CONTENT BASED METHODOLOGY-I

UNIT 10
Number Systems, Number Theory, Exponents and Logarithms

UNIT 11
Polynomials: Basic Concepts and Factoring

UNIT 12
Linear Equations, Inequations and Quadratic Equations

UNIT 13
Sets, Relations, Functions and Graphs
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Course: BES-143 Pedagogy of Mathematics

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- Unit 3 How Children Learn Mathematics
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- Unit 17 Mensuration and Coordinate Geometry
Block Introduction

The course BES-143: Pedagogy of Mathematics contains four blocks. This is the third block which is titled as Content based Methodology-I. In the first two blocks nature, aims, curriculum and approaches of teaching-learning & assessment tools of mathematics have been discussed. This is the block where the subject contents/concepts have been discussed together with the suggested methodology for the teaching-learning and evaluation. This block focuses on the basic concepts of algebra, polynomials and their factorization, solution of linear and quadratic equations, sets, relations and functions and various ways of transaction of these topics in the classroom. This block consists of four units.

Unit 10: Number System, Number Theory, Exponents and Logarithms
Unit 11: Polynomials: Basic Concepts and Factoring
Unit 12: Linear Educations, Inequations and Quadratic Equations
Unit 13: Sets, Relations, Functions and Graphs

Unit 10 provides a detailed account of the development of the number system and number theory. The need for different number systems and their interrelations are discussed. This unit focuses on the concepts of exponents and logarithms. Moreover, in this unit the concepts of HCF and LCM also find their places.

Everyone knows that mathematics is an art of problem solving. The representation of unknown quantities by alphabets paved the way for algebra. The algebraic calculations and procedures are the backbone of all problem-solving exercise. In algebra the unknown quantities are known as variables which can assume different values under different conditions. The variables are generally represented by the alphabets x, y, z, etc. Anyone who wants to pursue higher mathematics must be proficient in dealing with algebraic expressions. Thus, Unit 11 deals with the basic concepts of algebraic calculations, in particular the polynomials and their factorization.

In problem solving, when a real life situation is translated in the language of mathematics, we get equations and/or inequations which are solved to obtain the answer to the problem. So the Unit 12 discusses the methods of solving linear and quadratic equations along with their applications.

Apart from problem solving, the study of Mathematics at the higher level can not be pursued without the knowledge of functions. Unit 13 deals with the concepts of sets, relations and functions to the beginners.

For each of the concepts discussed in the four units of this block, we have suggested some teaching-learning strategies which may help your learners to understand them better. These teaching-learning strategies involve various ways of instructional transaction and interacting with children.
UNIT 10 NUMBER SYSTEMS, NUMBER THEORY, EXPONENTS AND LOGARITHMS

Structure

10.1 Introduction
10.2 Objectives
10.3 Number Systems
  10.3.1 Sets of Numbers
  10.3.2 Natural Numbers
  10.3.3 Zero and Integers
  10.3.4 Rational and Irrational Numbers
  10.3.5 Real Numbers – Operations and their Properties
  10.3.6 The Number Line
10.4 Exponents and Logarithms
  10.4.1 Exponents, Power and Root
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  10.4.3 Use of Logarithmic Tables
10.5 Elementary Number Theory
  10.5.1 Basic Concepts
  10.5.2 Prime Factorization: HCF, LCM
  10.5.3 Division Method for Finding HCF, LCM
  10.5.4 Applications of HCF and LCM
10.6 Let Us Sum Up
10.7 Unit End Activities
10.8 Answers to Check Your Progress
10.9 Suggested Readings

10.1 INTRODUCTION

Numbers form an essential element of our everyday thought and language. In present day life, every activity requires the use of numbers. Prices and weights of commodities, scores of games, travel time tables, etc., all are expressed in numbers. Man owes much of his success or progress to the art of computing with numbers.

The present course of Mathematics in schools introduces the students to the various properties of natural numbers, integers, rational numbers, and real numbers. They are essential for learning of arithmetic, algebra and geometry and their study begins from Class I and continues till Class VIII. It is, thus, assumed that the student who enters Class IX, i.e., the secondary stage, has a working knowledge of all these types of number systems.

This unit introduces the system of real numbers along with its sub-systems. The relationship among these systems is sketched so that there are no gaps in their treatment. Structure of natural numbers and different subset of natural numbers will be also discussed in this unit. We shall explain interesting aspects
of some subsets of the set of natural numbers. The concepts are explained and illustrated with examples drawn from familiar situations. It is expected that you will be able to plan their lessons more effectively after going through this unit.

10.2 OBJECTIVES

At the end of this unit, you will be able to:

• understand the system of real numbers and the relationships between its various sub-systems;
• understand algorithms and the routine processes of carrying out computations;
• illustrate with diagrams (on a number line) clearly showing the meaning and relationships between the four fundamental operations on real numbers;
• help students to acquire an in-depth knowledge of the system of real numbers and its application in day to day life;
• state and interpret the meanings of different subsets of natural numbers;
• explain basic concept of number theory;
• exemplify the various situations in which processes such as factorization, LCM and HCF are applied; and
• use various child-centred methods that help children to understand process of factorization, LCM and HCF.

10.3 NUMBER SYSTEMS

It may be noted that concept of ‘number’ is different from that of ‘numeral’. Thus, the teaching of the number system incorporates the idea of numeration, but is distinct from it. We shall assume the knowledge of the base-ten numeration system, while discussing number systems in this unit.

Further, we shall use the idea of ‘set’, which is basic and central to all mathematical operations. The language of sets simplifies and clarifies explanations. It is also assumed that before reaching the secondary stage, students have sufficiently worked with numbers and are familiar with basic ideas, at least intuitively. The basic knowledge of concepts such as one-to-one matching, counting, addition, subtraction, multiplication, division, fractions, decimals, etc. on the part of readers has been assumed. If necessary, teachers may recapitulate these ideas before teaching this Unit.

10.3.1 Sets of Numbers

Main Teaching Point: The relationship between different sets of numbers.

Teaching-Learning Process:

1) Let pupils recall that ‘fractions’, ‘negative numbers’, ‘zero’, ‘decimals’, etc., are different kinds of numbers. Ask them to give examples/instances to illustrate their use.

2) Encourage pupils to observe that these different kinds of numbers overlap and are related to each other. Ask them to arrange these numbers in order starting with the simplest. A diagram such as the following may be useful.
3) Discuss the above diagram with the help of questions such as:

- Are natural numbers positive?
- Is it true that \{fractions\} \cap \{integers\} = \emptyset?
- Why is zero on the dotted line separating positive and negative numbers?
- Is there a one-to-one correspondence between the set of positive integers and the set of negative integers?

**Methodology:** The discussion method is used to bring out the relationship between different sets of numbers using diagrams and their analysis. A quiz organized in the class can be an effective technique for recapitulation.

### 10.3.2 Natural Numbers

**Main Teaching Point:** Properties of natural numbers

**Teaching-Learning Process:**

The teaching of natural numbers can be approached in two ways, either by following the history of the development of number concept, or by simulating the same through activities in the classroom. In both the cases, the key idea to be explored is a one-to-one correspondence between objects in a group and notches or tally marks to keep a record of the sizes of group.

To any given finite set we attach a label, called a number. The same label is attached to all those sets which are in one-to-one correspondence with the given set. Through activities – playing with blocks, crayons and other objects – two primary number concepts, cardinal and ordinal may be developed. A cardinal number tells the size of a group and an ordinal number tells the place of a member of the group in the sequence of numbers (which is used in counting). The natural number is an idea (or a mental tool), which helps in telling ‘how many objects are in a group’ or the size of the group.

Natural Numbers (N) : 1, 2, 3, 4, 5, .... ................. are also called counting numbers. They start with the ‘unit’ or ‘one’ and are successively obtained by using the idea of ‘one more’ or the successor. The set of natural number has no largest number.
Counting is matching a given finite set and one of the natural number sets – \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, etc. in which every corresponds to an element in the given set. The process of counting the objects in any set will result in the same cardinal number, no matter how we arrange the objects.

**Methodology:** The discussion method is used. The properties of natural numbers are elicited from students using inductive reasoning and by giving illustrations.

### 10.3.3 Zero and Integers

**Main Teaching Point:** Properties of Integers

**Teaching-Learning Process:**

The number zero ‘0’ is introduced as (i) the number associated with the empty set \(\emptyset\) or \{ \}. This can be done through an activity. For example, a child could be given say 5 chocolates and asked to eat all of them one by one so that finally none is left. Since the empty set is a proper subset of every non-empty set, zero is less than any natural number \(k\), that is \(0 < k\), (ii) the starting point in arranging the objects in a sequence (on a line), and (iii) an answer to questions like \(6 - 6 = ?; 15 - 15 = ?\), which can be demonstrated through counting backwards or taking away. Zero completes counting both ways – ahead and backwards. The set of numbers consisting of zero and natural numbers is called the set of **whole numbers** (W).

**Whole Numbers:** \{0, 1, 2, 3, 4, 5, ………..\}

Zero is the smallest whole number. There is no greatest whole number.

A more inclusive set than W is the set of **Integers** (I). The set of integers can be introduced through activities involving (a) opposites such as profit and loss, rise in level and fall in level earning and spending etc. or (b) direction – high and low, east and west, north and south, etc. The integers consist of negative integers, zero and positive integers. Thus, the set of integers (I) may be presented as:

**I : \{ ……. ……. -4, -3, -2, -1, 0, +1, +2, +3, + 4 ……. ……. \}**

The number zero is not included in the set of positive integers or in the set of negative integers. The set of integers has no smallest or largest number. It extends to infinitely in both directions. The natural numbers are assigned the positive (+) sign and called positive integers. For every positive integer (+a), we have a negative integer (–a) such that:

\[ (+a) + (–a) = 0 \]

The extension to negative numbers can be demonstrated by presenting a pattern in subtraction.

<table>
<thead>
<tr>
<th>5</th>
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<th>5</th>
<th>5</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-2</td>
<td>-3</td>
<td>-4</td>
<td>-5</td>
<td>-6</td>
<td>-7</td>
<td>-8</td>
</tr>
</tbody>
</table>

| 4 | 3 | 2 | 1 | 0 | A | B | C |

Clearly A is one less than 0, B is one less than A and so on. Further A is such that when 1 is added to it, we get zero, B is such that when 2 is added to it, we get zero.
Thus, A is the opposite of 1 and B is the opposite of 2 etc. These can be represented by –1 and –2. The other negative integers may also be explained in the same way.

Thus, (1) + (–1) = 0, (+2) + (–2) = 0. But we know that 1 – 1 = 0 and 2 – 2 = 0, so we can replace +(–1) by –1 and +(–2) by –2. By convention, we do not put a sign with positive numbers +(+1) =1 and –(+1) = –1, etc.

The set of integers is closed under addition, subtraction and multiplication. But while trying division, we find \( \frac{3}{4}, \frac{15}{27}, \frac{200}{327} \) etc., are not integers, that the system which is not closed under division. We, therefore, extend number systems further and define “rational numbers” (Q).

Methodology: The discussion-cum-lecture method is used. The properties of integers are enlisted by giving illustrations and using inductive reasoning.

### Check Your Progress

**Notes:**

a) Write your answers in the space given below.

b) Compare your answers with those given at the end of the unit.

1) Is the sum, difference, product, quotient of two given integers always an integer?

2) Illustrate with examples the rules for multiplication of integers.

3) Show that the square of an even integer is even and the square of an odd integer is odd.
10.3.4 Rational and Irrational Numbers

Main Teaching Points:

a) Need for rational numbers
b) Need for irrational numbers

Teaching-Learning Process:

Numbers which can be expressed in the form $\frac{p}{q}$, where $p$ and $q$ are integers and $q \neq 0$, are called rational numbers ($Q$). Fractions are a subset of rational numbers. Rational numbers (or fractions) can be easily introduced through activities involving equal sharing or partitioning of sets. These are interpreted as a part of a whole; a part of a group; indicated as division and as a ratio. In fact, the word rational comes from the word ‘ratio’. The same number can be written in many ways as a ratio of integers. For example:

\[
\begin{align*}
3 &= \frac{+3}{+1} = \frac{+6}{+2} = \frac{-6}{-2} = \frac{-9}{-3} = \frac{+15}{+5} \\
2 &= \frac{4}{6} = \frac{6}{9} = \frac{8}{12}
\end{align*}
\]

Since any integer ‘n’ can be written as $n/1$, integers form a subset of the set of rational numbers. It can be easily demonstrated that the set of rational numbers is closed under addition, subtraction, multiplication and division (provided we do not divide by zero). It is also closed under the operation of finding the power of a number (i.e., squaring, cubing, etc.). But, it is not closed when we try to find roots – square root, cube root, etc. Sometimes, the root of a rational number is itself a rational number. Sometimes, it has a rational approximation, but cannot be written exactly in the form of a ratio.

i.e. $\frac{p}{q}, q \neq 0$

Such numbers are called irrational numbers.

The numbers etc. $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{12}$, are irrational numbers.

Another way of describing irrational numbers as distinct from rational numbers is to consider the decimal form of numbers. It is possible to prove that for each rational number the pattern of digits in the decimal representation either terminates, e.g., $\frac{1}{20} = 0.05$ or recurs, e.g., $\frac{1}{7} = 0.142857142857\ldots$

In case of irrational numbers the decimal representation neither terminates nor recurs, e.g., $2.131133133331\ldots$

For some numbers such as $\sqrt{2}, \sqrt{3}$ etc., it is easy to prove that they are irrational, while for other numbers such as $\pi$, it may be quite difficult. The proof for ‘$\sqrt{2}$ is irrational’ is of special importance as we use the method of contradiction: we assume that it is rational and show that this leads to a contradiction.

Assume $\sqrt{2} = \frac{m}{n}$ where $m$ and $n$ are coprime integers, that is they do not have a common integral factor (other than 1 or −1).
Now \( \sqrt{2} = \frac{m}{n} \Rightarrow 2 = \frac{m^2}{n^2} \Rightarrow m^2 = 2n^2 \) ..... (1)

\[ \therefore m^2 \text{ is even} \quad \Rightarrow m \text{ is even.} \]

(Since the square of an odd integer is odd and that of an even integer is even).

Now let \( m = 2p, \ p \in I \)

\[ m^2 = 4p^2 \quad \text{(from (1) above)} \]

\[ 2n^2 = 4p^2 \]

\[ n^2 = 2p^2 \]

\[ \therefore n^2 \text{ is even} \quad \Rightarrow n \text{ is even} \]

or \( n = 2q, \ q \in I \)

Hence, \( m \) and \( n \) are both even i.e., they have a common factor 2, which contradicts the initial assumption.

Hence, \( \sqrt{2} \) is an irrational number.

**Methodology:** Illustrations are given and mainly the lecture cum discussion method is used for showing the need for rational numbers, while the deductive method is used to prove that certain numbers are not rational.

---

**Check Your Progress**

**Notes:**

a) Write your answers in the space given below.

b) Compare your answers with those given at the end of the unit.

4) **Is the sum, difference, product, quotient of two given irrational numbers always a rational number?**

5) **Is the sum of two irrational numbers always irrational?**

6) **Is the product of two irrational numbers always irrational?**
10.3.5 Real Numbers – Operations and their Properties

Main Teaching Points:

a) Properties of real numbers
b) Concept of absolute value of a number

Teaching-Learning Process:

The union of the sets of rational numbers and irrational numbers is called the set of real numbers.

\[
\{ \text{Real numbers} \} = \{ \text{Rational numbers} \} \cup \{ \text{Irrational numbers} \}
\]

Most of the numbers used in multiplication at school stage are real numbers. But, there are still some roots that are not irrational numbers, e.g. \( \sqrt{-9}, \sqrt{-1} \).

These are different kinds of numbers, sometimes called non-real numbers. These are studied in senior secondary classes. The union of real and non-real numbers is called the set of complex numbers. The diagram below gives the sequence of development of the number system from natural numbers to real numbers.

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Number System</th>
<th>Closed under Operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Natural numbers (N)</td>
<td>addition, subtraction, multiplication</td>
</tr>
<tr>
<td>2.</td>
<td>Whole numbers (W)</td>
<td>addition, subtraction, multiplication</td>
</tr>
<tr>
<td>3.</td>
<td>Integers (I)</td>
<td>addition, subtraction, multiplication</td>
</tr>
<tr>
<td>4.</td>
<td>Rational numbers (Q)</td>
<td>addition, subtraction, multiplication</td>
</tr>
<tr>
<td>5.</td>
<td>Real numbers (R)</td>
<td>addition, subtraction, multiplication</td>
</tr>
</tbody>
</table>

The study of the properties of real numbers is one of the major aspect of curriculum at the secondary level. It is, therefore, necessary that the meaning of the operations of addition, subtraction, multiplication and division be explained through examples and their properties be inductively generalized.

Addition, subtraction, multiplication and division operations are binary since we carry them out on two numbers. Addition is counting forward, multiplication is repeated addition, subtraction is the inverse of addition and division is repeated subtraction. The properties of real numbers are listed in the next table.
The Properties for Real Numbers

1. Closure Laws  
   For all $a, b \in \mathbb{R}$  
   $a + b$ is a real number.  
   $ab$ is a real number.

2. Commutative Laws  
   $a + b = b + a$  
   $ab = ba$

3. Associative Laws  
   $(a+b) + c = a + (b + c)$  
   $(ab)c = a(bc)$

4. Identity Laws  
   There is a unique real number 0, such that  
   for all $a \in \mathbb{R}$, $a + 0 = a = 0 + a$  
   There is a unique real number 1, such that  
   for all $a \in \mathbb{R}$, $a \times 1 = a = 1 \times a$

5. Inverse Laws  
   For all $a \in \mathbb{R}$, there is a unique real number $-a$ such that  
   $a + (-a) = 0 = (-a) + a$  
   For all $a \in \mathbb{R}$ (except 0), there is a unique real number $\frac{1}{a}$ such that  
   $a \left( \frac{1}{a} \right) = 1 = \left( \frac{1}{a} \right) a$

6. Distributive Law  
   For all $a, b, c \in \mathbb{R}$  
   $a(b+c) = ab + ac$

The Order Laws for Real Numbers

7) The Trichotomy Law  
   If ‘a’ is a real number then out of the three statements given below, exactly one is true.  
   (i) $a = 0$, (ii) $a$ is positive, (iii) $-a$ is positive

8) For any two real numbers $a$ and $b$, exactly one of the following three statements is true:  
   (i) $a < b$, (ii) $b < a$, (iii) $a = b$

These laws are used to prove many results, which have been taught as rules. Two simple rules are proved here.

1) For any real number $a$, $a \cdot 0 = 0 = 0 \cdot a$  
   Proof: $1 + 0 = 1$ (identity law)  
   $a(1 + 0) = a \cdot 1$  
   By distributive law  
   $a \cdot 1 + a \cdot 0 = a \cdot 1$  
   But, identity law is  
   $a \cdot 1 + 0 = a \cdot 1$  
   Hence, by uniqueness of 0, $a \cdot 0 = 0$

2) For any two numbers $(-a) + (+a) = 0$. (Inverse law), multiplying by $(-b)$ we get  
   $(-b)(-a) + (-b)(+a) = 0$  
   Now, $(-b) \times (+a) = (-ba)$. (To be proved earlier)  
   $(-b)(-a) + (-ba) = 0$
By using inverse law we argue that:

\((-b) \times (-a)\) should be opposite of \((-ba)\).

Hence \((-b) \times (-a) = +ba\).

An important concept, which should be made clear through examples is the absolute value of a real number. Recall the idea of opposites and ask pupils to write additive inverses (using signed numbers).

The absolute value of numbers that are additive inverses of each other is the same. We use the symbol \( | | \) for absolute value. Thus,

\[
\left| \frac{-2}{3} \right| = \frac{+2}{3} ; | -5 | = | +5 | = +5 \quad \text{etc.}
\]

In general, for all \( x \),

\[
|x| = \begin{cases} 
  x, & \text{if } x \geq 0 \\
  -x, & \text{if } x < 0 
\end{cases}
\]

Note that the last line means that the absolute value of a negative number is a positive number \(|-3| = -(3) = +3\).

**Methodology used:** Properties are enlisted on the blackboard by putting questions to the students regarding them. The concept of absolute value is illustrated through examples.

### Check Your Progress

**Notes:**

a) Write your answers in the space given below.

b) Compare your answers with those given at the end of the unit.

7) Prove that:

a) \( a (b - c) = ab - ac \)

---

b) \(-[a + (-b)] = -a + b\)

---

c) \((a + b) (a + b) = a^2 + 2ab + b^2\)
8) Illustrate the meaning of
   a) Additive inverse
      .................................................................................................................
      .................................................................................................................
      .................................................................................................................
   b) Multiplicative inverse
      .................................................................................................................
      .................................................................................................................
      .................................................................................................................

9) If, a, b and c are real numbers and a > b, which is larger (a) a + c or b + c (b) ac or bc?
    .................................................................................................................
    .................................................................................................................
    .................................................................................................................
    .................................................................................................................
    .................................................................................................................

10) If a > b > 0, which is larger (a) $\sqrt{a}$ or $\sqrt{b}$, (b) $\frac{1}{a}$ or $\frac{1}{b}$?
     .................................................................................................................
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11) Is each of the following true or false?
    a) $\mathbb{R} = \mathbb{R} - \{0\} \cup \mathbb{R}^+$
    b) $\mathbb{R}^+ \cap \{0\} = \emptyset$
    c) $\mathbb{R}^- \cap \{0\} = \emptyset$
    d) $\mathbb{R}^+ \cap \mathbb{R}^- = \emptyset$
     .................................................................................................................
     .................................................................................................................
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10.3.6 The Number line

Main Teaching Point: Representation of real numbers on the Number line

Teaching-Learning Process:
In recent years the idea of Number line has been used as a very effective aid to familiarize pupils with the properties of the number system.

Draw a horizontal line and mark on it a set of evenly spaced points to represent the integers, this line is called the Number line.
Given a rational number, we can mark a point representing it by using a scale or a geometrical construction. The approximate rational value of an irrational number can be marked as a point by measurement, but the exact position of a point representing an irrational number can be marked by using a geometrical construction. Thus, for every real number we can mark a point on the number line. The order relation is also indicated by movement on the Number line as shown below:

\[\ldots -5 < -4 < -3 < -2 < -1 < 0 < 1 < 2 < 3 < 4 \ldots\]

Of the two consecutive numbers, the one on the right is greater than the one on the left. Thus, 8 is greater than 6. We write \(8 > 6\); 0 is greater than all the negative numbers. Sets of numbers can be shown by marking corresponding points.

