1\)
for all \(n \in \mathbb{N}\) and \(t \in \left[0, \frac{\pi}{2}\right]\) and also that for any \(n\), there exists some \(t,\)
\(0 \leq t \leq \frac{\pi}{2}\), such that \(f_n(t) = 1\).

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For any non-empty subset \(A\) of a metric space \((X, d)\), we consider another set
\(D(A)\) given by

\[D(A) = \{d(a_1, a_2) : a_1, a_2 \in A\}\]

Then \(D(A)\) is a non-empty set of non-negative real numbers and therefore has a
supremum (or least upper bound) in the extended real number system. The
supremum of this set is called the **diameter of \(A\)**.

**Definition 6:** Let \((X, d)\) be a metric space. The **diameter** of a non-empty
subset \(A\) of \(X\) which is denoted by \(d(A)\) is defined as the supremum of the set
\(D(A)\). That is,

\[d(A) = \sup \{d(a_1, a_2) : a_1, a_2 \in A\}\]

**Remark 7:** Note that \(d(A) \geq 0\) since \(D(A)\) is non-empty and consists of
non-negative numbers.

We have observed here that if \(A \neq \emptyset\), then \(d(A) \geq 0\). The diameter of a set may
not be a finite number. It is possible that \(d(A) = \infty\). You can easily see that for
the set \(Z\) in \(\mathbb{R}\) with the usual metric, \(d(Z) = \infty\).

So, what are the possible value of the diameter of a non-empty subset of \(X\)? It
could be a non-negative real number or it could be \(+\infty\).

If the diameter of a set is not infinity, it is called a **‘bounded set’**.
Note that any non-empty subset of a bounded set is bounded. Also a non-empty finite set is bounded. It is for these reasons that we term the empty set $\phi$ also to be a bounded set even though
\[
\sup\{d(x, y) : x, y \in \phi\} = \sup \phi = -\infty.
\]
There are metric spaces where any set is bounded. Can you give an example? Try these exercises now.

E9) If $(X, d)$ is a metric space with the metric $\rho$ given by
\[
\rho(x, y) = \min(1, d(x, y))
\]
then any non-empty subset of $X$ is bounded.

E10) Find $d(A)$ in the context of $\mathbb{R}$ with the standard metric when
i) $A = (0, 2)$
ii) $A$ is the set of rationals in $[1, 10]$.

E11) Give an example of a set $A$ together with an $x \notin A$ in a metric space $(X, d)$ satisfying $d(x, A) = 0$

E12) Give an example of an unbounded set in $\mathbb{R}^2$.

Now we shall discuss an important class of sets in a metric space $(X, d)$.

You are already familiar with the concept of a ball in Euclidean space $\mathbb{R}^n$. Let us for example, consider $\mathbb{R}^3$. Then the open ball of radius $r$ with centre $a = (a_1, a_2, a_3)$ is the set $B(a, r)$ given by
\[
B(a, r) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 < r^2\}
\]
Now we extend this definition to a metric space $(X, d)$.

**Definition 7:** Let $(X, d)$ be a metric space. Let $x \in X$ and $r > 0$. The subsets

\[
B_d(x, r) := \{y \in X : d(x, y) < r\}
\]
and
\[
B_d[x, r] := \{y \in X : d(x, y) \leq r\}
\]
are respectively called the open and closed balls centred at $x$ with radius $r$ with respect to the metric $d$. We use the notation $B_d(x, r)$ or $B_d[x, r]$ only when we want to emphasize that the metric under consideration is $d$. If no confusion arises, we denote $B_d(x, r)$ by $B(x, r)$ and $B_d[x, s]$ by $B[x, s]$.

**Example 15:** Consider $X = \mathbb{R}$ with the standard metric. Then we note that $B(x, r) = (x - r, x + r)$.

***

**Example 16:** Let $X = \mathbb{R}^2$ with the standard metric $d$. Then
\[
B(0, r) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < r^2\}
\]
This is because $p := (x_1, x_2) \in B(0, r)$ iff $d(p, 0) < r$ iff $d(p, 0)^2 < r^2$ iff $x_1^2 + x_2^2 < r^2$. More generally, if $q = (a_1, a_2)$, then $B(q, r)$ is the set of points inside the circle of radius $r$ with centre at $(a_1, a_2)$. 


If we look at the shape of the unit balls defined by the three metrics on \( \mathbb{R}^2 \) considered in Examples 4, 5 and 6, we will be able to find the difference between the metrics.

Note that in \( \mathbb{R}^2 \), the unit balls with respect to the three metrics are given

\[
\begin{align*}
B(0,1) &= \{ x \in \mathbb{R}^2 : d(0, x) \leq 1 \} \\
B_1(0,1) &= \{ x \in \mathbb{R}^2 : d_1(0, x) \leq 1 \} \\
B_\infty(0,1) &= \{ x \in \mathbb{R}^2 : d_\infty(0, x) \leq 1 \}
\end{align*}
\]

The following figure illustrates the balls.

![Figure 7](image)

**Example 17:** Consider \( C[0,1] \), the set of continuous real valued functions on \( [0, 1] \), with supremum metric,

\[
d(f, g) = \sup \{|f(x) - g(x)|, x \in [0, 1], f, g \in C[0,1] \}
\]

How will you visualize \( B(f, \alpha) \) for an \( f \in C[0,1] \) and \( \alpha > 0 \)? For instance, let us consider \( B(0, \alpha) \). Note that here 0 denotes the function which is identically 0 on \( [0, 1] \).

We first note that an element \( g \in C[0,1] \) lies in \( B(0, \alpha) \) if and only if (or iff) its graph lies in the region bounded by the lines \( y = \pm \alpha \) and \( x = 0 \) and \( x = 1 \) excluding the horizontal lines \( y = \pm \alpha \). Similarly, \( g \) lies in \( B(f, \alpha) \) iff its graph lies in the region bounded by the curves \( y = f \pm \alpha \) and \( x = 0 \) and \( x = 1 \) excluding the curves \( y = f \pm \alpha \) (see Fig. 8).

![Figure 8](image)
Example 18: Consider the $X = C[0, 1]$, with the metric $d$ defined by Riemann integral

$$d(f, g) = \int_0^1 |f(t) - g(t)| dt, \ f, g \in C[0, 1].$$

How do we visualize open balls in this situation? For instance, let us consider $B(0, 1)$.

We shall look at the geometric meaning of the integral illustrated by Fig.9. Then, $g \in B(0, 1)$ iff the area 'under the graph' of $|g|$ is less than 1. So, we can say that the function $g$ given in Fig. 9(a) lies in $B(0,1)$ whereas $\phi$ in Fig. 9(b) does not.

Proposition 3: Let $(X, d)$ be a metric space. Then given two distinct points $x, y \in X$, there exists $r > 0$ such that $B(x, r) \cap B(y, r) = \phi$ (See Fig.10). (This property is called (Hausdorff Property)).

Proof: Since $x$ and $y$ are distinct $d(x, y) > 0$. Let $0 \leq r \leq \frac{1}{2}d(x, y)$.

Let $V = B(x, r)$ and $W = B(y, r)$ (see Fig. 10). Then $V$ and $W$ are open in $X$ and $x \in V, y \in W$. Also $V \cap W = \phi$. Hence the result. $\square$
Try these exercises now.

E13) Find $B\left[2, \frac{1}{2}\right], B(2, 3)$ in $(\mathbb{R}, \rho)$ where $\rho$ is the metric given by

$$\rho(x, y) = \min\{1, |x - y|\}.$$ 

E14) Let $(X, d)$ be a metric space with $X \neq \emptyset$ and, $x \in X$ and $0 < r < s$.

Show that $B[x, r] \subset B[x, s]$. Give examples to show that it is possible to have

i) $B[x, r] = B[x, s]$

ii) $B[x, r] \supsetneq B[x, s]$

We now define open sets.

**Definition 8:** A subset $U \subset X$ of a metric space $(X, d)$ is said to be d-open if for each $x \in U$, there exists $r > 0$ such that $B(x, r) \subset U$. See Fig. 11.

If there is no confusion about which metric is being used, we shall simply refer to as an open set rather than as a d-open set.

Thus, we observe that to show that a set is open, we have to show that every point in it is the centre of an open ball fully contained in the set.

Since the empty set $\emptyset$ does not contain any point, we term the empty set as an open set.

Now if $x \in X, B(x, r) \subset X$ for any $r > 0$. Hence $X$ is open.

This means that in any metric space $(X, d)$, $\emptyset$ and $X$ are open sets.

What are the other open sets in a metric space $(X, d)$? As the name suggests, we expect that each open ball is an open set. How do we prove this? The next proposition shows this.

**Proposition 4:** In any metric space $(X, d)$, each open ball is an open set.

**Proof:** Consider the open ball $B(x_0, r)$ in $X$, and let $x \in B(x_0, r)$. Then $d(x, x_0) = r_1 < r$. 

---

*Fig. 11*
Now take \( r_s = r - r_1 \). Then \( r_s \) is a positive real number.

Now let \( y \in B(x, r_s) \). Then \( d(y, x) < r_s = r - r_1 \).

\[
\therefore d(y, x_0) \leq d(y, x) + d(x, x_0) < (r - r_1) + r_1 = r.
\]

So \( y \in B(x_0, r) \). Thus we have shown that \( B(x_0, r_s) \subset B(x_0, r) \) (See Fig.12). So we have proved that around every point of \( B(x_0, r) \) we have an open ball, fully contained in \( B(x_0, r) \). This implies that \( B(x_0, r) \) is open.

Now we shall prove a theorem which brings out the strong connection between open sets and open balls.

**Theorem 1:** Let \( X \) be a metric space. A subset of \( X \) is open if and only if it is a union of open balls.

**Proof:** Suppose \( A \) is a subset of \( X \), which is the union of a class \( \mathcal{U} \) of open balls of \( X \). Now if \( \mathcal{U} \) is empty, then \( A = \phi \), and is thus open.

Next, if \( \mathcal{U} \) is non-empty, then \( A \) is also non-empty.

Let \( a \in A \). Then \( a \) belongs to some open ball \( B(x_0, r) \) in \( \mathcal{U} \). As in the proof of the proposition above we can choose \( r_s \) s.t.

\[
B(a, r_s) \subset B(x_0, r) \subset A
\]

Thus, around every point of \( A \) we can find an open ball fully contained in \( A \). This shows that \( A \) is open.

On the other hand, suppose \( A \subset X \), and \( A \) is open. If \( A \) is empty, then it is the union of the empty class of open balls. If \( A \) is non-empty, then for every \( a \in A \) there is an open ball \( B(a, r_a) \) which is fully contained in \( A \). That is,

\[
a \in B(a, r_a) \subset A.
\]

So \( A = \bigcup_{x \in A} B(x, r_x) \) Let \( \mathcal{U} \) be the class of all such open balls \( B(x, r_x), x \in A \).

Then \( A = \bigcup_{x \in A} x \subset \bigcup_{x \in A} B(x, r_x) \subset A \). Thus \( A \) is the union of open balls in \( \mathcal{U} \).

Next we shall consider an interesting aspect of openness. Let us consider \( X = [0, 1] \) with the standard metric. In this metric space \([0, 1]\) is an 'open' set being the whole set. But we know that the set \([0, 1]\) as a subset of \( \mathbb{R} \) is not open in \( \mathbb{R} \). Do not let this confuse you. Note that when we consider \([0, 1]\) as a metric space, our attention is focussed only on the points in \([0, 1]\) and thus we do not take into consideration the other points of \( \mathbb{R} \). Using a similar argument, we get that \( B(0, 1/2) \) in this space is \([0, 1/2] \)(and not)\(-1/2, 1/2\) as in \( \mathbb{R} \). In fact,

\[
B(0, r) = X, \quad \text{if } r > 1,
\]

\[
= [0, r], \quad \text{if } r \leq 1
\]

This means that the openness of a set is a relative concept. It depends on the metric space to which it belongs. We further note that if \( r > 1 \) then \( D(B(0, r)) = 1 < 2r \) and if \( 0 < r \leq 1 \) then \( D(B(0, r)) = r < 2r \). This means that the diameter of a set and diameter of a ball in the usual sense are not the same.
Next we shall see how open balls w.r.t. to the induced metric differ from the open balls in the original space. Let $A$ be a nonempty subset of a metric space $(X, d)$. Let us continue to denote by the same letter $d$ the induced metric on $A$. Let $B_A(x, r)$ denote the open ball in the metric space $(A, d)$ with centre $a$ and radius $r$. Then $B_A(x, r) = B(x, r) \cap A$, where $B(x, r)$ is the open ball in $X$ centred at $x$ with radius $r$.

**Example 19:** Let $X$ be a discrete space. Then for $x \in X$, $\{x\} = B\left(x, \frac{1}{2}\right)$ and therefore is open. By the above theorem every subset is open.

***

Before proceeding further we want to make an important point.

Let us consider $\mathbb{R}$ with the standard metric. Then, it is easy to check that $\{x\}$ is not an open set in this metric space. Whereas if we consider $\mathbb{R}$ with the discrete metric, then $\{x\}$ is open.

Once we have understood what an open set is, we are interested in the collection of open sets. More precisely we would like to know whether the union or intersection of open sets are open or not. Our next theorem explains this.

**Theorem 2:** In a metric space $X$, the following are true
i) An arbitrary union of open sets is open, and
ii) a finite intersection of open sets is open.

**Proof:** Let $[A_i]_{i \in I}$ be a class of open sets in $X$ where $I$ denotes an index set and put $A = \bigcup_{i \in I} A_i$ and $B = \bigcap_{i \in I} A_i$

i) If $A = \emptyset$ then it is open. Suppose $A \neq \emptyset$. Let $x \in A$. Then there exist some $i_0 \in I$ such that $x \in A_{i_0}$. Since $A_{i_0}$ is open, there exists a $\delta > 0$ such that $B(x, \delta) \subseteq A_{i_0} \subseteq A$

This shows that $A$ is open.

ii) If $I = \emptyset$, then $B = X$, which is open. Also if $B = \emptyset$, then it is open.

Now suppose $I \neq \emptyset$ and $B \neq \emptyset$. We may take $i = 1, \ldots, n$. Let $x \in B$. Then $x \in A_i$ for each $i$. Since each $A_i$ is open, we can find $r_i > 0$, such that $B(x, r_i) \subset A_i$.

Let $r = \min\{r_1, r_2, r_3, \ldots, r_n\}$. Then $r > 0$

Further, $B(x, r) \subset B(x, r_i) \subset A_i$ for each $i = 1, \ldots, n$.

Therefore $B(x, r) \subset \bigcap_{i = 1}^n A_i = B$.

So, around every element of $B$ we can find an open ball fully contained in $B$. Thus, $B$ is open.

According to the theorem above, union of any arbitrary class (finite or infinite) of open sets is open. But for intersections, the class is taken to be finite. In fact, you will see in the example below that an infinite intersection of open sets need not be open.

**Example 20:** Consider $\mathbb{R}$ with the standard metric. Let us consider the class

$\mathcal{U} = \left\{-\frac{1}{n}, \frac{1}{n} \middle| n \in \mathbb{N}\right\}$ of open sets in $\mathbb{R}$. Then $\left[-\frac{1}{n}, \frac{1}{n}\right]$ is open for all $n \in \mathbb{N}$.
But \( \bigcap_{n \in \mathbb{N}} \left[ -1/n, 1/n \right] = \{0\} \) and is not open in \( \mathbb{R} \), since any \( B(0, r) = \left] -r, r \right[ , r > 0 \), will contain \( r/2 \), and so \( B(0, r) \not\subset \{0\} \).

Next we shall see another interesting feature of an open set.

**Observe that an interval \((a, b)\) is open if and only if all its points are interior points.** You can observe the same feature for discs in the plane also. How do we express this for metric spaces? Let us see.

**Definition 9:** Let \( X \) be a metric space and \( A \) be a subset of \( X \). A point \( x \in A \) is called an interior point if there exists an open ball \( B(x, r) \), fully contained in \( A \).

The set of all interior points of \( A \) is called the 'Interior of \( A \)', and is denoted by \( \text{Int}(A) \). Thus,

\[
\text{Int}(A) = \{ x \in A : B(x, r) \subset A \text{ for some } r > 0 \}.
\]

We can prove that \( \text{Int}(A) \) is clearly an open subset of \( A \) (see E16). Infact \( \text{Int} A \) is the largest open set in \( X \) contained in \( A \). How do you show this? For that we have to show that \( \text{Int} A \) contains all open subsets of \( X \) that are contained in \( A \).

To see that, let us take an open subset \( A_0 \) of \( X \) contained in \( A \). Then we have to show that \( A_0 \subset \text{Int}(A) \). Let \( x \in A_0 \). Since \( A_0 \) is open, there is an open ball \( B(x, r) \), such that

\[
x \in B(x, r) \subset A_0 \subset A.
\]

This means that \( x \) is an interior point of \( A \), i.e., \( x \in \text{Int}(A) \). So, \( A_0 \subset \text{Int}(A) \).

This shows that \( \text{Int} A \) is the largest open set.

This leads us to the following result.

**Proposition 5:** A set \( A \) in a metric space \((X, d)\) is open if and only if \( A = \text{Int} A \).

Next we shall restate the Hausdroff in term of open sets rather than open ball.