**Examples**

(i) \([-3, -2, 0, 1, 2]\)  
(ii) \(\{\text{Real numbers from } -3 \text{ to } 4\}\)

(iii) \(\{\text{Positive real numbers}\}\)  
(iv) \(\{\text{Multiples of } 2\}\)

• means point is included. 0 means point is not included. The thick part of line shows continuity between finite boundaries. Infinite sets are shown by an arrow at one end or both ends.

Again we associate numbers with displacements, +5 is a number which is 5 units away from 0 and on the positive (i.e. right) side of it. We show displacement with a segment and direction by putting an arrow head to the segment. Thus, any number is represented by a **directed line segment**. To represent the sum of two numbers on a Number line we draw the first arrow with its tail at the origin, the second with its tail at the head of the first.

**Examples**

(i) \(4 + 3 = 7\)  
(ii) \(4 + (-3) = 1\)

(iii) \(-4 + (-3) = -7\)  
(iv) \(-4 + 3 = -1\)

The segment representing the answer has been shown by a dotted line. There is no need to do subtraction on the Number line because the subtraction sum can be changed to an addition sum.

\[4 - 3 = 4 + (-3) = 1, \quad 4 - (-3) = 4 + 3 = 7\]

Multiplication can be represented as repeated addition. Division can be represented as repeated subtraction. Thus, the Number line is a useful aid to illustrate the properties of rational numbers. The Number line can also furnish an excellent model for portraying the meaning of directed numbers. The
negative number is designated as a direction-changing operator. Different directional situations can be described by different rules.

The irrational numbers such as $\sqrt{2}, \sqrt{3}$, can be represented with the help of geometrical constructions using the Pythagoras theorem. Draw a line segment of unit length. With this as the base erect a right-angle triangle of altitude 1 unit. The hypotenuse of this triangle will be of length $\sqrt{2}$. Now use this hypotenuse as base and on it erects another right-angle triangle of altitude 1. The hypotenuse of this triangle is $\sqrt{3}$ and so on.

Methodology: Recapitulating the method used for dividing a line segment into number of parts, the skill of plotting the numbers on a Number line is developed by the practice method using proper geometrical instruments.

Check Your Progress

Notes: a) Write your answers in the space given below.
    b) Compare your answers with those given at the end of the unit.

12) Pick out pairs of numbers from {$-3, -2, -1, 0, 1, 2, 3$}, construct addition sums and draw Number line graphs. Verify the commutative and associative laws.

13) Draw a Number line graph to represent-
    i) {Real numbers greater than $-5$ and less than $5$}
    ii) $x < 30$
    iii) $x > 30$

10.4 EXPONENTS AND LOGARITHMS

Computation is an important aspect of Mathematics. In many problems from real life, we find large numbers and that too connected by complicated terms, formed from monomials. Suppose we have a problem involving an expression like:

\[
\frac{53296 \times 32847 \times 5 \times 10^2}{243941 \times 27}
\]
The usual process of computation may take a long time and cause frustration. Why should numerical complexities be imposed upon the students? Many devices to facilitate the solving of such problems have been developed and logarithm is one of these.

10.4.1 Exponents, Power and Root

Main Teaching Points:

a) Properties of exponents
b) Power and root as inverse processes

Teaching-Learning Process:

Power Notation

Number = (Base)$^\text{Exponent}$

$16 = 2^4$

2 is the base

4 is the exponent

16 is 4$^{th}$ power of 2

The exponent tells how many times the base is taken as a factor. Successive multiplication leads to the idea of power.

$2^1 = 2 = 2$ first power of 2

$2^2 = 2 \times 2 = 4$ squared or second power of 2

$2^3 = 2 \times 2 \times 2 = 8$ cubed or third power of 2

$2^4 = 2 \times 2 \times 2 \times 2 = 16$ fourth power of 2

$\ldots$ $\ldots$ $\ldots$

$2^n = 2 \times 2 \times \ldots \times 2$ (n factors) = $2^n$ nth power of 2

$x^n$ means a variable/literal number for which a number can be substituted has been raised to nth power.

In teaching of powers in the beginning, n should be taken as a positive integer, then a negative integer, and lastly, a rational number.

For any real number x the following laws hold:

1) $x^m \times x^n = x^{m+n}$

2) $x^m/x^n = x^{m-n}$

3) $(x/y)^n = x^n/y^n$

4) $x^0 = 1$

5) $x^{-n} = 1/x^n$

We can also consider $x^{m/n}$ where $m/n$ is rational.

What does $x^{m/n}$, $n > 0$ mean

Now, let $y = x^{m/n}$ and so $y^n = x^m$ or $y = \sqrt[n]{x^m}$

It is always possible to find nth root, $\sqrt[n]{a^m}$ is the positive $n^{th}$ root of $a^m$. 
The inverse operation of taking a power of a number is that of taking the corresponding root. A square root is indicated by the radical symbol $\sqrt{}$. Thus, $\sqrt{16} = 4$. With higher roots, an index is placed on the radical to show the order of the root.

Square

$$\sqrt{27}; \sqrt[3]{16}; \sqrt[5]{243};$$ etc., represent 3rd root of 27, 4th root of 16 and 5th root of 243 respectively.

The students should be exposed to a large number of examples so that they grasp the methods of finding roots by observing the pattern. The easiest method is that of finding roots by prime factorisation. Another method for finding the square root is called ‘division method’ of ‘square root algorithm’ by which the successive digits of the root are found one by one. Sometimes, it is difficult to explain why the rule works. A proper demonstration with proper explanation at each step is, therefore, necessary.

In short, in the field and laboratory, roots are read from tables or on slide rule, or occasionally are computed by the rule or division method. Secondary level students should be acquainted with the table of roots and the methods of interpolation.

Methodology: Inductive method is used to arrive at the different properties of exponents and a number of illustrations are given so that students are able to grasp the pattern.

10.4.2 Indices and Logarithms

Main Teaching Points:

a) Indices and logarithms as inverse processes

b) Properties of logarithms

Teaching-Learning Process:

We have considered sets of powers of various numbers. To give meaning to logarithm, closer attention is given to the indices (or exponents) themselves. The logarithm of a number is the index to which the base is raised to get the number. For example, the logarithm of the number 243 to the base 3 is 5, because $3^5 = 243$. This is abbreviated as $\log_3 243 = 5$.

Each statement in the logarithmic form has its equivalent statement in the index form.

$\log_3 243 = 5 \quad 3^5 = 243$

$\log_2 64 = 6 \quad 2^6 = 64$

Early in the seventeenth century, Henry Briggs expressed each of the natural numbers from 1 to 20,000 and 90,000 to 100,000 in the form $\log_{10^n}$ where n was calculated correct to 14 decimal places. Adrian VIacq completed the table for natural numbers between 20,000 and 90,000. The calculations were the
basis of tables of logarithm. The first practical system of logarithms was published by Napier in 1614. The word logarithm is derived from the Greek word ‘logos’ meaning ratio.

In our calculations, we use logarithms to the base 10 or common logarithms only. In practice, therefore, we do not write the base, we simply write \( \log 10 = 1; \log 100 = 2 \) etc. while calculating.

Since logarithm is really the index corresponding to a number, we can write index laws using logarithmic notation. Thus:

i) \( \log xy = \log x + \log y \)

ii) \( \log x/y = \log x – \log y \)

iii) \( \log x^m = m \log x \)

iv) if \( x = a^n \), \( n = \log_a x \), then \( x = a^{\log_a x} \)

Since we use base 10, it is necessary to introduce the idea of expressing a number in standard form. Thus,

\[
\frac{5636}{1000} = 5.636 \times 10^3;
\frac{349}{1} = 3.49 \times 10^2
\]

\[
56.36 = 5.636 \times 10^1; .002 = 2.0 \times 10^{-3}
\]

Any number, ‘\( n \)’, can be expressed as \( n = m \times 10^p \), where \( p \) is an integer (positive, zero or negative) and \( 1 < m < 10 \). This is called the standard form of any number, ‘\( n \)’.

The index ‘\( p \)’ in the standard form gives the integral part of the logarithm. This is called the characteristic. The decimal part of the logarithm is taken from a table. This is called the mantissa.

**Methodology:** The discussion cum lecture method is used to make the concept understood by students. Many examples are given to illustrate \( b^n = m \iff \log_b m = n \). A suitable drill exercise may be provided.

### 10.4.3 Use of Logarithm Tables

**Main Teaching Points:**

a) Reading mantissa from logarithm tables

b) Use of logarithm tables for calculations

**Teaching-Learning Process:**

Sufficient practice is required in writing the characteristics and reading the mantissa from the table. Examples such as the following may be given:

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>61</td>
<td>.7853</td>
<td>.7860</td>
<td>.7868</td>
<td>.7875</td>
<td>.7882</td>
<td>.7889</td>
<td>.7896</td>
<td>.7903</td>
<td>.7910</td>
</tr>
</tbody>
</table>

To find \( \log 6.176 \), we note that \( \log 61 \) has the decimal number column headed by 7, where the decimal number is .7903. \( \log 6.176 \) is obtained by adding to .7903 the number 4, actually .0004, found by noting that 4 is the number headed by 6 in the difference column. This gives \( \log 6.176 \) as .7907. Since 6 lies between 1 and 10, \( \log 6.176 = .7907 \).
Log 61.76 = 1.7907 (61 lies between 10 and 100), log 617.6 = 2.7907 (617 lies between 100 and 1000). It should be emphasized that (a) the mantissa always lies between 0 and 1, and hence the mantissa is always positive, (b) to find the characteristic we have only to count the number of digits in the integral part of n and subtract 1 from it, provided n > 1. For n < 1, we count the number of zeros after the decimal place to the first non-zero digit and add 1 to it. The number so obtained with a negative sign gives the characteristic part of the logarithm.

The inverse process of finding the number whose logarithm is given is called finding the anti-logarithm. We use the symbol antilog to denote the phrase ‘the antilogarithm of ...’. We have a separate table for finding the antilogarithm. In using the table of antilogarithms, we disregard the characteristics and make sure that the fractional part of the given number is indeed positive. We then find the number by the use of the mantissa and finally insert a decimal point by rule of the characteristics.

In the teaching of logarithm, sufficient practice should be given in the reading of tables. The fact that the mantissa is always positive should be carefully emphasized. Separate examples for multiplication, division and finding roots should be worked out. Speed and accuracy should be the final goal in calculating with logarithm.

**Methodology:** Every student must possess a logarithm table when this is discussed in the class. To develop skill of using logarithm tables, sufficient practice should be provided. Basically, the drill method is required.

**Check Your Progress**

**Notes:**

a) Write your answers in the space given below each question.

b) Compare your answers with those given at the end of the unit.

14) Is \(2^5 = 5^2\)? Is \(3^2 = 2^3\)? Is the operation ‘to the power of’ commutative? If not, can you find the conditions for which the statement \(a^m = m^n\) is true.

15) Is the statement \(a^2 + a^2 = a^4\) true for all values of ‘a’? Can you find any value of ‘a’ (real number) that makes this statement true?

16) Find the values of the following logarithms by first writing down a statement in index form: e.g., for \(\log_2 32, 32 = 2^5, \log_2 32 = 5\).

(i) \(\log_2 512\) (ii) \(\log_{11} 1331\) (ii) \(\log_7 343\) (iv) \(\log_5 625\)
10.5 ELEMENTARY NUMBER THEORY

10.5.1 Basic Concepts

Main Teaching Points:

a) What is a factor?

b) Even and odd numbers and their properties.

Teaching-Learning Process: The unit should be introduced by reviewing fundamental operations on natural numbers in such a manner that a pattern strikes the imagination of pupils and they are able to formulate a new generalization which can be named as a new definition or property. The procedure to develop this ability is outlined in the Unit.

A teacher should ensure that:

1) Induction is planned for and encouraged.

2) Deduction is urged. Some additional reasoning is necessary to arrive at a generalization. Also, there are no stereotype patterns of formulating “proofs”. If the essential ideas which the student develops are correct and flawless in reasoning, his/her “proof” should be accepted as a good one.

Some activities which set the stage for deductive arguments are given for a few selected topics:

Activities

1) Ask: What is $3 \times 4$?

   Write the answer ‘$3 \times 4 = 12$’
   3 multiplied by 4 is 12
   3 is a divisor of 12

   Read it (as shown)
   4 is a divisor of 12
   in many ways
   3 is a factor of 12
   4 is a factor of 12
   12 is a multiple of 3
   12 is a multiple of 4

   Ask: What other numbers can be multiplied to get 12?

   Explain: In a product, each of the numbers is called a factor? A factor is a divisor.

2) Ask: Repeat the above process for some other natural numbers, such as 24, 30, 36, 42, 48, 64, (If necessary students may refer to multiplication tables) $6 \times 4$, $8 \times 3$, $12 \times 2$, $24 \times 1$ are the factorizations of 24.

   $6 \times 4 = 24$, $8 \times 3 = 24$
   $12 \times 2 = 24$, $24 \times 1 = 24$
   $5 \times 6 = 30$, $10 \times 3 = 30$
   $15 \times 2 = 30$, $30 \times 1 = 30$

   Explain: This process of finding a factorization is called ‘factoring’. Point out that factors of natural numbers may be found in several different ways. Give more exercises in factoring.
3) **Ask:** Does 3 have one as a factor? Do 5, 7, 9 and 11 etc. have one as a factor?

What about 24, 30, 36, etc.? Does each of them has 1 as a factor?

Bring out (by induction on the part of students) that every natural number has 1 as the factor.

**Ask:** What is the product of any natural number (say \( n \)) and 1? How can it be expressed in symbols? If \( n \) is any natural number then what is \( n \times 1 \)?

Demonstrate with examples that ‘\( n \times 1 = n \)’ becomes true whenever a numeral for any natural number is put in place of \( n \).

\[
30 \times 1 = 30, \quad 55 \times 1 = 55, \quad 82 \times 1 = 82.
\]

Bring out that every natural number has ‘itself’ as well as ‘one’ as its factor. The number \( n \) has at least two factors \( n \) and 1 or \( n = nx1 \).

4) **Ask:** Think of some number that has 2 as a factor. Write it on the blackboard. Bring out that this set of natural numbers is called the set of even numbers. An even number is one which has the factor 2. Write numerals for some large natural numbers, some even, some odd.

**Ask:** Can you tell me which of these are even? Observe the digit at the unit’s place.

Bring out that those numbers which have digit 2 at unit’s place such as 2, 4, 6, 8 or 0, are even and the others are odd.

**Ask:** Suppose we multiply a natural number by an even number. What do we get? Try a few examples and write them on the board. Help students to reach the conclusion (inductively) that such products are always even.

**Ask:** Can we be sure that this is always true? We tried it for only a few cases. Can we reason this out for all natural numbers?

Let students give their arguments. If none of the arguments produced is good enough, bring out the associative law and suggest them to try using it. Any even number has 2 as a factor, and therefore, \( 2 \times a \), is even where ‘\( a \)’ represents any natural number. Then \( (2 \times a) \times b = 2 \times (a \times b) \) for any natural number ‘\( b \)’. We see that this has 2 as a factor.

This informal deduction should be encouraged by using any valid arguments proposed by students. It must be insisted that they argue in such a way that what they say is true for all natural numbers and not just for a finite number of them.

Point out that this discussion shows that the set of even numbers is closed for multiplication operation.

In the same manner, encourage students to discover that the set of even numbers is closed under the operation of addition:

\[
(2.a) + (2.b) = 2.(a + b)
\]

Again use first induction, then informal deduction, to enable students to discover generalizations about operations with odd and even numbers so that the study of numbers becomes a more creative experience.
5) Addition of odd and even numbers
   
a) Even numbers added to even numbers:

\[
\begin{array}{cccc}
2 & 6 & 2 & 230 \\
+ & 4 & 4 & 512 \\
\hline
6 & 8 & 12 & 108 \\
\hline
18 & 18 & 850 & \\
\end{array}
\]

Elicit from pupils: It can be observed that the sum of any number of even numbers is also even.

b) The sum of two odd numbers:

\[
\begin{array}{cccc}
7 & 13 & \text{Ask pupils to find a proof.} \\
+ & 3 & 11 & \text{Proof: Any odd number is an even number plus one.} \\
\hline
10 & 24 & \\
\end{array}
\]

\[
\begin{array}{cccc}
7 = 6 + 1 & 13 = 12 + 1 \\
3 = 2 + 1 & 11 = 10 + 1 \\
8 + 2 & 22 + 2 & \text{(an even number)} \\
\end{array}
\]

Elicit from pupils: The sum of odd natural numbers is an even number. In the same manner get the generalization: \textbf{The sum of an odd and an even number is an odd number.}

6) Multiplication of an odd and an even number

Review the facts that:

a) Multiplication is a shorter form of addition.

b) Multiplication of an even number by an even number.

c) Multiplication of an odd number by an even number.

Let pupils observe that this amounts to adding an odd number to itself even number of times. Give examples.

Elicit: An odd number multiplied by an even number yields an even number.

d) Multiplication of an even number by an odd number

Let pupils observe that this is the same as adding an even number to itself odd number of times. Give examples.

\textbf{Elicit: An even number multiplied by an odd number yields an even number.}

Similarly, let pupils get the generalization. \textbf{The product of two odd numbers is an odd number.}

\textbf{Methodology:} Mainly the inductive method is used. The deductive method is used to write down the proofs of some properties of even and odd numbers.
Check Your Progress

Notes: a) Write your answers in the space given below each question.
   b) Compare your answers with those given at the end of the unit.

17. A number $x$ multiplied by any other number $y$ always gives a product as $y$. What is the value of $x$?
   ........................................................................................................................................
   ........................................................................................................................................
   ........................................................................................................................................
   ........................................................................................................................................
   ........................................................................................................................................

18. Prove that the sum of two consecutive numbers always gives an odd number.
   ........................................................................................................................................
   ........................................................................................................................................
   ........................................................................................................................................
   ........................................................................................................................................
   ........................................................................................................................................

19. Show that the product of two even numbers is divisible by 4.
   ........................................................................................................................................
   ........................................................................................................................................
   ........................................................................................................................................
   ........................................................................................................................................
   ........................................................................................................................................

10.5.2 Prime Factorization: HCF, LCM

Main Teaching Points:
   a) Prime factorization of a number
   b) Finding HCF and LCM using prime factorization

Teaching-Learning Process

1) **Ask:** Write the factors of the numbers 1 to 10. Think of numbers that have exactly two different factors: 1 and the number itself.
   Explain that these are called ‘prime numbers’. Explain that numbers with more than two different factors are called ‘composite numbers’.
   **Ask:** Have students examine prime numbers and find out how many of these are even.
   Explain that 1 is not a prime number since it does not have two different factors.

2) Write down numbers such as 36, 60, and 84. Have students write the various factorizations of these numbers. Use the factor tree and show how to complete the factorizations by continuing to factorize composite factors.
Ask students to repeat the process for several other numbers. Encourage them to tell what they notice.

Bring out that the final prime factors are the same except for their order (induction). Then, ask for a deductive argument that this is true for all natural numbers.

Explain: This is called the ‘unique factorization property of natural numbers’ – that every composite natural number can be factored into primes in only one way, except for order of presentation.

3) Explain – If two or more numbers have a certain number as a factor, we say the factor is common to the numbers. For example, 12 and 18 have 1, 2, 3 and 6 as common factors. Ask students to write common factors of different sets of numbers and pick out the ‘Highest or Greatest Common Factor’. Explain that this is called the Highest (or Greatest) Common Factor (H.C.F.).

Solve some more examples on finding common factors and picking out the highest common factor.

Point out that we can use prime factorization of each number and find out H.C.F. by multiplying all the common factors.

60 = (2 × 2 × 3) × 5
84 = (2 × 2 × 3) × 7
Common Factors are (2, 2, 3)
H.C.F. = 2 × 2 × 3 = 12

4) Ask students to recall the factors of some natural numbers. Explain that if one number is a factor of another then the second is called a ‘multiple’ of the first.

Ask students to write ‘multiples’ of several numbers, such as 3, 5, 7, 9, 12. Bring out the meaning of the common multiple of any two numbers.

Given two (or more) numbers, then any number which is a multiple of each of them is called a ‘common multiple’ of them.

Explain: Least Common Multiple (L.C.M.) of several numbers is the smallest number which is a multiple of each of them.

Multiples of 14 = 14, 28, 42, 56
Multiples of 21 = 21, 42, 63, 84
Common multiples = 42, 84, 126
L.C.M. = 42

Use prime factorization to find L.C.M.
If the numbers are 8, 14 and 21 and N is their L.C.M., then N is divisible by all the three numbers. We find a numeral in place of ? mark. N must have 2 × 2 × 2 as a factor. It should also have 2 × 7 as a factor and 3 × 7 as a factor.

We see 2 × 2 × 2 × 7 × 3 satisfy these conditions. Therefore,
L.C.M. = 2 × 2 × 2 × 7 × 3 = 168.
Methodology: Inductive reasoning is used to illustrate the method of finding H.C.F. and L.C.M. using prime factorization.

10.5.3 Division Method for Finding HCF, LCM

Main Teaching Point: To find HCF by division method.

Teaching-Learning Process: Euclid developed a method of finding H.C.F. that was based on the learning relation. His process is called ‘Euclidean Algorithm’.

The process should be demonstrated through different examples:

<table>
<thead>
<tr>
<th>Number 1</th>
<th>Number 2</th>
<th>Division Method for Finding H.C.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>84</td>
<td>270</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>84</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

Find H.C.F. of 84 and 270. Divide the smaller (84) into the larger (270). The remainder is 18. Now divide 18 into the first number 84. Repeat the process till the remainder is 0.

H.C.F. (84, 270) =
H.C.F. (18, 84) =
H.C.F. (12, 18) =
H.C.F. (6, 12) =

Explain the ‘why’ of the process. H.C.F. is the largest number that divides each of the two numbers. 270 = 84 \times 3 + 18. A divisor of both 270 and 84 must divide 18. 84 = 18 \times 4 + 12. A divisor of 84 and 18 must divide 12. 18 = 12 \times 1 + 6. A divisor of 12 and 18 must divide 6.

The last divisor is the H.C.F. ∴ H.C.F. = 6.

Methodology: The demonstration-cum-lecture method is used. The reason at every step should be discussed.

Check Your Progress

Notes:

a) Write your answers in the space given below each question.

b) Compare your answers with those given at the end of the unit.


21. Find the product of H.C.F. and L.C.M. in each of the following:
   i) 45 and 75
   ii) 140 and 490
22. By division method, show that two consecutive numbers are always coprime.

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10.5.4 Applications of HCF and LCM

Main Teaching Point: To solve word problems.

Teaching-Learning Process: A word problem is a device useful in teaching problem solving. These devices are used as a practice material upon which a student can apply the generalization or the principle already learnt. A student’s success is dependent on his recall and understanding of generalizations that suit the situation under consideration.

Ask students to read the problem, analyse the information given in the problem to locate what is known and to understand what is to be found out. Encourage students to translate the verbal statement into mathematical sentences to discover the procedure.

Example: A school had 851 boys and 629 girls. It divided the students into the largest possible number of separate classes for boys and girls which had an equal number of students. Find the number of classes.

Given: Number of boys = 851
Number of girls = 629

Analysis: Since the students are divided into largest possible equal classes, the size of each class is the H.C.F. of 851 and 629 i.e. 37.

Total number of classes \[ \frac{851 + 629}{37} = \frac{1480}{37} = 40 \]

Example: Six bells at intervals of 2, 4, 6, 8, 10, 12 seconds respectively. If they just now tolled together, when will they next toll together and how often will they toll together in 30 minutes?

Given: Six bells toll at intervals of 2, 4, 6, 8, 10 and 12 seconds respectively.

Analysis: The timings of tolling for each bell separately will have to be the multiples of their respective intervals. Hence, the timing of their tolling together is the L.C.M. of 2, 4, 6, 8, 10 and 12.
i.e.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>120 seconds</td>
<td>2</td>
</tr>
<tr>
<td>120 seconds = 2 minutes</td>
<td>2</td>
</tr>
<tr>
<td>No. of times they will toll together</td>
<td>3</td>
</tr>
<tr>
<td>In 30 minutes</td>
<td></td>
</tr>
</tbody>
</table>

\[
\frac{30}{2} + 1 = 16 \text{ times}
\]

L.C.M. = \(2 \times 2 \times 2 \times 3 \times 5 = 120\)

(Discuss why 1 more)

**Methodology:** The discussion method is used together with the Heuristic approach. It should be stressed that students translate the verbal statement into a mathematical sentence themselves.

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**Check Your Progress**

**Notes:**

a) Write your answers in the space given below each question.

b) Compare your answers with those given at the end of the unit.

24. Three cans of capacity 900 ml, 1.2 l and 1.5 l are to be filled with oil. What is the size of the largest container which will fill each of them a complete number of times.

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25. In a morning walk three persons step out together. If their steps measure 72 cm, 84 cm and 96 cm, at what distance will they again step together.

....................................................................................................................
....................................................................................................................
....................................................................................................................
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**10.6 LET US SUM UP**

The teaching of numbers starts with the arithmetic of natural numbers. This in turn helps in defining rational numbers and their arithmetic. By assuming that the rational numbers are ordered pairs \((p, q)\), where \(p\) and \(q\) are natural numbers, one can proceed to define real numbers in several ways. For example as a decimal, in which case both rational and irrational can be expressed as a sequence of decimal digits. One may, however, prefer to conceptualize real numbers as measuring numbers (magnitudes) based on the \(|l - l|\) correspondence with points of the Euclidean line. It may be noted that defining
real numbers in terms of rationals and irrationals is of great importance. It enables us free analysis from geometric intuition. In actual practice, the concept of real numbers may be taught as relative to (a) the structure of the real line and (b) its arithmetic and algebraic properties. In the former case (a), the real number is essentially treated as a geometric entity in the latter case (b), as the arithmetized entity based on the natural (via the rational) numbers. Through, the device of labeling points on a line with arithmetized real numbers the two concepts are usefully combined.

10.7 UNIT END ACTIVITIES

1) If \( n \in \mathbb{N} \), prove that \( n^2 + n \) is always even.

2) If \( n/13 < 5/17 \) find the greatest integral value for \( n \).

3) Find all fractions \( p/q \) between 0 and 1, where \( p \) and \( q \) are relatively prime natural numbers and \( 2 \leq q \leq 5 \). Find the mean value of all these fractions. Which is the largest and which is the smallest of these fractions?

4) Set up situations to show that:
   a) Each prime number greater than 3 is either 1 less or 1 more than a multiple of 6.
   b) There is an infinite number of twin primes (Twin primes are pairs of prime differing by 2 such as 3, 5, 5, 7; 11, 13; and so on).
   c) From 2 onwards, there is at least one prime number between any number and its double.

5) a) Ask pupils to examine the subtraction and division of odd and even numbers. Ask them to give generalizations by inspection and then work out a proof for the same.
   b) Have pupils test the effect of squaring on numbers.

6) Using the relation \( 84 = 60 + 24 \), give an argument to show that any divisor of 60 and 84 must also be a divisor of 24.

7) Show that any common multiple of a pair of numbers is a multiple of their L.C.M.

8) Determine two numbers, knowing their H.C.F. and their sum as given in the following table:

<table>
<thead>
<tr>
<th>Sum</th>
<th>72</th>
<th>360</th>
<th>552</th>
<th>420</th>
</tr>
</thead>
<tbody>
<tr>
<td>H.C.F.</td>
<td>9</td>
<td>18</td>
<td>24</td>
<td>12</td>
</tr>
</tbody>
</table>

9) Given any counting number value for \( n \) such as 50, find a formula for the sum of the first \( n \) numbers.
   \[ \text{Sum} = 1 + 2 + 3 + 4 + \ldots + n. \]
   Give the analysis of the nature of the problem and examine it to discover a pattern which should help in arriving at a formulation.

10) Euclid proved that whenever the natural number \( n \) is such that \( 2^{n+1} - 1 \) is a prime number, then multiplying this prime by \( 2^n \) produces a perfect number.

11) Explore Euler’s polynomial: \( n^2 - n + 41 \) to get prime numbers.
10.8 ANSWERS TO CHECK YOUR PROGRESS

1) The sum, the difference and the product of two integers is always an integer, but it is not necessary for the quotient of two integers to be an integer. For example \((-3) ÷ 2\) is not an integer.

2) The product of two positive integers is positive

\[
3 \times 5 = 15 \\
7 \times 2 = 14
\]

The product of two negative integers is positive

\[
(-3) \times (-5) = 15 \\
(-7) \times (-2) = 14
\]

The product of a positive and a negative integer is negative

\[
(-2) \times 7 = -14 \\
3 \times (-4) = -12
\]

3) Let \(2n\) be an even number.

\[
(2n)^2 = 4n^2
\]

which is divisible by two, and hence it is even. Let \(2n + 1\) be an odd number.

\[
(2n + 1)^2 = 4n^2 + 4n + 1
\]

\(4n^2\) and \(4n\) are both divisible by 2.

\[
\therefore 4n^2 + 4n + 1 \text{ is not divisible by 2, hence it is odd.}
\]

4) Yes, the sum, the difference, the product and the quotient of two rational numbers is always a rational number.

5) The sum of two irrational numbers is not always irrational. Ex. \(\sqrt{3} + \left(- \sqrt{3}\right) = 0\) and ‘0’ is rational.

6) The product of two irrational numbers is not always irrational.

For example \(\sqrt{2} \times \sqrt{4} = \sqrt{2 \times 4} = \sqrt{8} = 2\) which is rational.

7) a) \(a (b – c) = ab – ac\)

Proof: LHS \(a (b – c) = a (b + (-c)) = ab + a(-c) = ab + a(-c) = ab – ac = RHS\)

b) \([-a + (-b)] = -a + b\)

\[
\{a + (-b)\} + \{-a + b\} = a + (-b) + (-a) + b = a + (-a) + (-b) + b
\]

(by commutative and associative properties)

\[
= 0 + 0 = 0
\]

\[
\therefore -[a + (-b)] = -a + b
\]

c) \((a + b) (a + b) = a^2 + 2ab + b^2\)

Proof: LHS \((a + b) (a + b) = (a + b) a + (a + b) b. = a.a + b.a + a.b + b.b = a^2 + ab + ab + b^2\)

(by commutative property)

\[
= a^2 + 2ab + b^2 = RHS
\]
8) a) Additive inverse of ‘a’ is the number ‘b’ such that
   \[ a + b = 0 = b + a. \]
   Thus, the additive inverse of a is \(-a\).
   For example, the additive inverse of \(-5\) = \(-(-5) = 5\)

b) The multiplication inverse of \(a \neq 0\) is the number b such that
   \[ a \times b = 1 = b \times a \]
   \[ b = \frac{1}{a} \]
   Thus, the multiplicative inverse of a is \(\frac{1}{a}\)
   For example, the multiplicative inverse of \(\frac{-2}{3}\) is \(-\frac{3}{2}\).

9) a) \(a > b\) \(\Rightarrow a + c > b + c\)
    \[ a > b, c > 0 \quad ac > bc \]
    \[ a > b, c < 0 \quad ac < bc \]

10) a) \(a > b > 0\) \(\sqrt{a} > \sqrt{b}\)
    b) \(a > b > 0\) \(\frac{1}{a} < \frac{1}{b}\)

11) All are true.

12) Find \((-3) + 2\) and \(2 + (-3)\) using the number line

\[ 2 + (-3) = -1 \quad (-3) + 2 = -1 \]

The dotted arrow represents the sum.

For a positive number we move to the right and for a negative number
we move to the left.

\[ \therefore 2 + (-3) = (-3) + 2 \]

The commutative property is verified.

Similarly, the associative property can also be verified.

13) (i) Real numbers greater than \(-5\) and less than 5.

   \[ \begin{array}{c}
   \text{Real numbers greater than } -5 \text{ and less than } 5.
   \end{array} \]

(ii) \(x < 30\)

(iii) \(x > 30\)
No, the operation ‘to the power of’ is not a commutative operation.
\[ a^m = m^a \text{ if } a = m \]

No, it is true if \( a = 0 \)

(i) \( 8^3 = 512 \) \( \therefore \log_8 512 = 3 \)

(ii) \( 11^3 = 1331 \) \( \therefore \log_{11} 1331 = 3 \)

(iii) \( 7^3 = 343 \) \( \therefore \log_7 343 = 3 \)

(iv) \( 5^4 = 625 \) \( \therefore \log_5 625 = 4 \)

17) \( x = 1 \)

18) Let the two consecutive numbers be \( n \) and \( n + 1 \), sum = \( n + (n + 1) = 2n + 1 \) = even number + one \( \therefore \) sum is odd.

19) Let the two even numbers be \( 2n \) and \( 2m \) \((n, m \in \mathbb{N})\).

\[ \therefore \text{Product} = (2n) \times (2m) = 2 \times (n \times 2) \times m \text{ (associativity of multiplication)} \]

\[ = 2 \times (2 \times n) \times m \text{ (commutativity of multiplication)} \]

\[ = (2 \times 2) \times n \times m \text{ (associative property)} \]

\[ = 4nm \]

Thus, the product of two even numbers is divisible by 4.

20) \( 66 = 2 \times 3 \times 11 \) and \( 110 = 2 \times 5 \times 11 \)

\( \therefore \text{H.C.F.} = 22 \) and \( \text{L.C.M.} = 330 \)

21) i) H.C.F. of 45 and 75 = 15 and L.C.M. of 45 and 75 = 225

\[ \text{H.C.F.} \times \text{L.C.M.} = 15 \times 225 = 3375 \]

ii) H.C.F. of 140 and 490 = 70 and L.C.M. of 140 and 490 = 980

\[ \text{H.C.F.} \times \text{L.C.M.} = 70 \times 980 = 68600 \]

22) Let the number be \( n \) & \( n + 1 \).

\[ \frac{n}{n+1} \begin{array}{c} 1 \\ \hline n \\ \hline \end{array} \]

\[ \frac{n}{n+1} \begin{array}{c} -n \\ \hline 1 \end{array} \begin{array}{c} n(n \\ \hline -n \\ \hline 0 \end{array} \]

\( \therefore \text{H.C.F. of } n \text{ and } n + 1 = 1 \)

\( \therefore n \text{ and } n + 1 \text{ are coprime} \)

23) \( 65 \) \( 395 \) \( 6 \)

\[ \begin{array}{c} -390 \\ \hline 5 \end{array} \begin{array}{c} 65 \begin{array}{c} 13 \\ \hline -65 \\ \hline 0 \end{array} \]

H.C.F. = 5
24) Capacity of the first cane = 900 ml
Capacity of the second cane = 1.2 l = 1200 ml
Capacity of the third cane = 1.5 l = 1500 ml
The capacity of the largest container which can fill each one of them in complete number of times = H.C.F. of 900, 1200 and 1500.

H.C.F. of 900 and 1200

\[
\begin{array}{c|c}
900 & 1200 \\
-900 & -900 \\
300 & 900(3) \\
-900 & -900 \\
0 & \\
\end{array}
\]

H.C.F. of 900 and 1200 = 300

\[
\begin{array}{c|c}
300 & 1500 \\
1500 & 0 \\
\end{array}
\]

Capacity of the container = 300 ml.

25) They would again step together at a distance equal to the L.C.M. of their steps = L.C.M. of 72, 84 and 96.

= 2016 cm.
= 20.16 m

10.9 SUGGESTED READINGS


UNIT 11  POLYNOMIALS : BASIC CONCEPTS AND FACTORING*

Structure

11.1 Introduction
11.2 Objectives
11.3 Basic Concepts of Polynomials
11.4 Polynomials: Concepts and Definitions
   114.1 Polynomials in One Variable
   114.2 Polynomials in Two Variables
11.5 Operations on Polynomials
   11.5.1 Value of Polynomials and Zeroes of a Polynomial
   11.5.2 Addition of Polynomials
   11.5.3 Subtraction of Polynomials
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   11.5.5 Division of Polynomials
11.6 Factorization of Polynomials
   11.6.1 Basic Concepts
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   11.6.3 Method of Splitting the Middle Term
   11.6.4 Method of Splitting the Middle Term, (contd.)
11.7 Remainder Theorem
   11.7.1 Use of Remainder Theorem in Factorization
11.8 Let Us Sum Up
11.9 Unit End Activities
11.10 Points for Discussion
11.11 Answers to Check Your Progress
11.12 Suggested Readings

11.1 INTRODUCTION

This unit assumes that the students have already understood the basic ideas regarding dividend, divisor, quotient and remainder in arithmetic in earlier units. Also, they are familiar with the concept of numbers. They also know factorization of numbers into prime factors in arithmetic. However, a brief recapitulation by the teacher can help them to revise their basic concepts before you start actual teaching of this unit. The concepts to be learnt in this unit will be effectively learnt if they are related to the previous knowledge of the students. The concepts and techniques learnt in this unit would form the foundation for learning algebra later.

11.2 OBJECTIVES

At the end of this unit, you will be able to :

- clarify basic concepts of variables constant, algebraic expression and their numerical values, and zeroes of a polynomial;

* A few sections of this unit has been adopted from ES-342, IGNOU, 2000
• explain the meanings of degree of a polynomial, monomials and binomials, coefficients and constant terms;
• develop the skill of performing fundamental operations on polynomials;
• clarify the meaning and purpose of factorization of polynomials in algebra;
• develop skill to factorize quadratic polynomials by using basic algebraic identities;
• find ways of helping children to perform fundamental operations on polynomials;
• clarify the meaning of the statement of the remainder theorem;
• enable to compute remainder factorize quadratic and to polynomials x-a;
• enable to use remainder theorem to factorize polynomials; and
• use constructivist approach for teaching the concept and use of remainder theorem.

11.3 BASIC CONCEPTS OF POLYNOMIALS

Main Teaching Points

a) Meaning of a variable, terms, and expression
b) Value of an expression for given values of the variables

Teaching Learning Process

Teaching of algebra should begin with the relationship between arithmetic and algebra.

Variables

In algebra, we use alphabets to represent value, size and quantity in different situations. For example, the fact that the perimeter of a rectangle can be obtained by adding the length of its four sides, can be written algebraically as follows:

\[ P=2a+2b \quad \text{or} \quad P=2(a+b) \]

Where \( P \) represents the perimeter of the rectangle and 'a' and 'b' represent its length and breadth respectively. Here, a and b will take different values for different rectangles. Similarly, value of \( P \) will be determined by the values of a and b.

In algebra, the letters like \( P \), a and b, which are used to represent different elements and which take different values in different situations are referred to as unknowns or variables. Numbers and variables together act as alphabets of the language of algebra.

Terms

A combination of variables and numbers joined by operations of multiplication and division is known as a ‘term’. For examples:

\[ 3xy, -2x^2yz, \frac{7x}{y}, \frac{1}{2z}, 5x^3 \] are all terms. Terms are like words in English language.
**Algebraic Expression**

A combination of terms connected by operations of addition or subtraction is called an algebraic expression. For example:

\[2x + y, x^2 - 3xy, 3x^2 + 2x + 4, \frac{x}{y} + 4xy - 7\] are all algebraic expressions connected by multiplication and division are also algebraic expressions. Several algebraic expressions connected by multiplication and division are also algebraic expressions.

For example, \((x^2 + 5)(x + 2), \frac{2x + 3}{1 - y}, (x + 1) - (x + 2)(x + 3)\) are also algebraic expressions.

**Number of Variables in an Algebraic Expression.**

The expressions

\[2x^2 - 3x + 1, \frac{3x + 1}{2x - 3}, (2x + 5)^2\] are algebraic expressions in one variable x. Also, 3y+y^2 is an algebraic expression in one variable y. 2x + y, x^2 − xy, \(\frac{x}{y} + 4xy - 7\) are algebraic expressions in two variables x and y.

Similarly, 2x+y+z–1 is an algebraic expression in 3 variables x, y and z.

**Value of an Algebraic Expression**

An algebraic expression becomes a number when all its variables, such as x, y, z, are assigned definite numerical values and indicated operations are performed with those numerical values. Such a number is called the value of the algebraic expression.

For example, consider the algebraic expression 2x+y. When x=2 and y=3, the value of the expression is 2(2) + 3 = 7. Thus, 7 is the value of the expression.

Consider another algebraic expression \(3x^2 - 2xy + y^2\). It’s value, when x=-1 and y=2 is computed as follows:

\[3x^2 - 2xy + y^2 = 3(-1)^2 - 2(-1)(2) + (2)^2 = 3 + 4 + 4 = 11.\]

\(\therefore 11\) is the value of the expression when x=-1 and y=2.

The value of the expression will change if we change the values of x and y. **Methodology:** Discussion method to be used.

### 11.4 POLYNOMIALS : CONCEPTS AND DEFINITIONS

**Main Teaching Points**

Definition of

a) Polynomial

b) Monomial, Bionomial, and Trinomial
Consider several algebraic expressions. Write them such that in all the terms, variable is neither in the denominator nor under the root sign. Then ask the students to identify the algebraic expression in which powers of the variables are only positive integral numbers.

**Explain:**

An algebraic expression in which the variables in the terms are raised to positive integral powers only is called a polynomial.

**Example 1:** Consider the Following algebraic expressions:

(i) \(3x^2\)  
(ii) \(5x - 1\)  
(iii) \(x^3 - 2x + 3\)  
(iv) \(3x^2 - 2xy + 4y^2\)  
(v) \(x + \frac{1}{x}\) or \(x + x^{-1}\)  
(vi) \(x^3 - 3\sqrt{x} + 2\) or \(x^3 - 3x^{1/2} + 2\)  
(vii) \(\frac{3x + y}{x - y}\) or \((3x + y)(x - y)^{-1}\)  
(viii) \(\sqrt{x + y} = (x + y)^{1/2}\)

In algebraic expression (i) - (iv), the powers of the variables are 1, 2 or 3. These algebraic expressions are polynomials.

In expression (v), the power of variable is \(-1\)

In expression (vi), the power of variable is \(\frac{1}{2}\)

In expression (vii) and (viii) also, the power of variable is not a positive integer.

Therefore, the algebraic expressions (v), (vi), (vii) and (viii) are not polynomials.

Polynomials consisting of a single term are called monomials. For examples, \(2x\), \(3x^2\), \(-7y\), \(5\) are all monomials.

Polynomials having two terms are called Binomials. The expressions \(3x + 2y\), \(5x - 3\), \(7 - 2y\), and \(x^2 + 3xyz\) are all binomials.

Similarly, a trinomial is a polynomial having three terms. For examples, \(3x^2 + 2x + 1\), \(x^3 - xy + y^3\), \(2x^2 + 3x + 2y\), \(x + y + 2\) are all trinomials.

**Methodology:** Inductive method using different examples is used.

### 11.4.1 Polynomials in one Variable

**Main Teaching Points:**

a) Degree of the polynomial  
b) Standard form of a polynomial  
c) Constant polynomial  
d) Linear, Quadratic and Cubic polynomials.

**Teaching Learning Process**

The polynomial involving one variable is called a polynomial in one variable. Such polynomials \(x\), \(y\), \(s\) and \(t\) respectively may be written as:
Ask: What is the highest value of the exponent of the variable in each of the above polynomials. In the first polynomial there are three terms. The exponent of $x$ is highest in the term $3x^2$. It is 2. In $p(y)$, the highest exponent of the variable $y$ is 1.

Explain:
The highest exponent of the variable in a polynomial is called the degree of the polynomial. The degree of $p(x)$ is 2, the degree of $p(y)$ is 1, the degree of $p(s)$ is 2 and the degree of $p(t)$ is 3. The term which has the highest exponent of the variable is written first and then the other terms are written in the decreasing order of the exponent of the variable.

Explain:
The standard form of a polynomial of degree $n$ in one variable $x$ is:

$$p(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n.$$  

$a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants and $n$ is a positive integer.

**Linear Polynomial**  A polynomial of degree 1 is called a linear polynomial. For example,

$$3x + 2, \quad 2y - 1, \quad t + 3$$

are all linear polynomials.

**Quadratic Polynomial**  A polynomial of degree 2 is called a quadratic polynomial. For example,

$$x^2 - 2x + 3, \quad 3y^2 - 2y + 1, \quad s^2 + 3s + 5$$

are quadratic polynomials in variables $x$, $y$ and $s$ respectively.

**Cubic Polynomials**  A polynomial of degree 3 is called a cubic polynomial. For example,

$$x^3 - 2x^2 + 3x + 2, \quad x^3 - 1, \quad 4y^3, \quad s^3 + s^2 + s$$

are all cubic polynomials in variables $x$, $y$ and $s$ respectively.

**Constant polynomials**  A polynomial consisting of only constant term is known as a constant polynomial. For example,

$$p(x) = 3$$

is a constant polynomial. It is a polynomial of degree 0 as 3 can also be written as $3x^0$.

Methodology: After giving various examples, inductive reasoning is used to generalize the concepts.

### 11.4.2 Polynomials in Two Variables

**Main Teaching Points**

a) Degree of a term in case of more than one variable  
b) Linear and quadratic polynomials in the variables

**Teaching Learning Process**

Degree of a term with more than one variable is the sum of the exponents of all the variables $x$ and $y$. In the term $2xy^2z^2$ is $1 + 2 + 2 = 5$, because the exponents of the variables $x$, $y$ and $z$ are 1, 2 and 2 respectively.
Discuss the degree of terms by considering different examples of terms in several variables with positive integral exponents.