**Remark 8:** If \( x \) and \( y \) are distinct points of a metric space \( X \), then there exists disjoint open sets \( V \) and \( W \) in \( X \) such that \( x \in V \) and \( y \in W \).

So far we have been discussing open sets in a metric space. As we stated earlier, the concept of openness changes from one metric to another. In this connection we introduce you to another concept called equivalence of metrics.

**Definition 10:** Two metrics \( d_1 \) and \( d_2 \) on a set \( X \) are called equivalent if a set \( A \) is open in \( X \) w.r.t. \( d_1 \) if and only if it is open w.r.t. \( d_2 \).

Before considering examples we shall give a characterisation of the concept of equivalence of metrics.

**Theorem 3:** Two metrics \( d_1 \) and \( d_2 \) are equivalent if and only if given any \( x \in X \) and \( r > 0 \), there exist positive real numbers \( r_1 \) and \( r_2 \) such that

\[
B_{d_2}(x, r_1) \subset B_{d_1}(x, r), B_{d_1}(x, r_2) \subset B_{d_2}(x, r)
\]
The proof of this follows from the definition of open sets.

Let us see an example.

**Example 21:** Let us check whether the Euclidean metric $d$ on $\mathbb{R}$ and another metric $d_1$ defined on $\mathbb{R}$ by

$$d_1(x, y) = \min\{1, d(x, y)\}$$

are equivalent or not.

Let us take an $x \in \mathbb{R}$ and $r > 0$. Let $r_1 = \min\{1, r\}$.

Since $d_1(x, y) \leq d(x, y)$ for all $y \in \mathbb{R}$, we get that $B_d(x, r_1) \subseteq B_{d_1}(x, r_1)$.

Now let $y \in B_{d_1}(x, r_1)$. Then $d_1(x, y) < r_1$. So $d_1(x, y) < 1$. Thus $d_1(x, y) = d(x, y)$ and therefore $d(x, y) < t_1 \leq r$. This shows that $y \in B_d(x, r)$

$$\therefore B_{d_1}(x, r_1) \subseteq B_d(x, r).$$

Hence the two metrics are equivalent.

***

Try some exercises now.

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**E15** Let $(X, d)$ be a metric space and let $a \in X$ and $r > 0$. Can $B[a, r]$ be an open set? Justify your answer.

**E16** Show that $\text{Int } A$ is an open set.

**E17** Let $A_1$ and $A_2$ be open sets in metric spaces $(X_1, d_1)$ and $(X_2, d_2)$ respectively. Show that $A_1 \times A_2$ is open in the metric space $(X_1 \times X_2, d)$ where $d$ is the product metric defined in Section 1.2. Use this result to show that a rectangle $]a, b[ \times ]c, d[$ is open in $\mathbb{R} \times \mathbb{R}$.

**E18** Show that $d, D, \rho$ defined on the product space $X = X_1 \times X_2$ (see Section 1.2) are equivalent metrics on $X$.

In the next section we shall consider sets which are complements of open sets.
1.4 CLOSED SETS

A notion which is related to "openness" is "closedness". Here we shall define closed sets in metric spaces.

Before defining closed sets, we talk about 'limit points' of a set. We have the following definition.

**Definition 11:** Let \( A \) be a subset of a non-empty metric space \( X \). A point \( x \in X \) is called a **limit point** of \( A \) if every open ball centred at \( x \) contains a point of \( A \) other than \( x \). Limits points are also called "cluster points" or "accumulation points"

**Example 22:** Let \( A = [0, 1] \subset \mathbb{R} \). Then 1 is a limit point of \( A \), that is, every open interval around 1 contains points of \( A \), apart from 1. In fact, you will realize that every point of \( [0, 1] \) is its limit point.

\(* * * *

**Example 23:** Consider the set \( F = \{1, 1/2, 1/3, \ldots\} \) in \( \mathbb{R} \). Then 0 is a limit point of this set. In fact, 0 is its only limit point and it does not belong to \( F \). No other points of \( F \) are its limit point. For example, take \( 1/5 \in F \). The points of \( F \) close to \( 1/5 \) on right and left sides are \( 1/4 \) and \( 1/6 \) respectively.

Then \( 1/4 - 1/5 = \frac{1}{20} \) and \( 1/5 - 1/6 = \frac{1}{30} \).

Now, \( B \left( \frac{1}{5}, \frac{1}{30} \right) = \left( \frac{1}{5} - \frac{1}{30}, \frac{1}{5} + \frac{1}{30} \right) \) is an open interval around \( 1/5 \)
and it does not contain any point of \( F \) other than \( 1/5 \). So \( 1/5 \) cannot be a limit point of \( F \).

\(* * * *

**Example 24:** Let us consider \( B = [3, 4] \). All points of \( B \) are its limit points. In addition, 4 is also a limit point of \( B \).

\(* * * *

**Example 25:** Let \( X = \mathbb{R}^2 \) with the standard metric.

Let \( A \) be the union of \( x \) and \( y \)-axes; i.e. \( A = \{(x, y) \in \mathbb{R}^2 : xy = 0\} \). Then all the points of \( A \) are its limit points.

Let \( B \) be the set given by \( B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x \neq 0, y \neq 0\} \). Then \( (0, 0) \) is a limit of \( B \) which do not lie in \( B \).

\(* * * *

From these examples, we see that a point in a set may or may not be its limit point. On the other hand, a limit point of a set may or may not belong to it.

We now define closed sets.

**Definition 12:** A subset \( A \) of a metric space \( X \) is said to be a **closed set** if it contains all its limit points.

From the examples we discussed earlier, we get that \([1, 2]\) is a closed set in \( \mathbb{R} \). But \( \{1, 1/2, 1/3, \ldots\} \) and \([3, 4]\) are not closed in \( \mathbb{R} \). What about the set
Consider a discrete metric space $X$. In this space, any open ball with radius less than or equal to $1$ contains only its centre. It cannot contain any other point of $X$. This means no subset of $X$ can have a limit point. So, a discrete space is a metric space in which every subset is a closed set. When we compare this with $E(17)$, in which you have been asked to show that every subset of a discrete space $X$ is open, we get that in a discrete space every set is both open and closed in a discrete space.

Here we want to ask a question. What are the sets which are both open and closed in $\mathbb{R}$?

You will find an answer to this in Unit 4.

Unbelievable in the real sense, isn't it? This is what makes a set different from real objects like “Doors”!

**Theorem 4:** A subset $A$ of a metric space is closed, if and only if its complement $A^c$ is open.

**Proof:** Suppose $A$ is closed. This means it contains all its limit points. That is, no point of $A^c$ can be its limit point. Thus, if $x \in A^c$, then we can find an open ball $B(x, r)$ which does not contain any point of $A$, which means that it is fully contained in $A^c$. This proves that $A^c$ is open.

Now suppose $A^c$ is open. Let $x$ be a limit point of $A$ such that $x \notin A$. Then $x \in A^c$ and since $A^c$ is open, we can find an open ball $B(x, r)$ contained in $A^c$. That means $B(x, r)$ does not contain any point of $A$. This is not possible since $x$ is a limit of $A$. Thus all the limit points of $A$ lie in $A$. Hence $A$ is closed.

**Example 26:** Let $A = \{(x_1, x_2, 0) : (x_1, x_2) \in \mathbb{R}^2\}$. Then $A$ is closed in $\mathbb{R}^3$.

Let $x = (x_1, x_2, 0)$ where $(x_1, x_2) \in \mathbb{R}^2$. Then $x \in A$. For $r > 0$, put $y = \left(\frac{x_1}{2}, x_2, \frac{r}{2}\right)$. Then $y \in B(x, r)$. But $y \notin A$. So $B(x, r) \nsubseteq A$. Thus $x \notin \text{Int} A$ and therefore $A$ is not open. Hence $A$ is closed.

**Proposition 6:** Let $(X, d)$ be a metric space. Every closed ball $B[x, r]$ in $X$ is closed in $X$.

**Proof:** Suppose $y$ is a limit point of $B[x, r]$, and that $d(y, x) = r_1$. Let, if possible, $r_1 > r$. This means $r_2 = r_1 - r > 0$. Now consider $B(y, r_2)$. This open ball around $y$ must contain a point, say $z$, of $B[x, r]$. Then

\[
d(x, z) \leq r \quad \text{and} \quad d(y, z) < r_2.
\]

Hence, 
\[
r_1 = d(y, x) \leq d(y, z) + d(x, z) < r_2 + r = (r_1 - r) + r = r_1.
\]

This is a contradiction. Hence our assumption that $r_1 > r$ is not possible. Therefore $r_1 \leq r$, and hence $y \in B[x, r]$. That is $B[x, r]$ contains all its limit points. Therefore $B[x, r]$ is a closed set in $X$. \qed
**Theorem 5:** In a metric space $X$, the following results hold:

i) Any arbitrary intersection of closed sets is closed, and

ii) A finite union of closed sets is closed.

**Proof:** Let $\{A_i\}_{i \in I}$ be a collection of closed sets. Then $\{A_i^c\}_{i \in I}$ is collection of open sets. Thus $\bigcup_{i \in I} A_i^c$ is open and therefore by De Morgan's law,

$$\left( \bigcup_{i \in I} A_i^c \right)^c = \bigcap_{i \in I} A_i$$

is closed. Hence the result. We leave the proof of the second part of this theorem as an exercise for you to verify (Refer E19).

We now define a related concept.

**Definition 13:** Let $A$ be a subset of a metric space $X$. The union of $A$ with the set of its limit points is called the closure of $A$ and is denoted by $\overline{A}$.

Obviously, if $A$ is closed, then $A = \overline{A}$. In general, $A \subset \overline{A}$.

**Theorem 6:** If $A$ is a subset of a metric space $X$, then $\overline{A}$ is closed in $X$.

**Proof:** Suppose $x$ is a limit point of $\overline{A}$. If $x$ is in $A$, then $x \in A$. Let us assume that $x \notin A$. Since $x$ is a limit point of $\overline{A}$, given $r > 0$, the open ball $B(x, r)$ contains an element $y$ of $\overline{A}$ other than $x$.

Let $d(x, y) = r_1$ and let $r_2 = r - r_1$. Then $r_2 > 0$. Since $y \in \overline{A}$, the open ball $B(y, r_2)$ contains an element $z$ of $A$ other than $y$. Then

$$d(x, z) \leq d(x, y) + d(y, z)$$

$$< r_1 + r - r_1$$

$$= r.$$

This shows that $z \in B(x, r)$. Also $z \neq x$ simply because $x \notin A$. Thus we got that $x$ is a limit point of $A$ and therefore $x \in \overline{A}$.

Since $\overline{A}$ contains all its limit points, $\overline{A}$ is closed. Hence the result.

**Remark:** A set is $A$ closed, if and only if $A = \overline{A}$.

Using the concept of closure of a set in a metric space, we define another concept.

**Definition 14:** Let $A$ be a subset of a metric space $X$. If a point $x$ is such that every open ball around $x$ contains a point of $A$ and a point of $A^c$, then $x$ is called a boundary point of $A$. The set of all boundary points of $A$ is called the boundary of $A$ and is denoted by $\text{bdry}(A)$.

**Remark:** It follows immediately from the definition that $\text{bdry}(A) = \text{bdry}(A^c)$.

For example, the boundary of $[a, b]$ in $\mathbb{R}$ consists of the points $a$ and $b$. The boundary of $B(x, r)$ in $\mathbb{C}$ consists of the circle with centre $x$ and radius $r$. 

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**Metric Spaces**

We will be using De Morgan's law quite often without actually referring to it always.

**Theorem 5:**

i) Any arbitrary intersection of closed sets is closed, and

ii) A finite union of closed sets is closed.

**Proof:** Let $\{A_i\}_{i \in I}$ be a collection of closed sets. Then $\{A_i^c\}_{i \in I}$ is collection of open sets. Thus $\bigcup_{i \in I} A_i^c$ is open and therefore by De Morgan's law,

$$\left( \bigcup_{i \in I} A_i^c \right)^c = \bigcap_{i \in I} A_i$$

is closed. Hence the result. We leave the proof of the second part of this theorem as an exercise for you to verify (Refer E19).

We now define a related concept.

**Definition 13:** Let $A$ be a subset of a metric space $X$. The union of $A$ with the set of its limit points is called the closure of $A$ and is denoted by $\overline{A}$.

Obviously, if $A$ is closed, then $A = \overline{A}$. In general, $A \subset \overline{A}$.

**Theorem 6:** If $A$ is a subset of a metric space $X$, then $\overline{A}$ is closed in $X$.

**Proof:** Suppose $x$ is a limit point of $\overline{A}$. If $x$ is in $A$, then $x \in A$. Let us assume that $x \notin A$. Since $x$ is a limit point of $\overline{A}$, given $r > 0$, the open ball $B(x, r)$ contains an element $y$ of $\overline{A}$ other than $x$.

Let $d(x, y) = r_1$ and let $r_2 = r - r_1$. Then $r_2 > 0$. Since $y \in \overline{A}$, the open ball $B(y, r_2)$ contains an element $z$ of $A$ other than $y$. Then

$$d(x, z) \leq d(x, y) + d(y, z)$$

$$< r_1 + r - r_1$$

$$= r.$$

This shows that $z \in B(x, r)$. Also $z \neq x$ simply because $x \notin A$. Thus we got that $x$ is a limit point of $A$ and therefore $x \in \overline{A}$.

Since $\overline{A}$ contains all its limit points, $\overline{A}$ is closed. Hence the result.

**Remark:** A set is $A$ closed, if and only if $A = \overline{A}$.

Using the concept of closure of a set in a metric space, we define another concept.

**Definition 14:** Let $A$ be a subset of a metric space $X$. If a point $x$ is such that every open ball around $x$ contains a point of $A$ and a point of $A^c$, then $x$ is called a boundary point of $A$. The set of all boundary points of $A$ is called the boundary of $A$ and is denoted by $\text{bdry}(A)$.

**Remark:** It follows immediately from the definition that $\text{bdry}(A) = \text{bdry}(A^c)$.

For example, the boundary of $[a, b]$ in $\mathbb{R}$ consists of the points $a$ and $b$. The boundary of $B(x, r)$ in $\mathbb{C}$ consists of the circle with centre $x$ and radius $r$. 

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**Metric Spaces**

We will be using De Morgan's law quite often without actually referring to it always.
The following figure illustrates that the definitions of boundary, closure and interior coincides with our geometric intuition, for some specific subsets of $\mathbb{R}^2$ with the standard metric.

![Diagram](image)

But this is not the case always. We shall illustrate this with an example. Before that we establish the following result.

Now we prove a result relating to closure, interior and boundary.

**Proposition 7:** If $A$ is a subset of a metric space $X$ then

\[
\text{bdry}(A) = \overline{A} - \text{Int}(A).
\]

**Proof:** We will first prove that

\[
\text{bdry} A \subseteq \overline{A} - \text{Int} A
\]

Let $x \in \text{bdry} (A)$. Then it is immediate from the definition that $x \in \overline{A}$.

Also $x \notin \text{Int} A$ since every open ball $B(x, r)$ contains an element of $A^c$.

Therefore

\[
\text{bdry} A \subseteq \overline{A} - \text{Int} A. \tag{9}
\]

Now we have to show that $\overline{A} - \text{Int} A \subseteq \text{bdry} A$. Let $y \in \overline{A} - \text{Int} A$. Let $B(y, r)$ be an open ball. Since $y \in \overline{A}$, $B(y, r)$ contains an element of $A$. Suppose $B(y, r)$ does not contain any element of $A^c$. That means $B(y, r) \subseteq A$. So by definition $y \in \text{Int} A$ open set and $\text{Int} A$ is the largest open set contained in $A$, we get that $B(y, r) \subseteq \text{Int} A$ this is not possible since $y \notin \text{Int} A$. This shows that $B(y, r)$ intersects with $A^c$ and $A$. Therefore $y \in \text{bdry}(A)$. Thus we got that $\overline{A} - \text{Int} A \subseteq \text{bdry}(A)$. This together with (9) shows that

\[
\overline{A} - \text{Int} A = \text{bdry}(A).
\]  

**Example 27:** Let us find the interior, closure and boundary of the following sets in the given spaces

i) $B = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ in $\mathbb{R}^2$.

We shall first show that $\text{Int} B = \emptyset$. Let if, possible there exists $p = (x_0, y_0) \in \text{Int} B$. Then there exists $r > 0$ such that $B(p, r) \subseteq B$. Let
\[ q = \left(x_0, \frac{r}{2}\right). \] Then
\[ d(p, q) = \frac{r}{2} < r. \]

So \( q \in B(p, r) \). But \( q \notin B \), which is a contradiction. Thus \( \text{Int } B = \phi \).

Next we shall show that \( \overline{B} = B \). Let \( C \in \overline{B} \setminus B \). Let \( C = (\alpha, \beta) \) since
\[ C \in B, |\beta| > 0. \] Then consider \( B \left(C, \frac{|B|}{2}\right) \). Then \( B \left(C, \frac{|\beta|}{2}\right) \cap B = \phi \). So
\[ C \notin \overline{B} \), a contradiction. Thus \( \overline{B} = B \).

From the Proposition 7 above we get that \( \text{bdry } B = B \).

***

Here are some exercises for you.

E19) Complete the proof of Theorem 4.

E20) Show that any finite subset of a metric space is closed.