Explain: A linear polynomial in two variables \( x \) and \( y \) is a polynomial in which the maximum degree of any of its term is one. Consider various examples of polynomials in two variables \( x \) and \( y \) such that -

a) \( x^2 + y^2 \)

b) \( x + 3y + 2 \)

c) \( 2xy - 5 \)

d) \( 3x + 5y \)

e) \( 3x + 2xy^2 + y^3 \)

In polynomial (a), degree of each of its term is 2 so, it is not a linear polynomial whereas in polynomial (b) degree of the terms are 1,1 and 0 respectively. Since the highest degree of any of its term is 1, it is linear polynomial. Discuss that (c) and (e) are not linear polynomials but (d) is a linear polynomial. Similarly, explain that if in a polynomial, the highest degree of any of its term is two then it is a quadratic polynomial. Standard form of a linear polynomial in two variables is \( p(x, y) = ax + by + c \).

The quadratic polynomial in two variables is

\[ p(x, y) = ax^2 + bxy + cy^2 + dx + ey + f \]

where \( a, b, c, d, e \) and \( f \) are constants.

**Methodology:** After giving various examples, inductive reasoning is used to generalize the concepts.

While arithmetic is mainly concerned with the techniques of performing the four fundamental operations with numbers, algebra is concerned with general principles and properties of these operations. In other words, algebra generalizes arithmetical concepts.

## 11.5 OPERATIONS ON POLYNOMIALS

Before teaching the operations on polynomials, the teacher should clarify to his/her students that there is a lot of similarity between basic operations on integers and on polynomials. Like numbers, polynomials can also be added, subtracted, multiplied or divided. As in integers, the result of adding subtracting or multiplying polynomials is also a polynomial. But, when we divide one polynomial by another polynomial, we get a quotient and a remainder just as in the case of division in integers.

### 11.5.1 Value of a Polynomial and Zeroes of a Polynomial

**Main Teaching Points**

a) Value of a polynomial for given values of the variables

b) Meaning of zero of a polynomial

**Teaching Learning Process**

The value of a polynomial for given values of the variables is obtained by substituting these values of the variables in the polynomial and simplifying the numerical result.
Ask: How do you find the value of a polynomial for some assigned value of the variable?

Explain: Illustrate it by considering several examples as follows:

Let \( p(x) = 3x^3 - 5x^2 - 4x + 4 \) be a polynomial in \( x \). Find the value of \( p(x) \) for \( x = 0 \)

The value of \( p(x) \) for \( x = 0 \) is obtained by substituting \( x = 0 \) in the given polynomial and it is denoted by \( p(0) \).

\[
\therefore p(0) = 3(0)^3 - 5(0)^2 - 4(0) + 4 = 0 + 0 + 0 + 4 = 4
\]

Thus, the value of the given polynomial for \( x = 0 \) is 4.

Ask: Can you obtain the value of the polynomial for \( x = 1 \)?

Let them do and obtain the result.

\[
p(1) = 3(1)^3 - 5(1)^2 - 4(1) + 4 = 3 - 5 - 4 + 4 = -2
\]

Similarly, ask them to find out \( p(2) \).

\[
p(2) = 3(2)^3 - 5(2)^2 - 4(2) + 4 = 24 - 20 - 8 + 4 = 0
\]

Explain: The value of polynomial changes as the value assigned to \( x \) changes. Also, explain that the value of the polynomial for which \( p(x) \) becomes 0 is called the zero of the polynomial.

In the above example \( x = 2 \) is a zero of the given polynomial \( p(x) = 3x^3 - 5x^2 - 4x + 4 \) because for \( x = 2 \), the value of \( p(x) \) is zero.

There are as many zeroes of a polynomial as the degree of the polynomial. However, these may be real numbers or complex numbers. Let the students verify themselves that for the above polynomial other two zeros are \( x = -1 \) and \( x = \frac{-2}{3} \). The teacher should carefully select a few polynomials with one or more integral zeroes. Students should be asked to find the integral zeroes for the polynomials given to them.

Methodology: Through illustrations students should be taught to find the value of a polynomial and then inductively define zeroes of a polynomial.

11.5.2 Addition of Polynomials

Main Teaching Point

a) Similar terms or like terms

b) How to add two polynomials

Teaching Learning Process

Ask the students to add 5 pens and 3 pens.

5 pens + 3 pens = 8 pens.

Now, ask them to add 5 pens and 3 shirts. It can only be written as 5 pens + 3 shirts. In the first case, two things that we want to add are alike. So, we were able to add them. But in the second instance pens and shirts are different things and so we could not add them up. Similarly, in polynomials ask them to add \( 2x^2 \) and \( 3x^2 \) and let them tell that it is \( 5x^2 \). Again, ask them to add \( 2x^2 \) and \( 3x^4 \). Let them write it as \( 2x^2 + 3x^4 \).
Explain: Terms having the same variable part are called similar terms or like terms. We can add similar terms to get one single term. But when we have to add two terms which are not similar as \(2x^2\) and \(3x^4\), we can only express it as \(2x^2 + 3x^4\) (it is referred to as ‘indicated sum’).

Provide the students with several terms, and ask to add them up.

Illustration: Given: \(3x^2, 5x^3, 2x^3, 7x, 8x^2, -2x\)

Identify similar terms and add them up.

- \(3x^2\) and \(8x^2\) are similar, so, \(3x^2 + 8x^2 = 11x^2\)
- \(5x^3\) and \(2x^3\) are similar, so, \(5x^3 + 2x^3 = 7x^3\)
- \(7x\) and \(-2x\) are similar, so, \(7x + (-2x) = 5x\)

Explain: In the case of polynomials, similar terms in the polynomials are added and other terms are shown as indicated addition.

Example 2: Add the polynomials \(p(x) = x^3 + 2x^2 + 3\) and \(q(x) = 3x^2 + 5x + 2\)

In both the polynomials identity the similar terms and add them up and write the other terms as indicated addition.

In these two polynomials

- \(2x^2\) and \(3x^2\) are similar and therefore \(2x^2 + 3x^2 = 5x^2\)
- \(3\) and \(2\) are similar terms and \(3 + 2 = 5\)

So, \(p(x) + q(x) = (x^3 + 2x^2 + 3) + (3x^2 + 5x + 2)\)

\[= x^3 + (2x^2 + 3x^2) + 5x + (3 + 2)\]

\[= x^3 + 5x^2 + 5x + 5\]

Example 3: Add \(x^2 + 3x - 5\) and \(2x^2 - 2x + 3\)

\[(x^2 + 3x - 5) + (2x^2 - 2x + 3)\]

\[= (x^2 + 2x^2) + [(3x) - (2x)] + [(-5) + (3)]\]

\[= 3x^2 + x - 2\]

This method of addition is described as row method of addition. However, polynomials can also be added by writing one polynomial under the other polynomial such that similar terms are written one below the other, as in the method of number addition.

Explain: Write the first polynomial, then write the second polynomial such that similar terms are written in the same column, then add the two polynomials. Example 2 can be done as follows also:

\[
\begin{array}{c}
x^3 + 2x^2 + 3 \\
+ 3x^2 + 5x + 2 \\
\hline
x^3 + 5x^2 + 5x + 5 \\
\end{array}
\]

Example 3 can be done as follows:

\[
\begin{array}{c}
x^2 + 3x - 5 \\
2x^2 - 2x + 3 \\
\hline
3x^2 + x - 2 \\
\end{array}
\]

The teacher should give a few more similar examples to clarify the process of addition of polynomials to the students. In the end, a few problems may be
given as exercises. Students should be asked to practice both column method of addition and row method of addition.

**Methodology:** Lecture-cum-discussion method is used to illustrate that only similar terms can be added.

---

**Check Your Progress**

**Notes:** a) Write your answer in the space given below each question.

b) Compare your answer with the one given at the end of the unit.

Add the following polynomials by row method:

1) \[ p(x) = 2x^2 + 3x + 5 \]

\[ q(x) = 3x^2 - 4x - 7 \]

Add the following polynomials by column method:

2) \[ p(x) = x^3 - 2x - 3 \]

\[ q(x) = x^2 - 3x + 1 \]

Add the following polynomials by column method:

3) \[ p(t) = 2t^2 + t - 1 \]

\[ q(t) = 3t - 5 - 3t^2 \]

\[ r(t) = 1 - 3t - 3t^2 \]

4) \[ p(k) = k^4 + k^2 + 2 \]

\[ q(k) = k^2 + 2k + 1 \]

\[ r(k) = k^3 - 3k + 3 \]

---

### 11.5.3 Subtraction of Polynomials

**Main Teaching Point**

To subtract one polynomial from another.

**Teacher Learning Process**

**Explain:** The subtraction of polynomials involves the same principles and process as the subtraction of integers. Remind them that \[ 7 - (-3) = 7 + 3 = 10 \] and \[ 7 - (3) = 4. \]

Just as in the case of addition of polynomials, we subtract only like terms. For example, to subtract \(-3x\) from \(7x\). We write as \[ 7x - (-3x) = 7x + 3x = 10x \] and to subtract \(3x\) from \(7x\) we write as \[ 7x - (3x) = 7x - 3x = 4x \]. Note the sign of the term to be subtracted by row method or by column method. In both the ways, the sign of every term of the polynomial to be subtracted is changed.

**Example 4:** Subtract \(x - y\) from \(2x + 3y\)

**Row method:** \[ (2x + 3y) - (x - y) \]

\[ = 2x + 3y - x + y \]

\[ = [2x + (-x)] + [3y + y] \]

\[ = x + 4y \]
Column Method:

\[
\begin{align*}
2x + 3y \\
-(x - y)
\end{align*}
\]

Polynomials are written one below the other

\[
\begin{align*}
2x + 3y \\
-x + y
\end{align*}
\]

Sign of each term of the polynomial to be subtracted is changed

\[
\begin{align*}
x + 4y
\end{align*}
\]

The answer is the \(x + 4y\).

Example 5: Subtract \(2x^2 + 2y^2 - 6\) from \(3x^2 - 7y^2 + 9\)

Row method

\[
(3x^2 - 7y^2 + 9) - (2x^2 + 2y^2 - 6)
\]

\[
= 3x^2 - 7y^2 + 9 - 2x^2 - 2y^2 + 6
\]

\[
=(3x^2 - 2x^2) + (-7y^2 - 2y^2) + (9 + 6)
\]

Column Method

\[
\begin{align*}
3x^2 - 7y^2 + 9 \\
-(2x^2 + 2y^2 - 6)
\end{align*}
\]

(Changing the signs of the terms of polynomial to be subtracted.)

\[
\begin{align*}
3x^2 - 7y^2 + 9 \\
-2x^2 - 2y^2 + 6 \\
x^2 - 9y^2 + 15
\end{align*}
\]

The answer is \(x^2 - 9y^2 + 15\)

Let \(p(x) = 3x^2 - 4x + 7\)

\(q(x) = 2x^3 + x^2 - 5x - 3\)

and \(r(x) = 3x^2 - 2x + 1\)

Ask the students to compute

\(p(x) - q(x), q(x) - p(x), [ p(x) - q(x) ] - r(x)\) and \(p(x) - [ q(x) - r(x) ]\)

Then ask: Is \(p(x) - q(x) = q(x) - p(x)\)?

Let them tell that the two answers are different.

Explain: Subtraction is not commutative.

Ask: Is \([ p(x) - q(x) ] - r(x) = p(x) - [ q(x) - r(x) ]\)?

Explain: Subtraction is not associative. Point out that whereas addition of polynomials is both commutative and associative. The subtraction is neither associate nor commutative.

Methodology: Lecture-cum-discussion method is used. Sufficient practice should be provided so that the students develop the skill.
Check Your Progress
Notes: a) Write your answer in the space given below each question.
   b) Compare your answer with the one given at the end of the unit.

5) Perform the subtractions given below:
   i. \[3x - y - (2x - y)\]
   ii. \[a^2 - 3b + c^2 - (a^2 + 5b + 3c^2)\]

6) Subtract by row method:
   \((-6x - 8y) - (-3x + 8y)\)

7. Subtract \(x^2 - 7y^2 + 1\) from \(2x^2 + 3y^2 - 3\) by row method.

8. What should be subtracted from \(x^4 - 1\) to get \(3x^2 - 2x + 1 + x^4\)?

9. What should be added to \(x^3 + 3x^2 + 1\) to get \(x^5 - 2x^2 - 3x\)?

11.5.4 Multiplication of Polynomials

Main Teaching Points
a) To multiply a monomial by a monomial
b) To multiply a polynomial by a monomial
c) To multiply a polynomial by a polynomial

Teaching Learning Process

Three fundamental properties of operations are most commonly used in the process of multiplication of polynomials.

Commutative law which tells us \(xy = yx\)

Associative law Which expresses the fact that \((xy)z = x(yz)\) and

Distributive law Which states that \(x(y + z) = xy + xz\). When a monomial is multiplied by another monomial, laws of exponents are also useful.

In particular, \(x^m \cdot x^n = x^{m+n}\)

Explain: To multiply a monomial by another monomial, we arrange the numbers together and the variables together using commutative and associative properties. Then using law of exponents, find the product.
Example 6: Multiply \(3x^3y\) by \(5xy^2z\)

**Solution**

\[(3x^3y) (5xy^2z)\]

\[= (3 \times 5) (x^3 \cdot x) (y \cdot y^2) \cdot z\]

\[= 15x^4y^3z \quad \text{(using commutative and associative properties)}\]

Example 7: Multiply \(-2ay^2z\) by \(4xyz^2\)

**Solution**

\[(-2ay^2z) (4xyz^2)\]

\[= (-2) (4) (a) (x) (y^2 \cdot y) (z \cdot z^2)\]

\[= -8axy^3z^3\]

**Explain:** To multiply a polynomial by a monomial, we multiply each term of the polynomial by the monomial just as in distribution law.

Example 8: Multiply \(2x^2 - 3x - 5\) by \(4x\)

**Solution**

\[4x (2x^2 - 3x - 5)\]

\[= (4x) (2x^2) - (4x) (3x) - (4x) (5)\]

\[= (4 \cdot 2) (x \cdot x^2) - (4 \cdot 3) (x \cdot x) - (4 \cdot 5)x\]

\[= 8x^3 - 12x^2 - 20x\]

In actual practice, we straight away write the last step doing it orally without showing the intermediary steps.

For example: \(-2x (5x^2 - 3xy + 1) = -10x^3 + 6x^2y - 2x\)

**Explain:** To multiply a polynomial by another polynomial, we multiply each term of the second polynomial by each term of the first polynomial just as in distributive law.

Example 9: Multiply \(2x + 3y\) by \(2x^3 - xy + 2y^2\)

**Solution**

\[(2x + 3y) (2x^3 - xy + 2y^2)\]

\[= 2x(2x^3 - xy + 2y^2) + 3y (2x^3 - xy + 2y^2)\]

\[= (4x^4 - 2x^2y + 4xy^2) + (6x^3y - 3xy^2 + 6y^3)\]

\[= 4x^4 - 2x^2y + [4xy^2 + (-3xy^2)] + 6x^3y + 6y^3\]

\[= 4x^4 - 2x^2y + xy^2 + 6x^3y + 6y^3\]

Example 10: Multiply \(x^2 - 3x - 2\) by \(3x^2 + x - 1\)

**Solution**

\[(x^2 - 3x - 2) (3x^2 + x - 1)\]

\[= x^2(3x^2 + x - 1) - 3x (3x^2 + x - 1) - 2(3x^2 + x - 1)\]

\[= 3x^4 + x^3 - x^2 - 9x^3 - 3x^2 + 3x - 6x^2 - 2x + 2\]

\[= 3x^4 + (x^3 - 9x^3) + (-x^2 - 3x^2 - 6x^2) + (3x - 2x) + 2\]

\[= 3x^4 - 8x^3 - 10x^2 + x + 2\]

**Methodology:** The method of multiplication is illustrated through various examples and sufficient practice should be provided to develop the skill.
Check Your Progress

Notes: a) Write your answers in the space given below each question.
     b) Compare your answer with the one given at the end of the Unit

10) Find the following products:
    (2x + 3) · (x + 4)
    ..............................................................................................................................

11) (x³ − x² − 1) (2x² + x + 3)
    ..............................................................................................................................

12) (x + y) (x² − xy + 2y²)
    ..............................................................................................................................

13. (x − y) (x + y) (x² + y²)
    ..............................................................................................................................

11.5.5 Division of Polynomials

Main Teaching Points
a) To divide a monomial by a monomial
b) To divide a polynomial by a polynomial
c) Dividend = Divisor × Quotient + Remainder

Teaching Learning Process

Here we will be restricting ourselves to division of polynomials in one variable only. The process of division in the case of polynomials is the same as in arithmetic. The similarity between the two processes should be made clear to the pupils by giving suitable examples.

In case of numbers, we can only divide a bigger number by a smaller number. Similarly in case of polynomials a higher degree polynomial can be divided by a polynomial of lower degree.

Example 11: Divide 12x³ by 5x

Now, 12x³ ÷ 5x = \( \frac{12x^3}{5x} = \frac{12 \cdot x \cdot x \cdot x}{5 \cdot x} = \frac{12}{5}x^2 \)

or we can use the law of exponents : \( \frac{x^m}{x^n} = x^{m-n} \). Note that in integers, when a smaller integer is divided by a larger integer, the answer is not an integer.

For example, \( 3 ÷ 7 = \frac{3}{7} \), it is not an integer. Similarly, if we divide a monomial of smaller degree by a monomial of higher degree, then the answer is not a polynomial clearly, \( 3x ÷ x^3 = \frac{3x}{x^3} = \frac{3}{x^2} \); it is not a polynomial.
The process of dividing a polynomial by another polynomial is quite similar to long division for integers. Here, we write the dividend and the divisor both in decreasing powers of the variables and then divide as illustrated in the following example:

**Example 12:** Divide \(3x^2 - 9x + 6\) by \(3x\)

**Step 1:** both the polynomials should be written in decreasing powers of \(x\).

The polynomials are already in this form

\[
\begin{align*}
3x^2 & \quad \text{(Dividend)} \\
3x & \quad \text{(Divisor)} \\
\end{align*}
\]

**Step 2:** Divide the first term of the dividend by \(3x\) \(\frac{3x^2}{3x} = x\) write it as the first term of the quotient.

**Step 3:** Multiply the divisor by the quotient of step 2 and write it below the dividend and subtract to get \(-9x + 6\).

**Step 4:** Divide \(-9x\) by \(3x\) the result in \(-3\), write it as the second term of the quotient.

**Step 5:** Multiply the divisor by the quotient of step 4 and write it below the divided and subtract to get \(6\).

**Step 6:** Since the degree of \(6\) is smaller than the degree of \(3x\), the process stops. Thus, when \(3x^2 - 9x + 6\) is divided by \(3x\), the quotient is \(x - 3\) and the remainder is \(6\).

Ask the students to verify that \(3x^2 - 9x + 6 = 3x \times (x - 3) + 6\)

(Dividend = quotient \(\times\) divisor + remainder)

**Example 13:** Divide \(2x^3 - 3x^2 + 5x - 1\) by \(x - 2\)

**Step 1:** Write both the polynomials is standard form.

**Step 2:** Divide first term of dividend by the first term of divisor

\[
\begin{align*}
2x^3 & \quad \text{(Dividend)} \\
2x^3 & \quad \text{(Divisor)} \\
\end{align*}
\]

write it as first term of the quotient.

\[
\begin{align*}
2x^2 + x + 7 & \quad \text{(Quotient)} \\
2x^3 - 3x^2 + 5x & \quad \text{(Dividend)} \\
2x^3 & \quad \text{(Divisor)} \\
\end{align*}
\]

write it as first term of the quotient.
Step 3: Multiply the divisor by the quotient in step 2, write it below the dividend and subtract to get $x^2 + 5x - 1$

Step 4: Divide the first term of this polynomial by the first term of the divisor.

$$x^2 \div x = \frac{x^2}{x} = +x$$ (with proper sign) write it as the second term (with proper sign) of the quotient.

Step 5: Multiply the divisor by the quotient in step 4, write it below the new dividend and subtract to get $7x - 1$.

Step 6: Divide the first term of this new dividend by the first term of the divisor $7x \div x = \frac{7x}{x} = 7$ write it as the third term (with a proper sign) of the quotient.

Step 7: Multiply the divisor by the quotient in step 6 writes it below the new dividend and subtract to get 13.

Step 8: Since the degree of 13 is less than the degree of the divisor, the process stops.

When $2x^3 - 3x^2 + 5x - 1$ is divided by $x - 2$, the quotient is $2x^2 + x + 7$ and the remainder is 13.

Ask the students to verify that

Dividend = Quotient $\times$ Divisor + Remainder

or $2x^3 - 3x^2 + 5x - 1 = (x-2) (2x^2 + x + 7) + 13$

**Check Your Progress**

**Notes:**

a) Write your answer in the space given below.

b) Compare your answer with the one given at the end of the unit.

Divide and write the quotient and remainder in each of the following:

14) $(6m^2 - 17m - 3) \div (m - 3)$

15) $(8y^2 - 6y - 5) \div (2y + 1)$

16) $(27x^3 - 1) \div (3x - 1)$

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11.6 FACTORIZATION OF POLYNOMIALS

In the preceding section you were introduced to the methods of multiplying polynomials to obtain polynomials of higher degree. Now, we shall consider the reverse procedure, i.e., techniques of writing a polynomial as a product of two or more polynomials of lower degree.

11.6.1 Basic Concepts

Main Teaching Points

a) What is a factor?

b) Meaning of factorization of polynomials

Teaching Learning Process

Students have already learnt in arithmetic, the concept of factors and prime factorization of numbers. In algebra, the concept of factor is the same, given a polynomial, we express it as a product of immedeicate polynomials and the process is called the factorization of the polynomial.

For example, we know that

1) \((2x) \cdot (3x^2) = 6x^3\)
2) \((2xy) \cdot (x + y) = 2x^2y + 2xy^2\)
3) \((x^2 + x) \cdot (2x + 1) = 2x^3 + 3x^2 + x\)

In examples 1, \(6x^3\) is the product of \(2x\) and \(3x^2\). \(2x\) and \(3x^2\) are called the factors of \(6x^3\).

So, in these examples, we get—

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Polynomial</th>
<th>Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>(6x^3)</td>
<td>(2x) and (3x^2)</td>
</tr>
<tr>
<td>2.</td>
<td>(2x^2y + 2xy^2)</td>
<td>(2xy) and (x + y)</td>
</tr>
<tr>
<td>3.</td>
<td>(2x^3 + 3x^2 + x)</td>
<td>(x^2 + x) and (2x + 1)</td>
</tr>
</tbody>
</table>

Explain: If a polynomial can be expressed as a product of two or more polynomials of lower degree, then these polynomials of lower degree are called its factors.

Ask: In the above examples, which of the factors can be further factorized?

\(2x = 2 \cdot x\) (2 and \(x\) are further factors)

\(3x^2 = 3 \cdot x \cdot x\) (3, \(x\) and \(x\) are further factors)

Similarly,

\(2xy = 2 \cdot x \cdot y\) (3 more factors)
\(x^2 + x = x(x + 1)\) (2 more factors)

\(x + y\) and \(2x + 1\) cannot be factorized further.

Therefore,

\(6x^3 = 2 \cdot 3 \cdot x \cdot x \cdot x\) (it has 5 factors)
\(2x^2y + 2xy^2 = 2 \cdot x \cdot y \cdot (x + y)\) (it has 4 factors)
\(2x^3 + 3x^2 + x = x(x + 1) (2x + 1)\) (it has 3 factors)
Explain: These factors are such that they cannot be factorized further. A polynomial which can not be factorized is called **irreducible**.

Expressing a polynomial as a product of irreducible factors is similar to prime factorization of numbers.

**Methodology:** Discussion with inductive reasoning is used to clarify that concept.

### 11.6.2 Factoring a Quadratic Polynomial

**Main Teaching Points**

To factorize using algebraic identities

**Teaching learning Process**

Students have already learnt in lower classes

That

\[(x + a)^2 = x^2 + 2ax + a^2\] ...........................................(i)

\[(x - a)^2 = x^2 - 2ax + a^2\] .............................................(ii)

\[(x + a) \cdot (x - a) = x^2 - a^2\] ...................................................(iii)

A quadratic polynomial is of the form \(p(x) = ax^2 + bx + c\), where \(a, b\) and \(c\) are real numbers, \(a \neq 0\). The students may be led to discover that the right hand side expressions given in the above equations are quadratic polynomials and the left hand side expressions are their factors, since \((x + a)^2 = (x + a) \cdot (x + a)\) and \((x - a)^2 = (x - a) \cdot (x - a)\).

Note that the first term and the last term of the quadratic polynomials in the above equations are perfect squares. So, whenever a given quadratic polynomial in its standard form is such that the first and the last term are perfect square, then it might be possible to express it in any one of the above forms. The following examples may be given to clarify the method.

**Example 14:** Factorize \(x^2 + 10x + 25\)

The first and the last terms are \(x^2\) and 25. They are perfect squares. The quadratic polynomial may be written as-

\[x^2 + 10x + 25 = (x)^2 + 2(x) (5) + (5)^2\]

\[= (x + 5)^2 \] .................................................. (using identity(i))

**Example 15:** Factorize \(9x^2 + 12x + 4\)

The first and the last terms are \(9x^2\) and 4. Clearly they are perfect squares. The quadratic polynomial may be written as:

\[9x^2 + 12x + 4 = (3x)^2 + 2(3x) (2) + (2)^2\]

\[= (3x+2)^2\] ..................................................(using identity (i))

**Example 16:** Factorize \(25x^2 - 30x + 9\)

The first and the last terms are \(25x^2\) and 9, which are perfect squares. The quadratic polynomial may be written as

\[25x^2 - 30x + 9 = (5x)^2 - 2(5x) (3) + (3)^2\]

\[= (5x-3)^2\] ..................................................(using identity (ii))
Example 17: Factorize $9x^2 - 16$

The first and the last terms are $9x^2$ and $16$ which are perfect squares. The quadratic polynomial may be written as:

$$9x^2 - 16 = (3x)^2 - (2)^2$$

$$= (3x + 4)(3x - 4) \text{ \ldots \ldots \ldots \ldots \ldots (using \ identity \ (iii))}$$

After giving a few more similar examples, the teacher may give some exercises for practice.

Methodology: Method is illustrated using various examples and sufficient drill should be provided to develop the skill.

11.6.3 Method of Splitting the Middle Term

Main Teaching Point

To factorize a quadratic polynomial of the type $x^2 + (p + q)x + pq$ by splitting the middle term.

Teaching Learning Process

A polynomial of the form $x^2 + (p + q)x + pq$ can be factorized into linear factors by splitting the middle term as follow:

$$x^2 + (p + q)x + pq = x^2 + px + qx + pq$$

$$= x(x + p) + q(x + p)$$

$$= (x + p)(x + q)$$

Explain: If a given polynomial of the form $x^2 + bx + c$ is such that $b = p + q$ and $c = pq$ for some numbers $p$ and $q$, then the given polynomial can be factorized by splitting the middle term as shown above. The numbers $p$ and $q$ are determined by hit and trial. The method should be illustrated through examples as follows:

Example 18: Factorize $x^2 + 7x + 12$ Note that $p + q = 7$ and $pq = 12$ Ask the students to find two numbers $p$ and $q$ such that $p + q = 7$ and $pq = 12$ by hit and trial.

The pupils should be made to think as follows: Find two numbers whose product is 12 or break it up as product of two numbers. As $12 = 1 \times 12$ or $2 \times 6$ or $3 \times 4$

Out of these, 3 and 4, are such that $3 + 4 = 7$

$\therefore p$ and $q$ are 3 and 4.

Now, we can factorize the polynomial.

As follow: $x^2 + 7x + 12 = x^2 + (4 + 3)x + (4.3)$

$$= x^2 + 4x + 3x + 4.3$$

$$= x(x + 4) + 3(x + 4)$$

$$= (x + 4)(x + 3)$$

Which are the required factors?

Example 19: Factorize $x^2 + x - 6$.

Here, $p + q = 1$ and $pq = -6$. 
Now, break up $-6$ into product of the two numbers.

$-6 = 1 \times (-6) \text{ or } (-1) \times 6 \text{ or } 2 \times (-3) \text{ or } (-2) \times 3$.

Out of these $-2$ and $3$ are such that $(-2) + 3 = 1$.

$\therefore$ p and q are $-2$ and $3$.

$\therefore x^2 + x - 6 = x^2 + [(-2) + 3] x + (-2) (3)$

$= x^2 - 2x + 3x + (-2) (3)$

$= x (x - 2) + 3 (x - 2)$

$= (x - 2) (x + 3)$

Which are the required factors.

**Example 20:** Factorize $x^2 - 7x + 10$

Here, $p + q = -7$ and $pq = +10$.

Now, break up $10$ into product of the two numbers.

$10 = 1 \times 10 \text{ or } (-1) \times (-10) \text{ or } 2 \times 5 \text{ or } (-2) \times (-5)$.

Out of these $-2$ and $-5$ are such that $(-2) + (-5) = -7$.

$\therefore$ p and q are $-2$ and $-5$.

$\therefore x^2 - 7x + 10 = x^2 + [(-2) + (-5)]x + (-2) (-5)$

$= x^2 + (-2)x + (-5)x +(-2) (-5)$

$= x(x - 2) + (-5) (x - 2)$

$= (x - 2) (x - 5)$

Which are the required factors.

**Methodology:** The method is illustrated using several examples and then practice should be provided to develop the skill.

**11.6.4 Method of Splitting the Middle Term (Contd.)**

**Main Teaching Point**

To factorize a quadratic polynomial of the type $ax^2 + bx + c$ by splitting the middle term.

**Teaching Learning Process.**

When a quadratic polynomial is of the form $ax^2 + bx + c$ such as $2x^2 + 9x + 4$ or $3x^2 - 14x + 8$, the factors are of the form $px + q$ and $rx + s$. Multiplying $px+q$ and $rx + s$ we get $(px + q)(rx + s) = prx^2 + (ps + qr)x + qs$ comparing it with $ax^2 + bx + c$, we get $a = pr$, $b = ps + qr$ and $c = qs$.

It is also seen that $(ps) (qr) = (pr) (qs) = ac$.

$\therefore$ Coefficient of $x = b$

$= \text{ sum of two numbers } ps \text{ and } qr \text{ whose product is } ac$

So, the method of factoring $ax^2 + bx + c$ is similar to the one for the polynomial $x^2 + bx + c$ except for now we have to find two numbers whose sum is $b$ and their product is $ac$.

Consider the following examples:
**Example 21.** Factorize $2x^2 + 9x + 4$

Here, we have to find two numbers $p$ and $q$ such that $p + q = 9$ and $pq = 2 \times 4 = 8$

Now, break up 8 into product of two numbers $8 = 1 \times 8$ or $(-1) \times (-8)$ or $2 \times 4$ or $(-2) \times (-4)$

Out of these pairs 1 and 8 are such that $1 + 8 = 9$

\[ \therefore p \text{ and } q \text{ are } 1 \text{ and } 8. \]

\[ \therefore 2x^2 + 9x + 4 = 2x^2 + (1 + 8)x + 4 \]
\[ = 2x^2 + x + 8x + 4 \]
\[ = x(2x + 1) + 4(2x + 1) \]
\[ = (2x + 1)(x + 4) \]

Which are the required factors.

**Example 22:** Factorize $3x^2 - 14x + 8$

Here, we have to find two numbers $p$ and $q$ such that $p+q = (-14)$ and $pq = 3 \times 8 = 24$

Now, break up 24 into product of two numbers $24 = 1 \times 24$ or $2 \times 12$ or $3 \times 8$ or $4 \times 6$

Or $(-1) \times (-24)$ or $(-2) \times (-12)$ or $(-3) \times (-8)$ or $(-4) \times (-6)$

Out of these pairs, $-2$ and $-12$ are such that $(-2) + (-12) = -14$

\[ \therefore p \text{ and } q \text{ are } -2 \text{ and } -12. \]

\[ \therefore 3x^2 - 14x + 8 = 3x^2 + (-2-12)x + 8 \]
\[ = 3x^2 - 2x - 12x + 8 \]
\[ = x(3x - 2) - 4(3x - 2) \]
\[ = (3x - 2)(x - 4) \]

Which are the required factors.

**Example 23:** Factorize $6x^2 - 11x - 2$

Here, we have to find two numbers $p$ and $q$ such that $p + q = (-11)$ and $pq = 6 \times (-2) = -12$

Now, break up $-12$ into product of two numbers $-12 = 1 \times (-12)$ or $(-1) \times 12$ or $3 \times (-4)$ or $(-3) \times 4$

Or $2 \times (-6)$ or $(-2) \times 6$

Out of these pairs, $1$ and $-12$ are such that $1 + (-12) = -11$

\[ \therefore p \text{ and } q \text{ are } 1 \text{ and } -12. \]

\[ \therefore 6x^2 - 11x - 2 = 6x^2 + (1-12)x - 2 \]
\[ = 6x^2 + x - 12x - 2 \]
\[ = x(6x + 1) - 2(6x+1) \]
\[ = (6x + 1)(x - 2) \]

Which are the required factors.

**Methodology:** The method is illustrated by various examples and sufficient practice is required to develop the skill.
11.7 REMAINDER THEOREM

Main Teaching Points
a) Statement of remainder theorem
b) Use of remainder theorem to find remainder

Teaching Learning Process
You have already taught your students the process of dividing a polynomial by another polynomial. If a polynomial \( p(x) \) is divided by \( q(x) \) to give quotient \( s(x) \) and remainder \( r(x) \), then verify that:

\[
p(x) = q(x) \cdot s(x) + r(x)
\]

Also, remind them that degree of remainder \( r(x) \) is smaller than the degree of the divisor. Now, as a special case, \( p(x) \) divided by \( x - a \) to give quotient \( q(x) \) and remainder \( r(x) \), then-

\[
p(x) = (x - a) q(x) + r(x)
\]

and degree of \( r(x) \) is smaller than the degree of the divisor \( x - a \), which is 1.

\[\therefore\] degree of \( r(x) \) should be zero as \( r(x) \) is a constant polynomial \( (r) \), independent of \( x \).

\[\therefore\] we get \( p(x) = (x - a) q(x) + r \)
Ask: Find the value of the polynomial
\[ p(x) \] at \( x = a \)
\[ p(a) = (a - a) q(a) + r = r \]
This leads to an important theorem, known as remainder theorem which states as

'If \( p(x) \) is a polynomial and it is divided by \( (x-a) \), where \( a \) is a real number, the remainder will be \( p(a) \)'

Illustrate through examples, how remainder theorem is useful in finding the remainder without actual division.

**Example 24:** Find the remainder when \( 3x^2 + 2x + 1 \) is divided by \( x - 4 \).
Here, \( p(x) = 3x^2 + 2x + 1 \),
Comparing \( x - 4 \) with \( x - a \), we get \( a = 4 \).
According to remainder theorem,
Remainder, \( r = p(4) = 3(4)^2 + 2(4) + 1 \)
\[ = 48 + 8 + 1 = 57 \]

**Example 25:** Find the remainder when \( x^3 - 2x^2 + x - 5 \) is divided by \( x + 2 \).
Here, \( p(x) = x^3 - 2x^2 + x - 5 \),
Comparing \( x + 2 \) with \( x - a \), we get \( a = -2 \).
According to remainder theorem,
\[ \therefore \text{Remainder} = p(-2) = (-2)^3 - 2(-2)^2 + (-2) - 5 \]
\[ = -8 - 8 - 2 - 5 \]
\[ = -23 \]

**Methodology:** Deductive method is used to explain the meaning of the statement of the remainder theorem. Than various illustrations are given to find the remainder.

11.7.1 **Use of Remainder Theorem in Factorization**

**Teaching Points**

a) \( x - a \) is a factor of \( p(x) \) if \( p(a) = 0 \)
b) Factorizing a cubic polynomial using the factor theorem

**Main Teaching Process**

Ask: When \( p(x) \) is divided by \( x - a \), the remainder is \( p(a) \). If \( p(a) \) turns out to be zero, what do you conclude?

Explain: If the remainder is zero, then \( p(x) = (x - a) \cdot q(x) \) and therefore \( x - a \) is a factor of \( p(x) \).
\[ \therefore \text{If } p(a) = 0, \text{ for some number } a, \text{ than } x-a \text{ is a factor of } p(x). \text{ This statement is called the factor theorem.} \]

**Example 26:** Find if \( x - 3 \) is a factor of \( x^3 - 3x^2 + 4x - 10 \).
Here, \( p(x) = x^3 - 3x^2 + 4x - 10 \)
\[ p(3) = (3)^3 - 3(3)^2 + 4(3) - 10 \]
\[ = 27 - 27 + 12 - 10 \]
\[ = 2 \neq 0 \]
\( x - 3 \) is not a factor of \( p(x) \)
Example 27: Find the factor of $x^3 - x^2 + x - 1$.

Here, 
\[ p(x) = x^3 - x^2 + x - 1 \]
\[ p(1) = 1^3 - 1^2 + 1 - 1 = 0 \]

Now, divide $p(x)$ by $x - 1$ to get the other factor
\[
\begin{array}{c|ccccc}
\hline
& x^2 & +1 \\
\hline
x - 1 & x^3 & - x^2 & + x & - 1 \\
\hline
\hline
& x - 1 & + \\
\hline
& x - 1 & - + \\
\hline
& 0 & \\
\hline
\end{array}
\]

$\therefore x^3 - x^2 + x - 1 = (x - 1)(x^2 + 1)$

since, $(x^2 + 1)$ is irreducible

$\therefore$ These are the required factors.

Example 28: Find the factors of $x^3 + 3x^2 - 4x - 12$

Here, 
\[ p(x) = x^3 + 3x^2 - 4x - 12 \]
\[ p(1) = 1 + 3 - 4 - 12 = -12 \neq 0 \]

$\therefore (x - 1) \text{ is not a factor of } p(x)$

\[ p(-1) = (-1)^3 + 3(-1)^2 - 4(-1) - 12 \\
= -1 + 3 + 4 - 12 \\
= 7 - 13 = -6 \neq 0 \]

$\therefore (x + 1) \text{ is not a factor of } p(x)$

\[ p(2) = (2)^3 + 3(2)^2 - 4(2) - 12 \\
= 8 + 12 - 8 - 12 = 0 \]

$\therefore (x-2) \text{ is a factor of } p(x)$

Now, divide $p(x)$ by $(x - 2)$:
\[
\begin{array}{c|ccccc}
\hline
& x^2 & +5x & +6 \\
\hline
x - 2 & x^3 & +3x^2 & -4x & -12 \\
\hline
\hline
& 5x^2 & -4x & -12 \\
\hline
& 5x^2 & -10x & + \\
\hline
& 6x & -12 \\
\hline
& 6x & -12 & + \\
\hline
& 0 & \\
\hline
\end{array}
\]
\[ \begin{align*}
\therefore x^3 + 3x^2 - 4x - 12 &= (x - 2)(x^2 + 5x + 6) \\
&= (x - 2)[x^2 + (2 + 3)x + 2.3] \\
&= (x - 2)[x^2 + 2x + 3x + 2.3] \\
&= (x - 2)[x(x + 2) + 3(x + 2)] \\
&= (x - 2)(x + 2)(x + 3)
\end{align*} \]

These are the required factors.

**Example 29:** For what value of k is \( x^3 - kx^2 + 4x - 12 \) divisible by \( x - 3 \).

Here, \( p(x) = x^3 - kx^2 + 4x - 12 \)

Comparing \( x - 3 \) with \( x - a \), we get \( a = 3 \)

\[ \therefore \text{Remainder} = p(3) = (3)^3 - k(3^2) + 4(3) - 12 \]

\[ = 27 - 9k + 12 - 12 \]

\[ = 27 - 9k \]

For divisibility, \( p(3) = 0 \Rightarrow 27 - 9k = 0 \Rightarrow k = 3 \)

**Methodology:** Inductive logic is used to illustrate that \( x-a \) is a factor of \( p(x) \) if \( p(a) = 0 \)

---

**Check Your Progress**

**Notes:**

a) Write your answer in the space given below each question.

b) Compare your answer with the one given at the end of the Unit.

21) Find whether \( p(x) \) is divisible by \( q(x) \) for \( p(x) = x^3 - 2x^2 + x + 3 \) and \( q(x) = x - 4 \).

22) For what value of \( k \), \( q(x) \) will be a factor of \( p(x) \) in the following:

\( p(x) = x^3 + 2x^2 + kx - 2 \) and \( q(x) = x + 2 \)

---

**11.8 LET US SUM UP**

In this unit you have learnt some strategies and approaches to clarify certain important concepts such as constants, variables, algebraic expression, value of an algebraic expression and polynomials. You have also learnt how to teach effectively, the concepts of basic operations involving polynomials and impart
skills to perform these operations. You have also studied the techniques of factorizing quadratic polynomials. The use of remainder theorem is algebra to find the remainder without performing the division was also presented. It was also shown that the remainder theorem may be used to factorize polynomials of higher degree. In short, this unit provides you a foundation for further development in teaching algebra.

11.9  UNIT END ACTIVITIES

At the end of this unit you may:

1) Give your students some more exercises on basic operations with polynomials and factorization;
2) Evaluate the learning outcomes of your students after this unit has been taught;
3) Prepare lesson plans or develop teaching aids for teaching factorization and multiplication of polynomials; and
4) Visit your nearby schools and observe the teaching of polynomials to see whether the ideas and techniques learnt in this unit find a place in classroom teaching.

11.10  POINTS FOR DISCUSSION

1) Can we correlate the ideas involved in this unit with corresponding concepts in arithmetic?
2) Can we correlate the concepts of products and factors involved in this unit to the concepts of area and volume in mensuration and develop some novel teaching aids.
3) Can remainder theorem be used to develop a general technique of factorization?

11.11  ANSWER TO CHECK YOUR PROGRESS

1) 5x² – x – 2
2) x³ + x² + x – 2
3) –4t² + t – 5
4) k⁴ + k³ + 2k² – k + 6
5) (i) x  (ii) –8b – 2c²
6) –3x – 16y
7) x² + 10y² – 4
8) –3x² + 2x – 2
9) x³ – x³ – 5x² – 3x – 1
10) 2x² + 11x + 12
11) 2x⁵ – x⁴ + 2x³ – 5x² – x – 3
12) x³ + xy² + 2y³
13) x⁴ – y⁴
14) Quotient = -6m + 1, Remainder = 0
15) Quotient = 4y – 5, Remainder = 0
16) Quotient = 9x² + 3x + 1, Remainder = 0
17) (x + 3) (x + 5)
18) (x – 3) (x + 2)
19) 5(x + 2) (x + 6)
20) (9x – 4) (6x + 1)
21) Not divisible
22) k = –1
11.12 SUGGESTED READINGS


UNIT 12 LINEAR EQUATIONS AND INEQUATIONS AND QUADRATIC EQUATIONS*

Structure

12.1 Introduction
12.2 Objectives
12.3 Linear Equation in One Variable
12.4 Linear Equation in Two Variables
   12.4.1 Graph of a Linear Equation in Two Variables
   12.4.2 Graph of a Linear Equation Involving Absolute Values
   12.4.3 System of Linear Equations in Two Variables
   12.4.4 Methods of Solving System of Linear Equations
   12.4.5 Solution of Word Problems
12.5 Inequations
   12.5.1 Graphical Representation of Inequations
12.6 Quadratic Equation in One Variable
   12.6.1 Solution of a Quadratic Equation
   12.6.2 Relation between Roots and Coefficients
   12.6.3 Equations Reducible to Quadratic Form
   12.6.4 Solving Word Problems
12.7 Let Us Sum Up
12.8 Unit End Activities
12.9 Answers to Check Your Progress
12.10 Suggested Readings

12.1 INTRODUCTION

The word equation is within the comprehension of the students. They must be able to differentiate between an expression and an equation. Equations are of different types depending on the number of variables and the degree of variables. Besides, there are many situations which are represented by inequalities. The student is familiar with the solution of linear equations in one variable.

This unit gives various ways to teach linear equations in two variables and quadratic equations in one variable. Linear inequations and their graphical representation are also discussed.

12.2 OBJECTIVES

At the end of this unit, you will be able to:

• explain the distinction between linear equation in one variable and the one in two variables; a system of equations in two variables and quadratic equations;

* A few sections of this unit has been adopted from ES-342, IGNOU, 2000
- show graphically a linear equation in two variables;
- use various methods of solving systems of linear equations in two variables and quadratic equations;
- describe the difference between consistent and inconsistent systems of equations, both graphically and algebraically;
- inculcate problem solving skills, that are:
  i) translate word problems into mathematical models;
  ii) apply mathematical techniques to solve word problems; and
- show graphically the linear inequations; and
- plan and design learning activities that help to teach linear educations, inequations and quadratic equations.

### 12.3 LINEAR EQUATION IN ONE VARIABLE

**Main Teaching Points**

a) Recognizing a linear equation in one variable

b) Solution of a linear equation in one variable

**Teaching Learning Process**

Students are familiar with the terms equation, expression, variable and degree. As a prelude to further study of equations, you may find out whether students can discriminate between an expression and an equation.

You may present them with a number of expressions and equations and ask them to select those which are equations. Also ask them to find out the number of variables in these equations and write the degree of expression in each of the equation.