E21) Show that \( \overline{A} = \{x \in X : B(x, r) \cap A \neq \phi \text{ for every } r > 0\} \).

E22) If \( X \) is a metric space and \( A \) is a non-empty subset of \( X \), then show that
\( \overline{A} = \{x : d(x, A) = 0\} \).

In the next section we shall discuss continuous functions in metric spaces.

### 1.5 CONTINUOUS FUNCTIONS

The concept of continuous functions is one of the very basic concepts in Mathematical Analysis.

You are already familiar with this concept for the Euclidean spaces \( \mathbb{R} \), \( \mathbb{R}^2 \) and so on. In this section we shall formulate a definition of continuity for general metric spaces that will include these spaces. Then we shall study some properties.

You might be already familiar with continuous functions from \( \mathbb{R}^n \rightarrow \mathbb{R}^m \) with the standard metrics (see IGNOU Maths Course MTE-07). Here are some examples.

i) The function \( t \rightarrow (\cos t, \sin t) \) from \( \mathbb{R} \) to \( \mathbb{R}^2 \) is continuous on \( \mathbb{R} \),

ii) The function \( (x, y) \rightarrow (\cos x, \sin x \sin y, e^x \sin y) \) from \( \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) is continuous on \( \mathbb{R}^2 \).

Like these you must have seen many continuous functions from one Euclidean space to another Euclidean space. You know that in the case of real line \( \mathbb{R} \) there are different ways of defining continuity. For instance, "\( \epsilon - \delta \) definition" "convergent sequence definition" and so on. We shall try to generalise these definitions to metric spaces.

Before we do this we shall familiarise you with some notations.
Suppose \((X, d_1)\) and \((Y, d_2)\) are two metric spaces and consider a function \(f : X \to Y\). Let \(A\) be any subset of \(X\). Then the image of \(A\) by \(f\) is the set

\[ f(A) = \{f(x) : x \in A\} \]

In other words, \(f(A)\) is the set of the values that \(f\) assigns to the elements of \(A\).

Suppose that \(B\) is a subset of \(Y\). Then the inverse image of \(B\) denoted by \(f^{-1}(B)\), is given by

\[ f^{-1}(B) = \{x \in X \mid f(x) \in B\}. \]

Now we shall try to generalise the \(\varepsilon - \delta\) definition of continuity. To do this the best way is to recall the definition of continuity of functions on \(\mathbb{R}\).

Let \(f\) be a real-valued function defined on \(\mathbb{R}\). Then \(f\) is said to be continuous at \(x_0 \in \mathbb{R}\) if given \(\varepsilon > 0\), \(\exists \) a \(\delta > 0\) such that

\[ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon. \]

Also a function \(f\) is said to be continuous on a set \(A\) in \(\mathbb{R}\) if \(f\) is continuous at all points of \(A\).

We can easily extend this definition to general metric spaces by replacing the modulus \(|x - y|\) by the distance \(d(x, y)\)

This gives the following definition.

**Definition 15:** Let \((X, d_1)\) and \((Y, d_2)\) be two metric spaces and \(f : X \to Y\). Then \(f\) is said to be continuous at a point \(x_0 \in X\), if given \(\varepsilon > 0\) \(\exists\) a \(\delta > 0\) such that

\[ d_1(x, x_0) < \delta \Rightarrow d_2(f(x), (f(x_0)) < \varepsilon. \]

Equivalently, given \(\varepsilon > 0\) there exists a \(\delta > 0\) such that

\[ x \in B(x_0, \delta) \Rightarrow f(x) \in B(f(x_0), \varepsilon) \]

A function \(f\) is said to be continuous on \(X\) if it is continuous at each point of \(X\). Such functions are called **continuous functions**.

It is immediate from the definition that the following functions are continuous

**Example 28:** Let \(X = B(T)\) with the metric defined in Example 9. Let \(\phi : X \to X\) be given by \(\phi(f) = t^2\). Then \(\phi\) is continuous.

Let \(f_0 \in X\). Put \(g_0 = \phi(f_0)\)

Let \(M = \sup_{t \in T} f_0(t)\)

Consider any \(f \in X\) and put \(g = \phi(f)\).

We note that for \(t \in T\),

\[ g(t) - g_0(g) = (f(t) - f_0(t))^2 + 2(f(t) - f_0(t))f_0(t), \]

and thus
$g(t) - g_0(t) = |f(t) - f_0(t)|^2 + 2M|f(t) - f_0(t)|$

So,

$d(g, g_0) \leq d(f, f_0)(d(f, f_0) + 2M)$

Choose $\delta = \min \left\{ \frac{\epsilon}{2M + 1}, 1 \right\}$

Then for $d(f, f_0) < \delta$, we have $d(g, g_0) < \epsilon$.

So $\phi$ is continuous at $f_0$. Since $f_0$ is arbitrary, $\phi$ is continuous.

---

Many of you might have faced problems with $\epsilon - \delta$ definitions of limit and continuity in your earlier Calculus and Analysis courses. The following real-life situation may help you to have a better understanding.

**Situation 1:** Let $X$ be a set of ingredients or input and $Y$ is the output and $f$ is the process or procedure of a recipe which takes an input $x$ and produces an item (or output) $y$. The end user or consumer needs $y \in Y$. The manufacturer knows that to get $y$, he/she can put the input $x$ and by applying the process $f$ to $x$, she gets $y = f(x)$. But now in real-life, there is no guarantee that the manufacturer will give precisely the output required as there is no surety that the manufacturer will be able to put the input exactly $x_0$ or apply the process correctly. The customer knows these limitations and therefore is ready to accept the output if the output does not vary with his/her expectation too much. In this case the customer fixes some tolerance level and decides that if the manufacturers delivers the product within that tolerance level, it will be accepted. From the experience, the manufacturer finds a $\delta$ such that if the input is within $\delta$ distance from a fixed input $x_0$, the process will produce $f(x)$ within the tolerance level for $f(x_0)$. Note that in such situation the error level upto $\epsilon$ of the output that can be tolerated is given, then we find difference level upto $\delta$ of the input. Now the continuity of the process in fact tells us that given $\epsilon > 0$ we can find such a $\delta$, such that the process gives the desired result within the tolerance level $\epsilon$.

The above situation might have helped you to understand the $\epsilon - \delta$ definition of continuity in a better way. By now you must be clear about which one out of $\epsilon$ or $\delta$ in the definition of continuity, needs to be considered first. It is important to know this because this is what is used by scientists and engineers.

Now we shall consider some examples.

**Example 29:** Let $(X, d)$ be a metric space. For any fixed $a \in X$, define a function $f_a : X \to \mathbb{R}$ by

$f_a(x) = d(a, x)$

Then $f_a$ is continuous on $X$.

To see this, let us apply $\epsilon - \delta$ definition.

Let $x_0 \in X$. Let $\epsilon > 0$. Then we have to show that there exists a $\delta > 0$ such that

$d(x, x_0) < \delta \Rightarrow |f_a(x) - f_a(x_0)| < \epsilon$
But, by the definition of the function $f_a$

$$|f_a(x) - f_a(x_0)| = |d(a, x) - d(a, x_0)| \leq d(x, x_0)$$  \hspace{1cm} (10)

So we may choose $\delta = \epsilon$. Then from Eqn.(10) we get that

$$d(x, x_0) < \delta \Rightarrow |f_a(x) - f_a(x_0)| < \epsilon.$$  

This shows that $f_a$ is continuous on $X$.

***

Another interesting continuous function is illustrated in the next example.

**Example 30:** Let $A$ be a nonempty subset of a metric space $(X, d)$. We define a function on $X$ to $\mathbb{R}$ by

$$d_A(x) = d(x, A)$$

as given Definition 5 of Section 1.3. That is,

$$d_A(x) = \inf \{d(x, a) : a \in A\}$$

as given Definition 5 of Section 1.3. That is,

$$d_A(x) := \inf \{d(x, a) : a \in A\}, \quad x \in X.$$  

Then $d_A$ is continuous.

To show that $d_A$ is continuous, we observe that, for any arbitrary $x, y \in X$ and $a \in A$, the triangle inequality implies that

$$d(a, y) \geq d(a, x) - d(y, x)$$

\hspace{1cm} (11)

$$\geq d_A(x) - d(y, x), \quad \text{(by the definition of } d_A(x)\text{)}$$

Taking infimum on both sides of (12) for $a \in A$, we have

$$d_A(y) \geq d_A(x) - d(y, x).$$

Thus,

$$d_A(x) - d_A(y) \leq d(y, x).$$

If we interchange $x$ and $y$ in this inequality, we see that

$$\pm (d_A(x) - d_A(y)) \leq d(x, y) \text{ that is, } |d_A(x) - d_A(y)| \leq d(x, y).$$

From this inequality the continuity of $d_A$ can easily be shown.