**Explain:** Equations in which there is only one variable and the degree of the variable is also one, are called equations of degree one in one variable.

They are also called linear equations in one variable. Consider, a linear equation in one variable, say, \( x + 5 = 7 \).

Ask the students whether the given equation is true for \( x = 3 \)?

Students will be able to tell that it is true only for \( x = 2 \).

**Explain:** The value of the variable for which the given equation is true is called the solution of the equation.

Standard form of the linear equation in one variable is \( ax+b=0 \) and its solution is obtained as follows:

\[
ax + b = 0
\]

Adding \((-b)\) to both sides, we get

\[
ax + b + (-b) = 0 + (-b)
\]

\[
ax = -b
\]

Multiplying both sides by \( \frac{1}{a} \), we get
\[ \frac{1}{a}(ax) = \frac{1}{a}(-b) \]

\[ x = -\frac{b}{a} \] is the solution of the linear equation \( ax + b = 0 \).

**Methodology:** Discussion with various illustrations.

---

**Check Your Progress**

**Notes:** a) Write your answer in the space given below each question.

b) Compare your answer with the one given at the end of the unit.

1) Which of the following are equations?
   - i) \( 2x + 3 = 5 \)
   - ii) \( 3x - 5 \)
   - iii) \( 3x \neq 2x - 10 \)
   - iv) \( x > 10 \)
   - v) \( 3x + 2y = 7 \)
   - vi) \( 2x + 4y < 4 \)
   - vii) \( x^2 + 3x - 4 = 0 \)
   - viii) \( 4x - 3y \)

2) Which of the following are linear equations in one variable. Also write their solutions:
   - i) \( 3x + 2 = 14 \)
   - ii) \( 2x + 3y = 7 \)
   - iii) \( 2x - 3 = 7 \)
   - iv) \( x^2 + x = 6 \)
   - v) \( -2x + 7 = 0 \)
   - vi) \( 5x - 3 = 3x + 1 \)

---

**12.4 LINEAR EQUATION IN TWO VARIABLES**

An equation of degree one involving two variables is discussed in this unit. Such an equation has infinite solutions and its graph is a straight line. The methods of solving system of linear equations in two variables and consistency of equations are the main points which are dealt with here.

**12.4.1 Graph of Linear Equation in two Variables**

**Main Teaching Points**

a) Finding various solutions of a linear equation in two variables

b) Graph of a linear equation in two variables is a straight line
Teaching Learning Process

Through examples you should bring out inductively that an equation of degree one in two variables has infinite solutions and when plotted on a graph, they lie on a straight line.

After revising the solution of a linear equation in one variable, ask them to find the solutions of a linear equation in two variables. viz. \( x + y = 5 \).

**Explain:** Any set of values of \( x \) and \( y \), which satisfies the given equation, is called the solution of the equation.

Here, \( x = 1 \) and \( y = 4 \) is a solution of the given equation.

Ask them if they can tell any other solution?

Ask them if they can find a solution of the equation with \( x = 10 \)? Let them learn to find the value of \( y \) for a given value of \( x \).

Substituting the value of \( x \) in the given equation,

they get: \( 10 + y = 5 \) \( \Rightarrow y = -5 \)

\( \therefore x = 10 \) and \( y = -5 \) is a solution of the equation.

**Ask:** How many solutions of the given equation they can find?

**Explain:** Every linear equation in two variables has infinitely many solutions.

Ask them to write down 5 different solutions of the given equation and plot them on a graph.

Say, the 5 solutions of \( x + y = 5 \) are

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Observe that all the points lie on a straight line.

**Explain:** The graph of an equation of degree one in two variables is a straight line. That is the reason it is called a linear equation.

Also, ask them to find any other solution of the equation and plot it on the graph. Note that it also lies on the same line.

Ask them to take any point of the line say, \((5,0)\). Note that \( x = 5 \) and \( y = 0 \) is a solution of the equation.

**Explain:**

i) Every solution of the equation lie on this line and

ii) Every point of this line is a solution of the given equation

Consider, various examples from real life where two variables are linearly related.

**Example 1:** A train is moving with a uniform speed of 60 km/hr. Draw the time distance graph.
Read from the graph the distance travelled in 2.5 hours.
We know that Speed = Distance/Time
Denoting time by x and distance by y,
we get \(60 = \frac{y}{x}\) \(\Rightarrow 60x = y\) \(\Rightarrow 60x - y = 0\)

It is a linear equation in two variables x and y.

Find various solutions of the equation as shown below:

<table>
<thead>
<tr>
<th>Time in Hours (x)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance in km (y)</td>
<td>60</td>
<td>120</td>
<td>180</td>
<td>240</td>
<td>300</td>
</tr>
</tbody>
</table>

Thus, we plot the ordered pairs: (1,60), (2,120), (3,180), (4,240) and (5,300) and join the points marked by a line.

This line represents all the solutions of the given equation.

From the graph, note that when x=2.5, then value of y=150. Indeed you can verify that x=2.5 and y=150 is a solution of the equation 60x–y=0. Hence, distance travelled in 2.5 hours is 150km.

Fig. 12.2

Ask the students to plot the graph of the equation \(x = 5\).

As a linear equation in one variable x,
Its graph is a single point, as shown below:
Note: It is a line parallel to Y-axis.

Now, if we consider it as a linear equation in two variables \( x \) and \( y \), then it can be expressed as:

\[
x + 0 \cdot y = 5
\]

Which means that the value of \( x \) is 5 for each value of \( y \). So, the various solutions of the equation are as shown below:

| \( x \) | 5 | 5 | 5 | 5 | 5 |
| \( y \) | -1 | 1 | 2 | 3 | 4 |

Plot these points on a graph and join them. Now you can see through the graph of the equation \( x = 5 \) is a straight line parallel to the \( y \)-axis.

Similarly, draw the graphs of \( x = 3 \) and \( x = -2 \)

**Explain:** They are all straight lines parallel to the \( y \)-axis.

Note that the equation of \( y \)-axis is \( x = 0 \). Similarly let the graphs of \( y = 2 \), \( y = 4 \), \( y = -3 \) etc. be drawn and interpreted by the students.

Let them conclude that graph of \( y = \text{constant} \) is a line parallel to \( x \)-axis and the equation of \( x \)-axis is \( y = 0 \).

**Methodology:** Examples are discovered and the facts are derived inductively from the different examples.
12.4.2 Graph of Linear Equation Involving Absolute Values

Main Teaching Point
To draw the graph of a linear equation involving absolute value

Teaching Learning Process
The absolute value of a is written as |a| and it is defined as follows:

\[ |a| = \begin{cases} 
  a, & \text{if } a \geq 0 \\
  -a, & \text{if } a < 0 
\end{cases} \]

Let us study the graph of a linear equation involving absolute value.

**Example 2**: Draw the graph of \( y = 4 \).

By definition \(|y| = 4 \Rightarrow \begin{cases} 
  y = 4, & \text{when } y \geq 0 \\
  y = -4, & \text{when } y < 0 
\end{cases} \)

The graph of the given equation is, Therefore, the union of graphs of \( y = 4 \) and \( y = -4 \) as shown.

**Example 3**: Draw the graph of \(|2x - 3| = 4\)

By definition \(|2x - 3| = \begin{cases} 
  2x - 3, & \text{when } 2x - 3 \geq 0 \\
  -(2x - 3), & \text{when } 2x - 3 < 0 
\end{cases} \)

\[ \therefore |2x - 3| = 4 \Rightarrow \begin{cases} 
  2x - 3 = 4, & \text{when } 2x - 3 \geq 0 \\
  -2x + 3 = 4, & \text{when } 2x - 3 < 0 
\end{cases} \]

or \[ \begin{cases} 
  x = \frac{7}{2}, & \text{when } x \geq \frac{3}{2} \\
  x = -\frac{1}{2}, & \text{when } x < \frac{3}{2} 
\end{cases} \]

\[ \therefore \text{The graph is as shown. It consists of union of graphs of } x = \frac{7}{2} \text{ and } x = -\frac{1}{2}. \]
Example 4: Draw the graph of $y = 2|x|$.

By definition of $|x|$, 

$$y = 2|x|$$ can be written as 

$$y = \begin{cases} 2x, & \text{when } x \geq 0 \\ -2x, & \text{when } x < 0 \end{cases}$$

Ask students to point out the difference in the graph of the equation $y = 2x$ and $y = -2x$ when $x \geq 0$.

Which part of the graph of $y = 2x$ can not form a part of the latter graph?

The graph of $y = 2x$, when $x \geq 0$, will be part of the graph of the line $y = 2x$ in the first quadrant.

The graph of $y = -2x$, when $x < 0$, will be part of the graph of the line $y = -2x$ in the second quadrant.

The graph of $y = 2|x|$ is the union of two rays as shown in the figure.

Methodology: Discussion method is used.
Check Your Progress

Notes: a) Write your answer in the space given below each question.

b) Compare your answer with the one given at the end of the unit.

Draw the graphs of the following equations:
3) \( 2x + y = 5 \)

4) \( y = |x - 2| \)

12.4.3 System of Linear Equations in Two Variables

Main Teaching Points

a) Finding the solution of a system of linear equations in two variables graphically

b) Conditions for consistency of a system of linear equations in two variables

Teaching Learning Process

Two linear equations in two variables taken together with the help of the connective ‘and’ are called a system of linear equations in two variables. Plot the graph of each of the linear equations on the same XY–plane. The point of intersection of the lines is a solution of both the equations. This common solution is called the solution of the system of linear equations. If the lines coincide, then there are infinitely many common solutions and if they are parallel and do not intersect, then the system has no common solution.

Example 5 : Solve the system of linear equations graphically : \( 2x - y = 1 \) and \( x + 2y = 8 \)

\[
\begin{array}{c|c|c|c}
2x - y = 1 & x & 1 & 2 & 3 \\
 & y & 1 & 3 & 5 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
x + 2y = 8 & x & -2 & 0 & 4 \\
 & y & 5 & 4 & 2 \\
\end{array}
\]

Fig. 12.8
Draw the graph of both the equation in the same XOY plane.

Note that the two straight lines intersect at point P(2,3)

So, \( x = 2 \) and \( y = 3 \) is a solution of each of the two equations.

\[ \therefore x = 2 \text{ and } y = 3 \text{ is the solution of the system of linear equations.} \]

**Explain** A system of linear equations is said to be consistent if they have one and only one solution i.e. They have a unique solution. The graph is a two intersecting lines. Algebraically, the system of equations are in general expressed as:

\[
\begin{align*}
\begin{cases}
  a_1x + b_1y = c_1 \\
  a_2x + b_2y = c_2
\end{cases}
\end{align*}
\]

In this example, note that

\[
\frac{a_1}{a_2} = \frac{2}{1} \quad \text{and} \quad \frac{b_1}{b_2} = \frac{-1}{2} \quad \text{and so} \quad \frac{a_1}{a_2} \neq \frac{b_1}{b_2}
\]

\[ \therefore \text{System of equations is consistent having a unique solution if } \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \]

**Example 6** : Solve the following system of equations graphically:

\[
\begin{align*}
2x + y &= 6 \\
4x + 2y &= 6
\end{align*}
\]

\[
\begin{array}{c|c|c|c}
\hline
x & 0 & 1 & 2 \\
\hline
y & 6 & 4 & 2 \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\hline
x & 0 & 1 & 2 \\
\hline
y & 3 & 1 & -1 \\
\hline
\end{array}
\]

\[ \text{Fig. 12.9} \]
Linear Equations, Inequations and Quadratic Equations

Draw the graph of both the equations in the same XY–plane.
Note that the two straight lines are parallel. They do not intersect.
There is no common point and \( \therefore \) there is no common solution.
This system of linear equations has no solution.

**Explain:** A system of linear equations is said to be ‘inconsistent’ if it has no solution. The graph is a pair of parallel lines which do not intersect each other.

In this example note that

\[
\frac{a_1}{a_2} = \frac{2}{4} = \frac{1}{2}, \quad \frac{b_1}{b_2} = \frac{1}{2} \quad \text{and} \quad \frac{c_1}{c_2} = \frac{6}{6} = 1
\]

\[\therefore \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}\]

\[\therefore \quad \text{the system of equations’ is inconsistent having no solution if } \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}\]

Observe that according to first equation \( 2x+y=6 \), but according to the second equation \( 2x + y = 3 \).

\[\therefore \quad \text{If } (x, y) \text{ satisfies first equation then it will not satisfy the second equation and if } (x, y) \text{ is such that } 2x + y = 3 \text{ then } 2x + y \neq 6. \quad \text{The two equations contradict each other.}
\]

**Example 7** Solve \[
\begin{aligned}
\begin{align*}
x - 2y &= 1 \\
3x - 6y &= 3
\end{align*}
\end{aligned}
\] graphically

\[
\begin{array}{c|c|c|c}
x & 1 & 3 & 5 \\
y & 0 & 1 & 2
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
& 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\end{array}
\]

**Fig. 12.10**

The same set of values of \( x \) and \( y \) satisfy both the equations.

Therefore, the lines are coincident. Each point of this line satisfies both the equations. The system of equations,

Therefore, has infinitely many solutions.

**Explain:** A system of linear equations is said to be consistent, in particular dependent, if they have infinitely many solutions. The graph in this case is coincident lines.

Algebraically note that
\[
\frac{a_1}{a_2} = \frac{1}{3}, \quad \frac{b_1}{b_2} = \frac{-2}{-6} = \frac{1}{3} \quad \text{and} \quad \frac{c_1}{c_2} = \frac{1}{3}
\]

\[
\therefore \text{The system of equations is dependent (consistent) having infinitely many solutions, If } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}
\]

Note that the second equation can be obtained from the first equation by multiplying it by 3. \( \left( \frac{a_2}{a_1} = \frac{3}{1} \right) \). Hence, the second equation is not different from the first equation. It actually depends on the first equation.

**Methodology:** Induction method is used. Several examples are discussed and then the results are derived inductively.

### Check Your Progress:

**Notes:**

a) Write your answer in the space given below each question.

b) Compare your answer with the one given at the end of the unit.

Classify the following system of equations as consistent, dependent or inconsistent:

5) \[
\begin{align*}
2x + 3y &= 5 \\
4x + 6y &= 10
\end{align*}
\]

6) \[
\begin{align*}
3x + 4y &= 2 \\
6x + 8y &= -1
\end{align*}
\]

7) \[
\begin{align*}
4x - y &= 5 \\
2x - y &= 3
\end{align*}
\]

8) Solve \[
\begin{align*}
x + y &= 5 \\
x - y &= 3
\end{align*}
\]

graphically

---

### 12.4.4 Methods of Solving System of Linear Equations

**Main teaching Point**

Different methods of solving a system of two linear equations in two variables
Teaching Learning Process

1) Method of Elimination

This method consists essentially of eliminating one of the variables from both the equations and thus getting an equation in a single variable which can be solved to get the value of the variable. This value of one variable thus obtained is substituted in any of the two given equations to get the value of the second variable.

Develop with students’ participation by various methods of elimination. They are the following:

i) Method of Comparison

From each of the two equations, find the value of any one variable in terms of the other and equate them.

Example 8

Solve

\[
\begin{align*}
3x + 2y &= 8 \\
2x + 3y &= 7
\end{align*}
\]

Finding value of \(x\) from each equation, we get

\[
3x + 2y = 8 \Rightarrow x = \frac{8 - 2y}{3} \quad \text{…………………………….. (1)}
\]

\[
2x + 3y = 7 \Rightarrow x = \frac{7 - 3y}{2} \quad \text{…………………………….. (2)}
\]

Equating (1) and (2), we get

\[
\frac{8 - 2y}{3} = \frac{7 - 3y}{2} \Rightarrow 16 - 4y = 21 - 9y \Rightarrow 5y = 5 \Rightarrow y = 1
\]

Putting this value in equation (1), we get

\[
\frac{8 - 2(1)}{3} = 2
\]

\(\therefore\) the solution is \(x = 2\) and \(y = 1\).

Ask the students to solve the system of equations by finding the value of \(y\) from both the equations and then comparing them and verifying that they get the same result.

ii) Method of Substitution

Find the value of any one variable from and one of the equations in terms of the other and then substitute it in the second equation to eliminate that variable.

Example 9

Solve

\[
\begin{align*}
3x + 2y &= 8 \\
2x + 3y &= 7
\end{align*}
\]

\[
3x + 2y = 8 \Rightarrow x = \frac{8 - 2y}{3} \quad \text{…………………………….. (3)}
\]

Substituting in the equation (2), we get

\[
2 \left( \frac{8 - 2y}{3} \right) + 3y = 7 \Rightarrow 2(8 - 2y) + 9y = 21
\]

\[
\Rightarrow 16 - 4y + 9y = 21 \Rightarrow 5y = 5 \Rightarrow y = 1
\]

Putting this value in (3), we get

\[
x = \frac{8 - 2(1)}{3} = 2
\]

\(\therefore\) the solution is \(x = 2\) and \(y = 1\).
The equations can also be solved by finding the value of $y$ from (1) and substituting it in (2).

Ask the students to find the value of the variable $x$ from the second equation and substitute it in the first equation and then find the solution.

iii) **Method of Addition or Subtraction**

Make the coefficients of either $x$ or $y$ in the two equations equal by multiplying the equations by suitable constants and then eliminate the variable by addition or subtraction.

**Example 10** Solve \[
\begin{align*}
3x + 2y &= 8 \quad \text{(1)} \\
2x + 3y &= 7 \quad \text{(2)}
\end{align*}
\]

To make coefficients of $x$ equal in both the equations, multiply equation (1) by 2 and equation (2) by 3 to get:

\[
\begin{align*}
6x + 4y &= 16 \\
6x + 9y &= 21
\end{align*}
\]

Eliminate $x$ by subtracting equation (4) from (3):

We get $-5y = -5 \Rightarrow y = 1$

Substituting $y=1$ in equation (1), we get $3x + 2(1) = 8 \Rightarrow 3x = 6 \Rightarrow x = 2$.

$\therefore$ the solution is $x = 2$ and $y = 1$.

Ask the students to eliminate $y$ from the two equations by this method and then solve the equations to get the same answer.

2) **The Method of Cross Multiplication**

Let the equations be \[
\begin{align*}
a_1x + b_1y + c_1 &= 0 \quad \text{(1)} \\
a_2x + b_2y + c_2 &= 0 \quad \text{(2)}
\end{align*}
\]

Multiplying (1) by $b_2$ and equation (2) by $b_1$, we get

\[
\begin{align*}
 a_1b_2x + b_1b_2y + b_2c_1 &= 0 \quad \text{(3)} \\
 a_2b_1x + b_1b_2y + b_1c_2 &= 0 \quad \text{(4)}
\end{align*}
\]

Subtracting equation (4) from equation (3), we get

\[
(a_1b_2 - a_2b_1)x + (b_2c_1 - b_1c_2) = 0
\]

\[
x = \frac{(b_1c_2 - b_2c_1)}{(a_1b_2 - a_2b_1)} \quad \text{(5)}
\]

Similarly, we can find $y = \frac{(c_1a_2 - c_2a_1)}{(a_1b_2 - a_2b_1)}$ \(\text{(6)}\)

From (5) and (6), we get

\[
\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}
\]

Which can be written as below, where denominators are obtained by cross–multiplication as indicated:

\[
\begin{align*}
\frac{x}{(+) b_1} &= \frac{y}{(+) c_1} = \frac{1}{(+) a_1b_1} \\
&\quad \frac{(+) a_1}{(+) b_1} \\
\frac{(-) b_2}{(+) c_2} &= \frac{(+) a_2}{(+) b_2}
\end{align*}
\]
Explain to the students, the steps given below to use this method.

**Steps**

i) Rewrite the equations in such a way that all the terms are on the left–hand side and zero on the right–hand side. Write the variables $x$ and $y$ in order i.e first $x$, then $y$ and then constants.

Now equations are in the form:

$$
\begin{align*}
\begin{bmatrix}
a_1x + b_1y + c_1 = 0 \\
a_2x + b_2y + c_2 = 0
\end{bmatrix}
\end{align*}
$$

ii) Take the detached coefficients, write them in cyclic order starting with coefficients of $y$ as given below and work the arrows as shown:

```
\begin{array}{cccc}
b_1 & c_1 & a_1 & b_1 \\
b_2 & c_2 & a_2 & b_2
\end{array}
```

iii) Write the first cross–product below $x$, the second cross–product below $y$ and third cross–product below 1.

iv) Simplify to get the values of $x$ and $y$.

**Example 11** Solve

$$
\begin{align*}
3x + 2y &= 8 \\
2x + 3y &= 7
\end{align*}
$$

by the method of cross–multiplication.

**Step 1** Write equations as

$$
\begin{align*}
\begin{bmatrix}
3x + 2y - 8 = 0 \\
2x + 3y - 7 = 0
\end{bmatrix}
\end{align*}
$$

**Step 2** Take the detached coefficients and write them cyclic order starting with coefficients of $y$:

```
\begin{array}{cccc}
2 & -8 & 3 & 2 \\
3 & -7 & 2 & 3
\end{array}
```

**Step 3** Write the cross–products below $x$, $y$ and 1:

$$
\begin{align*}
x &= \frac{2(-7) - (3)(-8)}{(-8)(2) - (-7)(3)} = \frac{1}{3(3) - 2(2)} \\
y &= \frac{1}{5}
\end{align*}
$$

$$
\therefore \quad \begin{align*}
x &= \frac{1}{10} = 2 \\
y &= \frac{1}{5} = 1
\end{align*}
$$

**Step 4**

$$
\begin{align*}
\frac{x}{10} &= \frac{1}{5} \implies x = 2 \\
\frac{y}{5} &= \frac{1}{5} \implies y = 1
\end{align*}
$$

$$
\therefore \quad x = 2 \text{ and } y = 1 \text{ is the solution.}
$$

**Note:** Cross–multiplication method is applicable when the given system is consistent i.e.

$$
\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \quad \text{or} \quad a_1b_2 - a_2b_1 \neq 0
$$

**Methodology:** Discussion method help to involve students in telling different ways of elimination of a variable from two equations.
12.4.5 Solution of Word Problems

Main Teaching Point
Translating a word problem into a mathematical model

Teaching Learning Process
For solving word problems, the main emphasis should be on translating the problem into a mathematical model. Once we obtain two linear equations in two variables, they can be solved by any one of the methods studied earlier.

After reading the problem carefully, determine what are the unknown quantities and denote them by the variables $x$ and $y$. Again read the problem and try to write down relation between the variables and express it as a linear equation in two variables $x$ and $y$.

Example 12 A fraction becomes $2/3$ on subtracting 1 from both the numerator and the denominator. Express it as a mathematical model.