***

We give some exercises now.

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**E23** The **identity function** $i : X \to X$ defined by $i(x) = x \ \forall \ x \in X$ is continuous.

**E24** Let $(X, d_1)$ and $(Y, d_2)$ be two metric spaces. For any fixed $c \in Y$ the function $f : X \to Y$ defined by $f(x) = c$ for all $x \in X$ is continuous. ($f$ is called a **constant function**.)

**E25** Let $X = C[0, 1]$ with sup metric. Then the following functions are continuous.
i) $\psi : X \rightarrow \mathbb{R}$ given

$$\psi(f) = \int_0^1 f(t)dt, \ f \in C[0, 1]$$

is continuous.

ii) $\phi : X \rightarrow \mathbb{R}$ given by $\phi(f) = f\left(\frac{1}{2}\right)$ is continuous.

Next we shall discuss a characterisation of continuous function.

**Theorem 7:** Let $(X, d_1)$ and $(Y, d_2)$ be metric spaces. Let $c \in X$. Then the following are equivalent.

(a) $f$ is continuous at $c$.

(b) Given an open set $V$ containing $f(c)$ in $Y$, we can find an open set $U$ containing $c$ in $X$ such that $f(U) \subseteq V$.

**Proof:** We shall first prove that (a) $\Rightarrow$ (b)

Let $f : X \rightarrow Y$ be continuous at $c$. We have to show that $f$ satisfies (b).

Let $V$ be an open set containing $f(c)$ in $Y$. Since $V$ is open and $f(c) \in V$, there exists an $\varepsilon > 0$ such that the open ball $B(f(c), \varepsilon) \subseteq V$. Since $f$ is continuous, given $\varepsilon > 0$, $\exists \delta > 0$ such that

$$d_1(x, c) < \delta \Rightarrow d_2(f(x), f(c)) < \varepsilon.$$

Let $U = B(c, \delta)$. Then $U$ is an open set containing $c$. We claim that $f(U) \subseteq V$. To show this let us consider $b \in f(U)$. Then $y = f(x)$ for some $x \in U$. This implies that

$$d_1(x, c) < \delta \Rightarrow d_2(f(x), f(c)) < \varepsilon \ i.e. \ f(a) \in B(f(c), \varepsilon).$$

Thus we have shown that $f(U) \subseteq V$.

This shows that (a) $\Rightarrow$ (b).

Next we have to show that (b) $\Rightarrow$ (a).

Let $f$ be a function such that (b) holds. We have to show that $f$ is continuous at $c \in X$. Let $\varepsilon > 0$ be given. Now we apply the condition (b) to the open set $B(f(c), \varepsilon)$. Then we get that $\exists$ an open set $U$ containing $c$ such that

$$f(U) \subseteq B(f(c), \varepsilon).$$

Since $U$ is open and $c \in U$, $\exists$ a $\delta > 0$ such that $B(c, \delta) \subseteq U$. This together with the fact that $f(U) \subseteq B(f(c), \varepsilon)$, shows that

$$d_1(x, c) < \delta \Rightarrow d_2(f(x), f(c)) < \varepsilon \hspace{1cm} (13)$$

Thus we have shown that given $\varepsilon > 0$ $\exists \delta > 0$ such that (13) holds. This shows that $f$ is continuous. □

From the above theorem we can easily establish the following theorem.

**Theorem 8:** Let $X$, $Y$ be metric spaces. Then a map $f : X \rightarrow Y$ is continuous if and only if for every open set $V \subseteq Y$, its inverse image $f^{-1}(V)$ is open in $X$.

The proof of this theorem is left as an exercise for you.
Remark 11: This characterisation of continuous function is very important (or useful) because this does not involve the notion of "distance function (metric)" directly; it mainly considers sets which are open. Indeed this criterion is used in defining continuity in more general settings like topological spaces.

Now we want you to try some exercises based on the discussions above.

E26) Prove Theorem 8.

E27) Show that the result in Theorem 7 is true if we replace open sets by closed sets.

E28) Let \((X, d_1), (Y, d_2)\) and \((Z, d_3)\) be three metric spaces. Let \(f : X \to Y\) be continuous at \(x \in X\) and \(g : Y \to Z\) be continuous at \(y = f(x)\). Then composite map \(g f : X \to Z\) is continuous at \(x \in X\).

E29) Let \(f : (X, d_1) \to (Y, d_2)\) be continuous. Let \(A \subset X\). Show that the restriction \(f|_A\) is a continuous function from \((A, d_1)\) to \((Y, d_2)\).

E30) Let \(p_1 : \mathbb{R}^2 \to \mathbb{R}\) and \(p_2 : \mathbb{R}^2 \to \mathbb{R}\) be the functions such that \(p_1(x, y) = x\) and \(p_2(x, y) = y\). \((p_1\) and \(p_2\) are called coordinate projections). Show that \(p_1\) and \(p_2\) are continuous on \(\mathbb{R}^2\).

E31) Let \((X, d)\) be a metric space and \(f : X \to \mathbb{R}\) be a continuous function. Then show that for each \(t \in \mathbb{R}\) the sets

\[ V_t = \{ x \in X : f(x) < t \} \]

\[ W_t = \{ x \in X : f(x) > t \} \]

are open sets in \(X\).

Next we shall state some basic properties of continuous functions which you can verify by yourself.

**Theorem 9:** Let \(f, g : X \to \mathbb{R}\) be continuous at \(a \in X\). Then the functions \(f + g, f - g, f \cdot g\) defined by

\[(f + g)(x) = f(x) + g(x), x \in X\]
\[(f - g)(x) = f(x) - g(x), x \in X\]
\[f \cdot g(x) = f(x) \cdot g(x), x \in X\]

are all continuous on \(X\) at \(a \in X\). If \(g(x) \neq 0\) for all \(x \in X\), then \(\frac{f}{g}\) is also continuous at \(a \in X\).

Now we shall prove a theorem which shows that two mutually disjoint closed sets in a metric space can be separated by a continuous function. The theorem is called Urysohn's Lemma.

**Theorem 10:** (Urysohn's Lemma): Let \(A, B\) be two disjoint non-empty closed subsets of a metric space \(X\). Then there exists a continuous function \(f : X \to \mathbb{R}\) such that \(0 \leq f \leq 1\), \(f = 0\) on \(A\) and \(f = 1\) on \(B\).
Before we start proving this we recall that \( d_A \) and \( d_B \) are continuous on \( X \) which vanish on \( A \) and \( B \) respectively. So various combinations of them can work as the required function. For example, any \( \alpha > 0 \) and consider \( f \) given by,

\[
f(x) = \frac{\alpha d_A(x)}{\alpha d_A(x) + d_B(x)}, \quad \alpha > 0
\]

**Proof:** Let \( f(x) = \frac{d_A(x)}{d_A(x) + d_B(x)} \). We first show that \( f \) is defined for all \( x \in X \).

Let \( x \in X \). For that we note that \( d_A(x) + d_B(x) \neq 0 \). For, otherwise, each of the terms, being nonnegative, must be zero. But then this means that \( x \in A \) and \( x \in B \) (recall that by E21, \( d(x, A) = 0 \) iff \( x \in \overline{A} \)). This is a contradiction since \( A \) and \( B \) are disjoint. Thus \( d_A(x) + d_B(x) \neq 0 \) for each \( x \in X \) and we conclude that \( f(x) \) makes sense for each \( x \in X \).

Since \( d_A \) and \( d_B \) are continuous real-valued functions on \( X \) by Theorem 8, \( f \) is continuous on \( X \). Clearly, \( 0 \leq f \leq 1 \).

Also if \( x \in A \), then \( d_A(x) = 0 \) so that \( f(x) = 0 \). If \( x \in B \), then the denominator of \( f(x) \) is \( d_A(x) + d_B(x) = d_A(x) \neq 0 \) and therefore \( f(x) = 1 \) for \( x \in B \). Hence \( f \) is a required continuous function.

Try this exercise now.

**E32)** Deduce the following result from Urysohn's lemma.

Let \( A \) and \( B \) be nonempty disjoint closed subsets of a metric space \((X, d)\). Then show that there exist open sets \( U \supset A \) and \( V \supset B \) such that \( U \cap V = \emptyset \).

Next we shall consider another interesting property.

**Lemma (Gluing Lemma):** Let \( X \) and \( Y \) be metric spaces.

Let \( \{A_i : i \in I\} \) be a family of non-empty open sets such that \( \bigcup_{i \in I} A_i = X \). Let \( \{f_i : i \in I\} \) be a family of functions satisfying

i) for each \( i \in I \), domain of \( f_i = A_i \) and \( f_i : A_i \rightarrow Y \) is continuous, and

ii) for each \( i, j \in I \), \( f_i(x) = f_j(x) \) for all \( x \in A_i \cap A_j \).

Then the function \( f : X \rightarrow Y \) defined by setting \( f(x) := f_i(x) \) if \( x \in A_i \) is well-defined and continuous on \( X \).

**Proof:** Let \( B \subset Y \) be open. Then we have to show that \( f^{-1}(B) \) is open in \( X \). We observe that for any fixed, \( i \), \( f^{-1}(B) \cap A_i = f_i^{-1}(B) \cap A_i \).

This is because \( x \in f^{-1}(B) \cap A_i \) iff \( x \in A_i \) and \( f(x) \in B \), iff \( x \in A_i \) and \( f_i(x) \in B \), iff \( x \in A_i \cap f_i^{-1}(B) \).

Since \( f_i : A_i \rightarrow Y \) is continuous, the set \( f_i^{-1}(B) \cap A_i \) is open in \( A_i \). But, \( A_i \) is open in \( X \). So \( f^{-1}(B) \cap A_i \) is open in \( X \). Now

\[
f^{-1}(B) = f^{-1}(B) \cap X = f^{-1}(B) \cap (\bigcup_{i \in I} A_i) = \bigcup_{i \in I} (f_i^{-1}(B) \cap A_i).
\]

Thus it follows that \( f^{-1}(B) \) is the union of open sets \( \{f_i^{-1}(B_i) \cap A_i, i \in I\} \) and hence is open. This proves the continuity of \( f \) on \( X \).
It is natural to ask whether such a continuous gluing is possible for closed sets. The following example shows that it is not possible in general.

For \( n \in \mathbb{N} \), let us take

\[
A_n = \left\{ \frac{1}{n} \right\} \\
A_0 = \{0\}
\]

Now let \( X = \bigcup_{n \geq 0} A_n = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \) considered as a subspace of \( \mathbb{R} \).

Then each \( A_n \) is closed in \( X \).

For each \( n \geq 0 \), define a function \( f_n \) on \( A_n \) given by

\[
f_n \left( \frac{1}{n} \right) = n, \text{ for } n > 0
\]

and

\[
f_0 (0) = 4.
\]

Then for each \( n \), \( f_n \) is a continuous function considered as a function on the closed subspace \( A_n \). Since \( A_n \)'s are mutually disjoint \( f_n \)'s can be glued together to give the function \( f \) on \( X \) given by

\[
f \left( \frac{1}{n} \right) = n, n > 0
\]

and

\[
f(0) = 4.
\]

This function \( f \) is not continuous at \( 0 \).

This shows that continuous gluing is not possible for infinitely many functions defined and continuous on infinitely many closed sets considered as subspaces of \( X \). However, it is possible for finitely many functions as the following theorem shows.

**Theorem 11:** Let \( X \) and \( Y \) be metric spaces. Let \( A_1, A_2, \ldots, A_n \) be a finite family of closed sets such that \( \bigcup_{i=1}^{n} A_i = X \). Let \( \{f_i, i = 1, \ldots, n\} \) be a finite family of functions satisfying

1) for each \( i = 1, \ldots, n \), domain of \( f_i \) = \( A_i \) and \( f_i : A_i \to X \) is continuous.

2) for \( i, j = 1, \ldots, n \), \( f_i(x) = f_j(x), x \in A_i \cap A_j \).

Then the function \( f : X \to Y \) defined by setting \( f(x) = f_i(x) \) if \( x \in A_i \) is well-defined and continuous on \( X \).

**Proof:** The proof of this is very similar to the earlier proof, except that we use the characterization of continuity by means of inverse images of closed sets. Let \( C \subset Y \) be a closed set. We shall show that \( f^{-1}(C) \) is closed in \( X \). As in the earlier case, we find that \( f^{-1}(C) \cap A_i = f_i^{-1}(C) \cap A_i \) and that it is closed in \( A_i \) and
hence in $X$. We then express $f^{-1}(C)$ as a finite union of closed sets of the form as follows:

$$f^{-1}(C) = f^{-1}(C) \cap X = f^{-1}(C) \cap (\bigcup_{i \in I} A_i) = \bigcup_{i \in I} (f^{-1}(C) \cap A_i).$$

This completes the proof.

We consider an example to illustrate the lemma.

**Example 31:** Let $X = [0, 1]$ considered as a subspace of $\mathbb{R}$. Let $Y$ be any metric space. Let $f, g : [0, 1] \to Y$ be two continuous functions. Assume that $f(1) = g(0)$.

Put $A_1 = \left[0, \frac{1}{2}\right]$ and $A_2 = \left[\frac{1}{2}, 1\right]$ considered as closed subspaces of $X$. Then $X = A_1 \cup A_2$.

Let $f_1(t) = f(2t), t \in A_1$

$f_2(t) = g(2t - 1), t \in A_2$

Then $f_1, f_2$ are both continuous in their respective domains.

Then by the above lemma the function $h$ defined by

$$h(t) = \begin{cases} f_1(t), & t \in A_1 \\ f_2(t), & t \in A_2 \end{cases}$$

is continuous on $X$.

We in fact note that the function $h$ is given by

$$h(t) = \begin{cases} f(2t), & 0 \leq t \leq \frac{1}{2} \\ g(2t - 1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

and can be called the join of $f_1$ and $f_2$.

***

Now we define uniform continuity for a general metric space on the same lines as for the real line.

**Definition 16:** Let us consider a function $f : (X, d_1) \to (Y, d_2)$. We say that $f$ is uniformly continuous on $X$ if for a given $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x_1, x_2 \in X$ are such that $d_1(x_1, x_2) < \delta$, we have $d_2(f(x_1), f(x_2)) < \epsilon$.

**Remark 12:** It follows immediately from definition that a uniformly continuous function is continuous. It is also immediate that if $d_1$ is the discrete metric then by taking $\delta = 1$ in definition 14 $f$ is uniformly continuous.

**Remark 13:** Here you may recall that a continuous function is uniformly continuous on a closed bounded interval. However a real valued continuous function defined on other kinds of subsets may or may not be uniformly continuous. For instance you may recall the following:
Let,

\[ f(x) = \frac{1}{x}, \quad 1 < x < \infty \]

\[ g(x) = x, \quad x \in \mathbb{R} \]

\[ \phi(x) = \frac{1}{x}, \quad 0 < x < 1 \]

\[ \psi(x) = \sin \frac{1}{x}, \quad 0 < x < 1 \]

Then \( f \) and \( g \) are uniformly continuous whereas \( \phi \) and \( \psi \) are continuous but not uniformly continuous.

**Example 32:** The function \( f_u \) given in Example 29 is uniformly continuous.

From the proof of Example 29 it follows that, for \( x_1, x_0 \in X \)

\[ |f(x) - f_u(x_0)| \leq d(x, x_0) \]

So \( f_u \) is uniformly continuous.

***

You can try some exercises now.

---

E33) Let \( X \) be a metric space with metric \( d \) and let \( a \) be a point of \( X \). Show that the real-valued function

\[ x \mapsto d(x, A), \quad x \in X, \]

is uniformly continuous on \( X \).

E34) Show that the functions \( p_1 \) and \( p_2 \) given in E 30 are both uniformly continuous.

E35) Let \( f \) and \( g \) be two real-valued uniformly continuous functions on a metric space \( (X, d) \). Is the product function \( fg \) given by

\[ fg(x) = f(x) \cdot g(x), \quad x \in \mathbb{R} \]

is uniformly continuous? Justify your answer.

---

We shall see in Unit 3 which is devoted to the generalisation of the concept of compactness you have met in context of real line, that a continuous function on a compact metric space is uniformly continuous.

But there are metric spaces where continuity and uniform continuity are equivalent concepts. Discrete metric space is an example for this. There are other metric spaces such as compact metric spaces where the two concepts are equal. You will come across this in Unit 3 of this block.

The notion of uniform continuity is of considerable interest in contexts involving approximating functions by other functions as is done in areas like approximation theory. Before we discuss more about this concept, we need to
consider another important concept of “convergence”. In the next unit we shall do this.

With this we come to an end of this unit.

### 1.6 SUMMARY

In this unit we have

1) introduced you to the concepts of a metric and metric space.

2) discussed various metrics that can be defined on
   i) \( \mathbb{R}^n \)
   ii) \( C[a, b] \) - Set of all continuous real-value function defined on \([a, b]\)
   iii) \( B(X) \) - set of all bounded real-valued functions on a metric space \( X \)
   iv) \( M(n, \mathbb{R}) \) - set of all \( n \times n \) real matrices.

3) defined induced metric and product metric.

4) defined and discussed open and closed checking sets.

5) introduced of the concepts “int \( A \)”, “Bdry \( A \)” “closure of \( A \) i.e. \( \bar{A} \)”.