Answer: Ask the students to tell the two unknown quantities in this statement. They can easily conclude that they are numerator and denominator.

Suppose, numerator = $x$ and denominator = $y$.

Then, fraction is $\frac{x}{y}$.

So, according to given statement

$$\frac{x-1}{y-1} = \frac{2}{3} \Rightarrow 3x - 3 = 2y - 2 \Rightarrow 3x - 2y = 1$$

Which is a linear equation in two variables $x$ and $y$.

Example 13 The cost of 3 chairs and 2 tables is Rs 4200. Express it as a linear equation in two variables.

Answer: In this example, we do not know the cost of 1 chair and the cost of 1 table. These are the unknown quantities. Suppose them as $x$ and $y$.

Then according to the given statement, we get

$$3x + 2y = 4200$$

This is a linear equation in two variables $x$ and $y$. Consider a few more such examples of translating a given statement into a mathematical equation.

Now, let us solve a complete problem using this method.

Example 14 The difference between the ages of father and the son is 20 years. After 5 years, father will be twice as old as his son. Find their present ages.

In this problem the unknown quantities are the present ages of the father and the son.

Let the present age of the father be $x$ years and the present age of the son be $y$ years.

Now, it is given that the difference between the ages of father and the son is 20 years.

$\therefore$ The difference between $x$ and $y$ is 20.

$\therefore x - y = 20 \quad -(1)$ (This is the first linear equation in $x$ and $y$)
After 5 years, father will be twice as old as his son.
After 5 years, father will be $x+5$ years old and the son will be $y+5$ years old

\[
\therefore x + 5 = 2(y + 5) \Rightarrow x + 5 = 2y + 10 \Rightarrow x - 2y = 5 \quad -(2)
\]

(This is the second linear equation in $x$ and $y$)

We have obtained two linear equations in two variables $x$ and $y$ i.e.

\[
x - y = 20 \quad -(1)
\]
\[
x - 2y = 5 \quad -(2)
\]

Now, solve equations $(1)$ and $(2)$ by any one of the methods studied earlier to find the values of $x$ and $y$.

Subtracting $(2)$ from $(1)$, we get $y = 15$

Substituting this value of $y$ in equation $(1)$

We get $x - 15 = 20 \Rightarrow x = 20 + 15 = 35$.

\[
\therefore \text{age of father} = 35 \text{ years}
\]

And age of son = 15 years.

Similarly, more examples of word problems be solved with the help of the students.

**Methodology:** Heuristic approach is more suitable for solving word problems. However, discussion should be encouraged and the teacher should guide the students towards logical thinking.

**Check Your Progress**

**Notes:**

a) Write your answer in the space given below each equation.

b) Compare your answer with the one given at the end of the unit.

9) A fraction becomes $\frac{1}{2}$ on adding 1 to both the numerator and the denominator. On adding 2 to the denominator, the fraction becomes $\frac{1}{3}$. Find the fraction.

----------------------------------------------------------------------------------------------------------------------
----------------------------------------------------------------------------------------------------------------------
----------------------------------------------------------------------------------------------------------------------

10) The difference between the ages of a father and his son is 30 years. Last year (one year ago) the age of the father was four times that of his son. Find their present ages.

----------------------------------------------------------------------------------------------------------------------
----------------------------------------------------------------------------------------------------------------------
----------------------------------------------------------------------------------------------------------------------

11) A boat travels 30 km downstream in one hour and comes back in 1.5 hours. Find the speed of the boat in still water and the speed of the stream.

----------------------------------------------------------------------------------------------------------------------
----------------------------------------------------------------------------------------------------------------------
----------------------------------------------------------------------------------------------------------------------
12.5 INEQUATIONS

It is not always the case that we are faced with problems involving equations. There are also situations when we have to deal with inequations. Consider the problem:

Raju went to a general store to purchase sugar. He found that it was available in 250g and 500g packets. He could not carry more than 2 kg of sugar. How many packets of each type could he buy?

Suppose he buys $x$ packets of 250g and $y$ packets of 500g sugar. Since total weight should not exceed 2000g.

- $250x + 500y \leq 2000$ or $x + 2y \leq 8$.

This is a linear inequation in two variables $x$ and $y$.

In general, a linear inequation in two variables $x$ and $y$ is in one of the following forms:

- $ax + by < c$
- $ax + by > c$
- $ax + by \leq c$
- $ax + by \geq c$

where $a, b, x, y$ are real numbers and $a, b$ are not simultaneously equal to zero. An ordered pair which satisfies an inequation has infinitely many solutions.

12.5.1 Graphical Representation of Inequation

Main Teaching Point
To represent an inequation graphically

Teaching Learning Process
Represent graphically: $2x + y \leq 4$, where $x, y \in \mathbb{R}$.

Let the students draw the graph of the equation $2x + y = 4$.

Straight line represent the solution set of the equation $2x + y = 4$.

Ask: In how many sets, the points in the plane are divided.

Explain There are three sets of points

i) Set of points which lie on the line.

ii) Set of points which lie on one side of the line, say, Region I.

iii) Set of points which lie on the other side of the line, say, Region II.

$2x + y = 4$

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>4</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>
Consider, the points (0,5), (2,3) and (3,0) of Region I.

They all satisfy the inequation $2x+y>4$. Consider, the points (0,2), (-1,0) and (1,0) of Region II.

They all satisfy the inequation $2x+y<4$.

Explain that all the points of Region (I) satisfy the inequation $2x+y<4$.

∴ Region (I) represents the solution set of the inequation $2x+y>4$, just as the straight line represents the solution set of the equation $2x+y=4$. Similarly, all the points of Region (II) satisfy the inequation $2x+y<4$.

∴ Region (II) represents the solution set of the inequation $2x+y<4$.

Therefore, the straight line together with Region (II) represent $2x+y≤4$.

We only shade the required region. To represent the solution set of $2x+y<4$, the line is not included and hence it is drawn dotted and only Region II, will be shaded.
Example 15 Represent graphically: $|x| \leq 3$.
First draw the graph of $|x| = 3$ i.e. $x = 3$ or $x = -3$.
Note that all the points between the two lines represent $|x| \leq 3$.
∴ Shaded region together with the two straight lines represent the solution set.
Note that $|x| \leq 3$ means $x \leq 3$ when $x \geq 0$ or $x > -3$ when $x < 0$.

![Graph of $|x| \leq 3$](image)

**Methodology:** Discussion method is suitable as students already know how to draw the graph of an equation.

**Check Your Progress**

Notes: a) Write your answer in the space given below each question.

b) Compare your answer with the one given at the end of the unit.

Draw the graphs of the following linear inequations in two variables $x$ and $y$:

12) $|y| \geq 4$

13) $x \geq 2$

14) $y - x \geq 3$

---

**12.6 QUADRATIC EQUATIONS IN ONE VARIABLE**

An equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$ is a quadratic equation in one variable $x$. Students must be able to recognize the quadratic equations form a given set of equations as follows:

(i) $3x^2 + 7x + 2 = 0$

(ii) $2x^2 - 3x = 5$

(iii) $x^2 + 5 = x^3 - 1$

(iv) $2x^2 + \frac{1}{x^2} = 3$
(v) \( x^2 = x + \frac{2}{x} \)
(vi) \( 3x^2 - 5x + 8 = 3x^2 + 2 \)

Only (i) and (ii) are quadratic equations.

You should discuss with the students why others are not quadratic equations.

12.6.1 Solution of a Quadratic Equation

Main Teaching Points

a) Factor method
b) Completing the square method

Teaching Learning Process

a) Factor Method

Express the given quadratic equation in the form \( p(x) = 0 \), where \( p(x) \) is a quadratic polynomial. Factorize \( p(x) \) to get \( p(x) = q_1(x) \cdot q_2(x) \), where \( q_1(x) \) and \( q_2(x) \) are linear polynomials, so that

\[ p(x) = 0 \Rightarrow q_1(x) \cdot q_2(x) = 0 \Rightarrow q_1(x) = 0 \text{ or } q_2(x) = 0 \]

(using \( ab = 0 \Rightarrow a = 0 \text{ or } b = 0 \)).

Solve each of the two linear equations to find the solution set of the quadratic equation.

Example 16 Solve: \( x^2 + 6x + 5 = 0 \)

Factorizing the left hand side, we get

\( (x+1)(x+5) = 0 \)

\( \Rightarrow x+1 = 0 \quad \text{or} \quad x+5 = 0 \)

\( \Rightarrow x = -1 \quad \text{or} \quad x = -5 \).

\( \therefore \{ -1, -5 \} \) is the solution set of the quadratic equation \( x = -1 \) and \( x = -5 \) are also called the roots of the equation.

b) Method of Completing the Squares

To solve \( ax^2 + bx + c = 0 \), \( a \neq 0 \)

Multiplying both sides by \( 4a \), we get

\( 4a^2x^2 + 4abx + 4ac = 0 \)

\( \therefore 4a^2x^2 + 4abx = -4ac \)

Adding \( b^2 \) to both sides, we get

\( 4a^2x^2 + 4abx + b^2 = b^2 - 4ac \)

Note that L.H.S. becomes a perfect square

\( \therefore (2ax+b)^2 = b^2 - 4ac \)

\( 2ax + b = \pm \sqrt{b^2 - 4ac} \)

\( 2ax = -b \pm \sqrt{b^2 - 4ac} \)

\( \therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)
**Explain:** This is called a quadratic formula and it is used to solve a quadratic equation \( p(x)=0 \), where \( p(x) \) is not easily factorizable. Instead of using this formula, we can use the steps involved in completing the square process.

Moreover, \( b^2-4ac \) is called the discriminant. Why?

This is because, it discriminates between different types of roots:

i) If \( b^2-4ac>0 \), the roots are real and unequal.

ii) If \( b^2-4ac=0 \), the roots are real and equal.

iii) If \( b^2-4ac<0 \), the roots are not real.

**Example 17** Solve \( 2x^2 - 3x - 6 = 0 \)

Here \( a = 2 \), \( b = -3 \) and \( c = -6 \).

\[
\begin{align*}
\therefore x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(3) \pm \sqrt{(-3)^2 - 4(2)(-6)}}{2(2)} \\
\therefore x &= \frac{3 \pm \sqrt{57}}{4}
\end{align*}
\]

Or \( 2x^2 - 3x - 6 = 0 \)

Multiplying by \( 4a = 4 \times 2 = 8 \), both sides, we get

\( 16x^2 - 24x - 48 = 0 \)

Adding \( b^2 = (-3)^2 = 9 \), both sides, we get

\( 16x^2 - 24x + 9 = 48 + 9 \)

LHS becomes a perfect square.

\[
\begin{align*}
\therefore (4x - 3)^2 &= 57 \\
4x - 3 &= \pm \sqrt{57} \\
4x &= 3 \pm \sqrt{57}
\end{align*}
\]

\[
\therefore x = \frac{3 \pm \sqrt{57}}{4}
\]

**Methodology:** Deductive method is used to derive the quadratic formula.

**12.6.2 Relation between Roots and Coefficients**

**Main Teaching Points**

a) Sum of roots = \( \frac{-b}{a} \)

b) Product of roots = \( \frac{c}{a} \)

**Teaching Learning Process**

Ask students to find the roots of the following equations and complete the following table:

<table>
<thead>
<tr>
<th>Equations</th>
<th>Roots</th>
<th>Sum of Roots</th>
<th>1</th>
<th>2</th>
<th>(-b/a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>i) ( x^2 - 7x + 6 = 0 )</td>
<td>1,6</td>
<td>7</td>
<td>1</td>
<td>-7</td>
<td>7</td>
</tr>
<tr>
<td>ii) ( x^2 - 3x - 4 = 0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>iii) ( 2x^2 - 5x - 7 = 0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>iv) ( 3x^2 - 11x + 10 = 0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>v) ( 2x^2 - 3x - 5 = 0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
From this table, let them reach the conclusion that Sum of roots $= \frac{-b}{a}$

Again, ask them to complete the following table:

<table>
<thead>
<tr>
<th>Equations</th>
<th>Roots</th>
<th>Sum of Roots</th>
<th>c</th>
<th>a</th>
<th>$\frac{c}{a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>i) $x^2 - 7x + 6 = 0$</td>
<td>1, 6</td>
<td>6</td>
<td>6</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>ii) $x^2 - 3x - 4 = 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>iii) $2x^2 - 5x - 7 = 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>iv) $3x^2 - 11x + 10 = 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>v) $2x^2 - 3x - 5 = 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From this table students will reach the conclusion that

Product of roots $= \frac{c}{a}$

The above results can also be obtained algebraically as follows:

The roots of the quadratic equation $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

ie $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

and $x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

∴ Sum of Roots $= x_1 + x_2 = \frac{-2b}{2a} = -\frac{b}{a}$

Product of Roots $= x_1x_2$

$$= \left[ \frac{b}{2a} + \left( \frac{\sqrt{b^2 - 4ac}}{2a} \right) \right] \left[ \frac{b}{2a} - \left( \frac{\sqrt{b^2 - 4ac}}{2a} \right) \right]$$

$$= \frac{b^2}{4a^2} - \frac{b^2 - 4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

∴ If, $\alpha, \beta$ are the equation $ax^2 + bx + c = 0$, then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Methodology: Inductive method is used to explain the formulae and then they are derived using a deductive logic.

Check Your Progress

Notes: a) Write your answers in the space given below each question.

b) Compare your answers with the one given at the end of the Unit.

15) If $\alpha$ and $\beta$ are the roots of the equation $ax^2 + bx + c = 0$, find the values of the following:

(i) $\frac{1}{\alpha} + \frac{1}{\beta}$

(ii) $a^2 \beta + a\beta^2$

........................................................................................................................................

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16) For what values of p are the roots of the following equations equal?
   i) $3x^2 - 5x + p = 0$
   ii) $2px^2 - 8x + p = 0$

17) Form a quadratic equation whose roots are:
   i) 2 and 2
   ii) 3 and $-\sqrt{3}$

12.6.3 Equations Reducible to Quadratic Form

Main Teaching Point
How to transform an equation which is not quadratic to a quadratic form in different situations.

Teaching Learning Process
There are equations which are not quadratic but suitable transformation reduces them to a quadratic equation. Such equations are reducible equations.

Example 18 Solve: $2x^4 - 5x^2 + 3 = 0$

This is not a quadratic equation as its degree is 4. Putting $x^2 = z$, we get

$2z^2 - 5z + 3 = 0$

This is a quadratic equation. Now, it can be solved as a normal quadratic equation.

Factorizing LHS we get

$(2z-3) (z-1) = 0$

$2z-3=0$ or $z-1=0$

$z=\frac{3}{2}$ or $z=1$

$x^2=\frac{3}{2}$ or $x^2=1$

$x= \pm \sqrt{\frac{3}{2}}$ or $x=\pm 1$

Example 19 Solve $x + \frac{1}{x} = \frac{13}{6}$, $(x \neq 0)$

This is not a quadratic equation.
Linear Equations, Inequations and Quadratic Equations

Multiplying by $x$, both sides, we get:

$$x^2 + 1 = \frac{13x}{6} \Rightarrow 6x^2 + 6 = 13x \Rightarrow 6x^2 - 13x + 6 = 0$$

This is a quadratic equation.

Factorizing LHS we get: $(3x - 2) (2x - 3) = 0$

$$\Rightarrow 3x - 2 = 0 \text{ or } 2x - 3 = 0 \Rightarrow x = \frac{2}{3} \text{ or } x = \frac{3}{2}$$

**Example 20** Solve: $\sqrt{25 - x^2} = x - 1$

This is not a quadratic equation.

Squaring both sides, we get

$$25 - x^2 = x^2 - 2x + 1 \Rightarrow 2x^2 - 2x - 24 = 0 \Rightarrow x^2 - x - 12 = 0$$

This is a quadratic equation.

Factorizing LHS we get: $(x - 4) (x + 3) = 0$

$$\Rightarrow x - 4 = 0 \text{ or } x + 3 = 0 \Rightarrow x = 4 \text{ or } x = -3.$$  

**Explain:** A linear equation $x - 1 = 0$ has only one solution, i.e., $x = 1$.

Now, $x - 1 = 0 \Rightarrow x = 1$

On squaring both sides, we get $x^2 = 1$ or $x^2 - 1 = 0$.

It is a quadratic equation and it has two roots $x = \pm 1$. These are the roots of the quadratic equation but not of the initial linear equation. The root $x = -1$ is called an extraneous root. So, whenever we do squaring both sides, we must look out for extraneous roots by verifying the roots from the initial equation.

In this example, initial equation is $\sqrt{25 - x^2} = x - 1$

Check whether $x = 4$ is a root of this equation. By substituting $x = 4$, we get

$LHS = \sqrt{25 - 16} = 3$, $RHS = 4 - 1 = 3$.

$\therefore LHS = RHS$

$\therefore x = 4$ is a root of this equation.

Now, verify whether $x = -3$ is its root.

$LHS = \sqrt{25 - 9} = 4$, $RHS = (-3) - 1 = -4$.

$LHS \neq RHS$

$\therefore x = -3$ is not its root.

It is an extraneous root.

$\therefore x = 4$ is the only root of the given equation.

**Example 21** Solve $\sqrt{3x + 10} + \sqrt{6 - x} = 6$

This is not a quadratic equation.

Keeping only one root on LHS, we get

$\sqrt{3x + 10} = 6 - \sqrt{6 - x}$

Squaring both sides, we get

$3x + 10 = 36 - 12 \sqrt{6 - x} - x$
Writing root term on LHS, we get

\[ 12\sqrt{6} - x = 32 - 4x \text{ or } 3\sqrt{6} - x = 8 - x \]

Again squaring both sides, we get:

\[ 9(6 - x) = 64 + x^2 - 7x + 10 = 0 \]

This is a quadratic equation.

Factorizing LHS, we get:

\[ (x - 5)(x - 2) = 0 \implies x - 5 = 0 \text{ or } x - 2 = 0 \implies x = 5 \text{ or } x = 2 \]

Here, students can verify that both are roots of the given equation.

**Example 22** Solve: \[ 3\left(x^2 + \frac{1}{x^2}\right) - 16\left(x + \frac{1}{x}\right) + 26 = 0 \]

This is not a quadratic equation.

Let \( x + \frac{1}{x} = y \)

On squaring both sides, we get:

\[ x^2 + \frac{1}{x^2} + 2 = y^2 \quad \text{or} \quad x^2 + \frac{1}{x^2} = y^2 - 2 \]

Putting these values in the given equation we get:

\[ 3(y^2 - 2) - 16y + 26 = 0 \quad \text{or} \quad 3y^2 - 16y + 20 = 0 \]

It is a quadratic equation and it can be solved as usual.

Let students solve it and obtain \( x = 1, x = 3 \) or \( x = \frac{1}{3} \) as its roots.

**Methodology:** Deductive method is used for teaching the transformation in different situations.

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**Check Your Progress**

Notes:  
(a) Write your answer in the space given below each question.  
(b) Compare your answer with the one given at the end of the Unit.

Solve the following equations:

18) \[ \sqrt{x - 2} + \sqrt{x + 1} = 3 \]

---

19) \[ \left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + 6 = 0 \]
12.6.4 Solving Word Problems

Main Teaching Point
To translate the given word problem into a mathematical model

Teaching Learning Process
There are a few problems which when translated into a mathematical model form a quadratic equation. The main emphasis should be on translation of the language of the problems into an appropriate quadratic equation in one variable.

Example 23
Find two natural numbers whose difference is 3 and sum of their squares is 117.
Let the one natural number be $x$. Since the difference of two numbers is 3.
$\therefore$ The other natural number is $x+3$
Since sum of their squares is 117
$\therefore x^2 + (x+3)^2 = 117$
On simplification, it leads to the quadratic equation
$x^2 + 3x – 54 = 0$
It can be solved as usual to get $x = –9$ or $x = 6$, as its roots.
Since $x = –9$ is not a natural number
$\therefore$ the only solution is $x = 6$
$\therefore$ One number = 6
The other number = 6+3 = 9.

Example 24 The product of two consecutive odd numbers is 15. Find the numbers.
Let the two consecutive odd numbers be $(2x + 1)$ and $(2x + 3)$.
Therefore, $(2x + 1)(2x + 3) = 15 \Rightarrow 4x^2 + 8x – 12 = 0
\Rightarrow x^2 + 2x – 3 = 0
It is a quadratic equation.
Therefore $(x + 3)(x – 1) = 0 \Rightarrow x + 3 = 0$ or $x – 1 = 0
\Rightarrow x = –3$ or $x = 1$ when $x = –3$
The consecutive odd numbers are $–5$ and $–3$.
When $x = 1$
The consecutive odd numbers are 3 and 5.

Methodology: Heuristic method is used to solve word problems. The teacher should guide the students towards logical thinking.
12.7 LET US SUM UP

In this unit, you have learnt different methods of solving a system of two linear equations in two variables and solving a quadratic equation in one variable. You have also learnt to translate the given word problem into a mathematical model and solving them by using your knowledge of solving equations. The relations between roots of a quadratic equation and its coefficients also helps us in simplifying the solution which is otherwise quite lengthy. A linear inequation in two variables has infinitely many solutions which can be easily represented by a graph.

12.8 UNIT END ACTIVITIES

Discuss with the students the solution of the following types of questions and encourage them to use the knowledge acquired in this unit and apply logical thinking in solving such problems:

1. Solve:
   \[
   \begin{align*}
   ax + by &= a^2 + b^2 \\
   \frac{x}{a} + \frac{y}{b} &= 2
   \end{align*}
   \]

2. Solve:
   \[
   6 \left( y^2 + \frac{1}{y^2} \right) - 25 \left( y - \frac{1}{y} \right) + 12 = 0
   \]

3. Solve:
   \[
   \left( \frac{2x + 1}{x - 1} \right)^4 - 10 \left( \frac{2x + 1}{x - 1} \right)^2 + 9 = 0
   \]

4. Find a quadratic equation whose roots are reciprocals of the roots of the equation \( x^2 - 5x - 14 = 0 \)

5. Find a quadratic equation whose roots are squares of the roots of the equation \( ax^2 + bx + c = 0 \) \((a \neq 0)\)

6. Find a quadratic equation whose one root is 2 and sum of the roots is –4.

7. Write the nature of the roots of the equation \( 2x^2 - 5x - 1 = 0 \), without actually finding the roots.

8. Find the value of k, so that the equation \( kx^2 + 3kx + 9 = 0 \) has real and equal roots.

9. Draw the graph of \( y \leq |x| \)

12.9 ANSWERS TO CHECK YOUR PROGRESS

1. (i), (v) and (vii) are equations.