6) defined and discussed the concept of continuity for a function defined on a metric space and with examples.

7) explained Urysohn’s lemma and glueing lemma.

8) defined uniform continuity and looked at related examples.

### 1.7 HINTS/SOLUTIONS

E1) Try by yourself

E2) Try by yourself

E3) Hint : Use Cauchy-Schwarz inequality

E4) i) Try it by yourself.
   ii) Hint : Condition 1 fails for \( x \neq 0 \). Therefore \( d \) is not a metric.
   iii) Hint : Triangle inequality is not satisfied for \( x = 1, y = 0 \) and \( z = -1 \).

E5) Hint : Condition 1 is not satisfied for \( x = (0, 0) \) and \( y = (0, 1) \).

E6) i) Hint : To prove the triangle, the following observation helps. Let us take \( x, y, z \in X \). Then we have

\[
\frac{1}{1 + d(x, y)} \leq 1 - \frac{1}{1 + d(x, y) + d(y, z)}
\]

Hence the result.
ii) Conditions 1 and 2 are immediate from the definition of \( \rho \). To check the triangle inequality we take any let \( x, y, z \in X \). Since \( \rho(x, z) \leq 1 \) the required inequality

\[
\rho(x, z) \leq \rho(x, y) + \rho(y, z)
\]

is satisfied if either \( \rho(x, y) = 1 \) or \( \rho(y, z) = 1 \). The only other possibility is that \( \rho(x, y) < 1 \) and \( \rho(y, z) < 1 \). In this case both \( \rho(x, y) = d(x, y) \) and \( \rho(y, z) = d(y, z) \).

So

\[
\rho(x, z) \leq d(x, z) \leq d(x, y) + d(y, z) = \rho(x, y) + \rho(y, z).
\]

Hence the condition 3 is satisfied. Thus \( \rho \) is a metric on \( X \).

E7) Hint: Use Remark 4
E8) Try it by yourself
E9) Try it by yourself
E10) Hint:
   i) To get the answer \( d(A) = 2 \), the observation that for \( 0 < a < 1 \), helps.
   ii) To get the answer \( d(A) = 9 \), the observation that for \( n \in N, 1 + \frac{1}{n}, 10 - \frac{1}{n} \in A \), helps.
E11) Hint: Try the following
   i) \( A = (1, 2), x = 1 \)
   ii) \( A = \left\{ 1, \frac{1}{2}, \ldots \right\}, x = 0 \)
E12) Hint: Try \( Z \times Z \) and \( \{3\} \times R \) for unbounded sets in \( R^2 \).
E13) Hint: Try to get the answers given below:
   i) \[
   \begin{bmatrix}
   3 & 5 \\
   2 & 3
   \end{bmatrix}
   \]
   ii) \( R \)
E14) Try it by yourself.
E15) Try it by yourself
E16) Let \( x \in \text{Int } A \). Then \( \exists r > 0 \) such that \( B(x, r) \subset A \). Let \( y \in B(x, r) \) put \( r_1 = r - d(y, x) \), then \( B(y, r_1) \subset B(x, r) \subset A \). So \( y \in \text{int } A \). Thus we have \( B(x, r) \subset \text{Int } A \). Since \( x \in A \) is arbitrary we conclude that \( \text{Int } A \) is open.
E17) To show that \( A_1 \times A_2 \) is open in \( X_1 \times X_2 \). Let \( x = (x_1, x_2) \in A_1 \times A_2 \). Then \( x_1 \in A_1 \) and \( x_2 \in A_2 \). Since \( A_1 \) is open, \( \exists r_1 > 0 \) such that \( B(x_1, r_1) \subset A_1 \). Similarly \( \exists r_2 > 0 \) such that \( B(x_2, r_2) \subset A_2 \). Let \( r = \min(r_1, r_2) \). Then \( B(x, r) \) is an open ball with centre \( x \). Also for any \( y = (y_1, y_2) \in B(x, r) \), we have \( d_p(x, y) < r \). This implies that \( d(x_1, y_1) < r \) and \( d(x_2, y_2) < r \). That means \( y_1 \in B(x_1, r_1) \subset A_1 \) and \( y_2 \in B(x_2, r_2) \subset A_2 \) and therefore \( (y_1, y_2) \in A_1 \times A_2 \). This shows that \( B(x, r) \subset A_1 \times A_2 \). Thus \( A_1 \times A_2 \) is open.
E18) Hint: \( d \leq D \leq 2d \)
\[
\rho \leq D \leq \sqrt{2} \rho
\]
\[
d \leq \rho \leq \sqrt{2} d
\]
E19) To get the reverse inclusion, consider any \( x \in X \) such that \( d(x, A) = 0 \). Let \( r > 0 \) by the definition of \( a(x, A) \), then exists \( y \in A \) with \( d(x, y) < r \). Thus \( B(x, r) \cap A \neq \emptyset \). So \( x \in \overline{A} \). Thus we have \( \{ x : d(x, A) = 0 \} \subseteq A \). This completes the proof.

E20) We first show that \( \{ x \} \) is closed. Let \( A = \{ x \} \). Then it is enough to show that \( A^c \) is open. Let \( y \in A^c \). Then \( d(x, y) > 0 \). Let \( d(x, y) = r \), for some real number \( r \) and \( r_1 = \frac{r}{2} \). Now consider \( B(y, r_1) \). Then \( y \in B(y, r_1) \) and \( B(y, r_1) \subseteq A^c \). This shows that \( A^c \) is open and therefore \( \{ x \} \) is closed. Then by Theorem 4, any finite set is closed.

E21) Let \( x \in \overline{A} \). We will show that \( d(x, A) = 0 \). Let, if possible, \( d(x, A) > 0 \). Let \( r = \frac{a}{2} \) and consider \( B(x, r) \). Since \( x \in \overline{A} \), \( \exists \) an \( y \in B(x, r) \cap A \).

So

\[
d(x, A) \leq d(x, y) < r < d(x, A).
\]

This is a contradiction. Thus \( d(x, A) = 0 \). Thus we have \( \overline{A} \subseteq \{ x : d(x, A) = 0 \} \).

Let \( B(x, r) \) be a ball containing \( x \). To show that \( B(x, r) \cap A \neq \emptyset \). Let \( B(x, r) \cap A = \emptyset \). This implies that \( d(x, y) > r \) for all \( y \in A \) which contradicts the fact that \( d(x, A) = 0 \). Therefore \( B(x, r) \cap A \neq \emptyset \). Hence \( x \in \overline{A} \).

E22) Try by yourself.

E23) Let \( \{ A_i \}_{i \in I} \) be a finite collection of closed sets. Then \( \{ a_i^c \}_{i \in I} \) is an open set. Then \( \bigcup_{i=1}^{n} F_i = \bigcap_{i=1}^{n} F_i^c \). Each \( F_i^c \) is open. Hence the result.

E24) Try by yourself.

E25) Hint:

i) Use the inequality

\[
\left| \int_0^1 f(t) \, dt \right| \leq \int |f(t)| \, dt
\]

for \( f \in C[0, 1] \).

ii) Try it by yourself.

E26) Take \( x \in f^{-1}(V) \). Then \( y = f(x) \in V \). Since \( V \) is open, it contain an open ball \( B(y, \epsilon) \) for some \( \epsilon \). Since \( f \) is continuous given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( x' \in B(x, \delta) \Rightarrow f(x') \in B(y, \epsilon) \). This means that

\[
B(x, \delta) \subseteq f^{-1}(B(y, \epsilon)) \subseteq f^{-1}(V).
\]

Therefore \( f^{-1}(V) \) is open in \( X \).

E27) Hint: Use the fact and that a set is open if and only if its complement is closed.

E28) Hint: If \( A \) is open in \( Z \), then \( g^{-1}(A) \) is open in \( Y \) and therefore \( f^{-1}(g^{-1}(A)) \) is open in \( X \). But

\[
f^{-1}(g^{-1}(A)) = (gof)^{-1}(A)
\]
Hint: The function $f|_A$ equals the composite of the inclusion map $j: A \to X$ and the map $f: X \to Y$. Therefore it is continuous.

E30) Hint: Use the fact that, for $A \subseteq \mathbb{R}$,
\[
p_1^{-1}(A) = A \times \mathbb{R} \\
p_2^{-1}(A) = \mathbb{R} \times A
\]

E31) Hint: Note that $V_t = f^{-1}\{(-\infty, t)\}$ and $W_t = f^{-1}(t, \infty)$ and $(\infty, t), (t, \infty)$ are open sets in $\mathbb{R}$.

E32) Hint: Try
\[
U = f^{-1}\left(0, \frac{1}{2}\right) \\
V = f^{-1}\left(\frac{1}{2}, 1\right)
\]

E33) Hint: A careful look on the last line in the Example 30 helps.