2. i) Linear in one variable; \( x = 4 \)
   ii) Linear in two variables.
   iii) Linear in one variable; \( x = 5 \)
   iv) Quadratic in one variable
   v) Linear in one variable; \( x = 7/2 \)
   vi) Linear in one variable; \( x = 2 \)
5. Dependent
6. Inconsistent
7. Consistent
8. From the graph, the solution is $x = 4, y = 1$

9. $\frac{3}{7}$
10. Father’s age = 41 years, Son’s age=11 years
11. Speed of the boat = 25 km/hour
    Speed of the stream = 5 km/hour

12.
13.

14.

15. (i) \(-\frac{b}{c}\)  
    (ii) \(-\frac{bc}{a^2}\)

16. (i) \(\frac{25}{12}\)  
    (ii) \(± 2\sqrt{2}\)

17. (i) \(x^2 - 4x + 4 = 0\)  
    (ii) \(x^2 + (-3 + \sqrt{3})x - 3\sqrt{3} = 0\)

18. \(x = 3\)

19. \(x = 1\)

12.10 SUGGESTED READINGS


UNIT 13  SETS, RELATIONS, FUNCTIONS AND GRAPHS*

Structure

13.1  Introduction
13.2  Objectives
13.3  Sets
  13.3.1  Introduction to Sets
  13.3.2  Subsets and Universal Set
  13.3.3  Operations on Sets
  13.3.4  Applications of Union and Intersection of Sets
  13.3.5  Cartesian Product of Sets
13.4  Relations
13.5  Functions
13.6  Graphs
13.7  Let Us Sum Up
13.8  Unit End Activities
13.9  Answers to Check Your Progress
13.10  Suggested Readings.

13.1  INTRODUCTION

Mathematics has been defined as the study of sets with structures. Sets provide a language which can be used in the study of Mathematics. This language is sometimes more expressive than the ordinary language. Set language illustrates the use of mathematical symbolism in formulating proofs and illustrating the structures in mathematical logic, probability, Boolean algebra, switching circuits etc.

The concepts of relation and function are fundamental in Mathematics, Sciences and Social Science. In Social Science, the set consists of human beings and the relationships exist between the members of the set of human beings such as father, mother, son, daughter etc. A function essentially shows how one quantity varies with variation of one or more of other quantities. For example, it may show how the area of a rectangle varies with its length and breadth or how the volume of a sphere varies with its radius. The graph of a function gives us a visual picture of the behavior of the function. It can help us in understanding the properties of the function.

13.2  OBJECTIVES

At the end of this unit, you will be able to:

•  explain the meaning of basic terms used in set theory;

* A few sections of this unit has been adopted from ES-342, IGNOU, 2000
apply the concepts of sets for solving daily life problems;

• analyse relations and functions;

• understand the behavior of different functions by drawing their graphs;

and

• help students to acquire an in-depth knowledge of sets, relations functions and graphs and its application in day-to-day life.

13.3 SETS

The idea of sets was developed towards the end of the 19th century. George Boole (1815-1864) and George Cantor (1845-1918) were the two mathematicians credited with the development of the idea of sets. Cantor is considered to be the founder of the set theory.

13.3.1 Introduction of Sets

Main Teaching Point

Basic concepts and definitions related to sets: sets and its elements, notations, roster and set builder forms, equal and equivalent sets, finite and infinite sets.

Teaching-Learning Process

Set is a concept to be explained and not to be defined. The word set is synonyms with the words, collection, group, bunch, chain etc. and an element is synonym of the words object, member etc. Hence, a collection of objects in mathematical language is called a set of elements. But, every collection of objects is not a set. Why? It should be well-defined in the sense that we should be able to decisively say whether any particular object belongs to the set or it does not belong to the set.

Explain: A set is a well-defined collection of objects.

Ask: Is, the collection of all tall boys in the class; a set?

The idea of tallness is vague. Given any student of the class, we cannot say whether he is tall or not. Hence, it is not well defined and it is not a set.

Ask: Is ‘the collection of all boys in a class whose height is above 160 cm, a set?’

Here, it is well defined that any student whose height is above 160 cm is considered as tall and he is a member of the set. So, in this case, it is a set.

Ask: Is ‘the collection of all odd numbers from 1 to 9, a set?’

Here, we can decisively say that 1, 3, 5, 7, 9 are the members of this set. Hence, it is a set.

Explain: Sets are denoted by capital alphabets and all the members are listed and enclosed within parenthesis. The above set can be written as A = {1, 3, 5, 7, 9}. The number ‘3 is a member of set A’ is written as 3 ∈ A. It is also read as ‘3 belongs to the set A’ or 3 is an element of the set A. This form of writing a set when all the elements are listed is called the tabular form or Roster form of expressing a set.
Ask: Consider the set $B=\{a, e, i, o, u\}$. If, $x \in B$, then what can you tell about $x$?

Explain: If $x \in A$, then we can say that $x$ is a vowel of English alphabet. Hence, $x$ is a member of set $B$, if $x$ is a vowel of English alphabet, we write, $B=\{x : x$ is a vowel of English alphabet$\}$. This form of writing a set, where some rule is used to define the members of the set, is called the set builder form. Give some exercises to write the sets given in 'set builder form' into Roster form and vice-versa, so that students become accustomed to using the language of sets.

**Equal and Equivalent Sets:**

Ask: Is the team consisting of Anil, Arun and Ravi same as the team consisting of Ravi Anil and Arun? Teams are same; only members are listed in different order.

Explain: Any two sets with the same elements are equal, irrespective of the order in which the elements are listed.

$$\{a, b, c\}, \{a, c, b\}, \{b, c, a\}, \{c, a, b\},$$

are all equal sets.

Ask: Are the sets $\{a, b, c\}$ and $\{p, q, r\}$ equal?

Write answer: No, as they have different elements.

Ask: Can we associate the members of the set $A=\{a, b, c\}$ with the members of the set $B=\{p, q, r\}$ such that no elements of $A$ are associated with the same element of $B$?

Write answer: Yes, $a \leftrightarrow p, b \leftrightarrow q, c \leftrightarrow r$.

Explain: This association of elements is called a one-to-one correspondence. Two sets are called equivalent sets, if we can establish a one-to-one correspondence between them. Equivalent sets need not be equal, but equal sets are always equivalent.

**Finite and Infinite Sets, Empty Set:**

Ask: How many elements are there in the following sets?

i) Set of natural numbers less than 10;

ii) Set of all natural numbers;

iii) Set of natural numbers between 5 and 6.

Write the answers:

i) 9 elements

ii) Countless elements

iii) No elements.

Explain: In (i), we can count the number of elements in the set and we call it a finite set. In (ii), we cannot count the number of elements in the set, so it is called an infinite set. In (iii), there is no element in the set. It is called an empty set, null set or void set.

It is denoted by $\{\}$ or alphabet $\emptyset$. (called 'phi').

**Methodology:** Discussion method is used along with a number of illustrations.
13.3.2 Subsets and Universal Set

Main Teaching Points:

a) Relation between subsets and universal set

b) Power set

Teaching Learning Process:

Let S be the set of all the students of a school.

Let A be the set of all the students in class VI of that school.

Let B be the set of all the students in the Cricket Team of that school.

Explain: Set A is such that every element of A is an element of the set S. Also, set B is such that every element of B is an element of the set S. Sets A and B are called the subsets of the set S and the set S is considered as the Universal set for the sets A and B.

Set A is a subset of Set B, if every element of set A is in Set B. In mathematical notations, it is expressed as:

\[ A \subset B, \text{ if and only if } a \in A \Rightarrow a \in B. \]

Ask: Write down all the subsets of A = \{1, 2, 3\}.

Explain:

Every set is a subset of itself and also empty set is a subset of every set.

So, the set of all the subsets is

\[ \{ \{ \} , \{ 1 \} , \{ 2 \} , \{ 3 \} , \{ 1, 2 \} , \{ 1, 3 \} , \{ 2, 3 \} , \{ 1, 2, 3 \} \} \]

The number of subsets is \(2^3 = 8\).

Ask: Write all the subsets of B = \{1, 2\} and C \{1\}.
**Explain:** The number of subsets of B are $4 = 2^2$ and the number of subsets of C are $2 = 2^1$.

Let the students inductively conclude that if a set has $n$ elements, then the number it of subsets is $2^n$. The set of all the subsets of a given set is called a Power set. Thus, the set of all subsets of a set of $n$ elements has $2^n$ elements.

**Ask:** Which of the following sets is the largest set?

(i) $A = \text{Set of all the students of the school.}$

(ii) $B = \text{Set of the students of class IX of the school.}$

(iii) $C = \text{Set of the students of class X of the school.}$

**Explain:** $B$ and $C$ are subsets of $A$ and $A$ is the largest set, which contains all the other sets. This largest set is called the Universal set.

**Explain:** Set $R$, of real numbers is a Universal set for the sets of rational, integers, irrational and natural numbers.

**Explain:** The Universal set is not unique, whereas the null set is unique.

**Methodology:** Discussion method is mainly used. Inductive method is used to show that the total number of subsets in a set of $n$ elements is $2^n$.

---

**Check Your Progress:**

**Notes:**

a) Write your answers in the space given below each question.

b) Compare your answer with the one given at the end of the unit.

2) Write the power set of $\{ a, b, c \}$.

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**13.3.3 Operations on Sets**

**Main Teaching Points**

a) Union and Intersection of Sets

b) $n (A \cup B) = n (A) + n (B) - n (A \cap B)$

c) Complement of a Set

**Teaching Learning Process**

In arithmetic we have operations of addition and multiplication. Similarly, in sets we have operations of union and intersection denoted by the symbols ‘∪’ and ‘∩’ respectively.

Sets are lists of objects and union of sets is like making a single list of all the elements in both the lists.

**Ask:** Team for English debate is the set $A = \{ \text{Ravi, Anil, Ashok} \}$ and the team for Hindi debate is the set $B = \{ \text{Ashish, Vimal, Anil} \}$. Write a set representing the list of students, who are either in English debate or in Hindi debate.

**Answer:** $C = \{ \text{Ravi, Anil, Ashok, Ashish, Vimal} \}$. Note that Anil is common in both the teams, and hence, is listed only once in $C$. 

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Explain: The name of Anil is written only once. Set C is called the union of sets A and B. We write:

\[ C = A \cup B \]

\( A \cup B \) contains all the elements, which are either in A or in B. In Mathematical notation:

\[ x \in A \cup B \iff x \in A \text{ or } x \in B. \]

Ask: If \( A = \{1, 2, 3, 4, 5\} \) and \( B = \{3, 4, 5, 6, 7\} \)

Write \( A \cup B = \{1, 2, 3, 4, 5, 6, 7\} \)
\( B \cup A = \{3, 4, 5, 6, 7, 1, 2\} \)
\( A \cup B = B \cup A \) [Since both have the same elements listed in different order.]

Commutative law holds good for union of sets.

We can also use Venn Diagrams to illustrate commutativity of the operation of union. Similarly, using Venn Diagrams and considering different examples, illustrate the following properties of union of sets:

a) Associative law: \( A \cup (B \cup C) = (A \cup B) \cup C \),
b) \( A \cup \emptyset = A \)
c) \( A \cup \emptyset = A \)
d) \( A \subseteq B \Rightarrow A \cup C \subseteq B \cup C \)

Ask: Set \( A = \{ \text{Ravi, Anil, Ashok, Ram}\} \) represent the students, who play cricket and the set \( B = \{ \text{Ravi, Vimal, Naveen, Ram}\} \) represent the students who play hockey. Write a Set C of students, who play both Cricket and Hockey.

Answer: \( C = \{ \text{Ravi, Ram}\} \)

Explain: Set C is the set of elements, which are common in both the sets A and B. Set C is called the intersection of the Sets A and B, written as:

\[ \therefore C = A \cap B \]

\( A \cap B \) contains all the elements, which are in set A and also in set B. In mathematical notation:

\[ x \in A \cap B \iff x \in A \text{ and } x \in B. \]

Ask: Write \( A \cap B \) and \( B \cap A \), when

\( A = \{1, 2, 3, 4, 5\} \) and \( B \{3, 4, 5, 6, 7\} \)

Answer: \( A \cap B = \{3, 4, 5\} \) and \( B \cap A = \{3, 4, 5\} \).

\[ \therefore A \cap B \text{ is same as } B \cap A \]

or \( A \cap B = B \cap A \)

Commutative law holds for intersection of sets.

Illustrate the following properties of intersection of sets by considering different examples and using Venn Diagrams:

i) Associative law: \( (A \cap B) \cap C = A \cap (B \cap C) \),

ii) \( A \cap \emptyset = \emptyset \)

iii) \( A \cap U = A \)
iv) \[ A \subset B \iff A \cap C \subset B \cap C \]

v) Distributive Laws:

\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

**Explain:** In number system we have only one distributive law i.e. \( a \times (b + c) = (a \times b) + (a \times c) \), but in sets, we have two distributive laws with the help of Venn Diagrams illustrate that two sets \( A \) and \( B \) are disjoint, if there is no element common between them i.e. when \( A \cap B \) is an empty set.

**Explain:** \( A = \{a, b, c, d, e\} \) and \( B = \{a, e, i, o, u\} \),

is \( n(A \cup B) = n(A) + n(B) \)?

**Answer:**

\[ A \cup B = \{a, b, c, d, e, i, o, u\} \]
\[ n(A \cup B) = 8, \quad n(A) = 5 \quad \text{and} \quad n(B) = 5 \]
\[ \therefore n(A \cup B) \neq n(A) + n(B) \]

**Explain:** That, in \( n(A) + n(B) \), the common elements \( \{a, e\} \) have been counted twice, both in \( A \) and \( B \), whereas in \( n(A \cup B) \), they are counted only once. So, we can conclude that:

\[ n(A \cup B) = n(A) + n(B) - n(A \cap B) \]

**Explain:** If \( \bar{U} \) is the universal set and \( A \) is any subset of \( \bar{U} \), then complement of the set \( A \) is the set of elements of \( \bar{U} \), which are not in \( A \). Complement of a set \( A \) is denoted by \( A' \) and \( A' = \{ x \in \bar{U} : x \notin A \} \) or \( x \in A \iff x \notin A' \).

Consider, the whole class as the universal set and let \( A \) be the set of students, who drink tea, then \( A' \) is the set of students in the class who do not drink tea. In this way, the class is divided in two groups \( A \) and \( A' \), which are such that together they constitute the whole class, no student is common in the two groups i.e. they are disjoint and the two groups complement each other.

Mathematically,
\[ (i) \quad (A')' = A \quad (ii) \quad A \cup A' = \bar{U} \quad (iii) \quad A \cap A' = \emptyset \]

**Methodology:** Intuitive logic and discussion is used to get the relation \( n(A \cup B) = n(A) + n(B) - n(A \cap B) \). Use of Venn Diagrams is made to illustrate various properties of union, intersection and complement sets. Teacher should take care to compare the properties of algebra of sets with the properties of algebra of real numbers at every step of his discussion on sets.

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**Check Your Progress:**

**Notes:**

a) Write your answer in the space given below each question.

b) Compare your answer with the one given at the end of the Unit.

3. What do the shaded positions in the following diagrams represent?

(i) ![Diag1](image1.png)

(ii) ![Diag2](image2.png)

(iii) ![Diag3](image3.png)
13.3.4 Application of Union and Intersection of Sets

Main Teaching Point

Word problems based on \( n(A \cup B) = n(A) + n(B) - n(A \cap B) \)

Teaching Learning Process

Solving word problems is the best way of understanding the knowledge assimilated. To solve a word problem, students should first read the problem carefully. Analyse the given data and clearly understand what is to be obtained. Encourage students to translate the problem into a mathematical language. Then solve the problem by applying the already learnt formulae and concepts.

Example: 1 In a class of 35 students, 25 play football and 20 play cricket. If each student plays at least one of the two games, how many students play both cricket and football and how many play only cricket?

With the help of students, analyse and translate the problem into a mathematical module.

Let \( A \) be the set of students, who play football, then \( n(A) = 25 \).

Let \( B \) be the set of students who play cricket, then \( n(B) = 20 \).

Since each student plays at least one of the games, \( n(A \cup B) = 35 \).

Now, we are required to find out how many students play both cricket and football i.e. we have to find \( n(A \cap B) \).

Now, we know that

\[
35 = 25 + 20 - n(A \cap B)
\]

\[
\therefore n(A \cap B) = 45 - 35 = 10
\]

The number of students, who play Cricket only

= number of students, who play cricket – number of students, who play both

= 20 – 10 = 10.

This can be better explained with the help of Venn Diagrams.

Example: 2 In a committee, 30 people speak Hindi, 35 speak English and 15 speak both Hindi and English. How many people speak at least one of the two languages?

Analyse: The set \( A \) of people speaking Hindi has 30 elements i.e. \( n(A) = 30 \). The set \( B \) of people speaking English has 35 elements i.e. \( n(B) = 35 \).

\( A \cap B \) is the set of people, who speak both Hindi and English i.e. \( n(A \cap B) = 15 \).

The set of people, who speak at least one of the two languages is \( A \cup B \), so we have to find \( n(A \cup B) \).

\[
\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B)
\]

\[
= 30 + 35 - 15 = 50.
\]

Methodology: Heuristic approach is the best to solve the word problems. In solving these problems, Venn Diagrams are also useful.
13.3.5 Cartesian Product of Sets

Main Teaching Point

a) Cartesian product of sets
b) \( n(A \times B) = n(A) \times n(B) \).

Teaching Learning Process:

Let the set \( A = \{a, b, c\} \) be the set of roads a, b and c between city P and city Q and let \( B = \{x, y\} \) be the set of roads between city Q and city R.

Suppose, we take road a to go from P to Q and the road x to go from Q to R, and represent this route from P to R as \((a, x)\).

Ask: Write in this way all possible routes that can be taken to go from P to R.

Answer: \((a, x), (a, y), (b, x), (b, y), (c, x), (c, y)\)

Ask: Similarly write the possible routes for going from R to P.

Answer: All possible routes from R to P are \((x, a), (x, b), (x, c), (y, a), (y, b), (y, c)\).

Ask: What is the difference between \((a, x)\) and \((x, a)\)?

Answer: \((a, x)\) means we first take road a and then road x, whereas \((x, a)\) denotes that we first take road x and then road a.

Explain: (i) \((a, x)\) and \((x, a)\) are called ordered pairs

(ii) \((a, x) \neq (x, a)\)

(iii) The set \(\{(a, x), (a, y), (b, x), (b, y), (c, x), (c, y)\}\) is the set of all ordered pairs from the set A to the set B. It is called cross-product or Cartesian product of the sets A and B.

\[
\therefore A \times B = \{(x, y) : x \in A, \ y \in B\},
\]

(iv) \(B \times A = \{(y, x) : x \in A, \ y \in B\},\)

(v) Since \((x, y) \neq (y, x),\) note that \(B \times A \neq A \times B\).

(vi) Note that \(n(A \times B) = n(A) \times n(B) = n(B \times A)\)

Ask: What is \(R \times R\) and \(R \times R \times R\), where \(R\) is the set of real numbers?

Explain: \(R \times R\) represents the Euclidean plane or XY – plane and \(R \times R \times R\) represents the Euclidean three dimensional space. The word Euclidean indicates that the idea was first conceived of by the Greek Mathematician, Euclid. \(R \times R\) is also written as \(R^2\) and \(R \times R \times R\) is written as \(R^3\).

Methodology: Discussion is the best way to deliver the meaning of an ordered pair and defining \(A \times B\).
13.4 Relations

Main Teaching Point

What is a relation?

Teaching Learning Process

In real life, we have seen that two persons may be related to each other and may not be related to each other. When they are related, we have seen different types of relations between them such as ‘is a brother of’, ‘is father of’, ‘is friend of’, ‘is neighbor of’ and so on. Similarly, given any two sets A and B, some relationship of the elements of A may exist with the elements of B.

Let

\[ A = \{ a, b, c, d \}, \quad B = \{ 0, 1, 2, 3 \} \]

We represent a relationship from A to B by the following arrow diagram:

![Diagram](image)

**Fig. 13.2**

In this diagram, arrows are used to represent as to which element of B is related to any element of A.

**Ask:**

(i) a is related to which element of B?

(ii) What element of A are related to 2 in B?

**Answer:**

(i) a is related to 2 in B.

(ii) a and c in set A are related to 2 in set B. If this relation from set A to the set B is denoted by R, then a is related to 2 can be written as aR2 or simply by an ordered pair (a, 2).

**Ask:** Write all the ordered pairs representing the relation of elements of the set A with the elements of the set B.

**Answer:** 

\[(a, 2), (b, 1), (b, 3), (c, 1), (c, 2), (c, 3)\]

**Explain:** Relation \( R = \{(x, y) : x \in A, y \in B\} \) is a relation R from set A to set B. It is the set of ordered pairs \((x, y)\) with \(x \in A\) and \(y \in B\).

In the above example

\[ R = \{(a, 2), (b, 1), (b, 3), (c, 1), (c, 2), (c, 3)\} \]

Note that relation R from A to B is a subset of \(A \times B\).

**Ask:** In the above relation, write the set of all the first elements of the ordered pairs in R along with the set of all the second elements of the ordered pairs in R.
Answer: (i) \( \text{dom } R = P = (a, b, c) \subset A \)

(ii) \( \text{ran } R = Q = (1, 2, 3) \subset B \)

Explain: \( P \) is called the domain of \( R \), denoted by \( \text{dom } R \). It is the subset of the set \( A \) consisting of elements of \( A \), which are related to some elements in \( B \). Similarly, \( Q \) is called the range of the relation \( R \) denoted by \( \text{ran } R \). It is the subset of set \( B \) consisting of elements of \( B \), which are related to some elements of \( A \).

Explain: If \( R \) is a relation from \( A \) to \( A \), it is called a relation on \( A \).

Example 3: Let \( A = \{ 2, 3, 4, 5, 6, 7, 8, 9, 10 \} \)
Let \( R \) be a relation on \( A \) defined by \( xRy \). If \( x \) divides \( y \), write \( R \).

Answer: \( R = \{ (2, 2), (2, 4), (2, 6), (2, 8), (2, 10), (3, 3), (3, 6), (3, 9), (4, 4), (4, 8), (5, 5), (5, 10), (6, 6), (7, 7), (8, 8), (9, 9), (10, 10) \} \)

Methodology: Discussion is used to arrive at the mathematical meaning of relation.

13.5 Functions

Main Teaching Point

What is a function?

Teaching Learning Process

A function from set \( A \) to set \( B \) is a relation from the set \( A \) to the set \( B \) such that every element of \( A \) is related to one and only one element of \( B \).

Let \( A = (a, b, c, \ldots) \) and \( B = (1, 2, 3, 4) \)

Ask the students to write down relations from \( A \) to \( B \) and discuss with them whether they are functions or not and why?

For example, consider the relations:

\[ f_1 = \{ (a, 2), (b, 1), (c, 4), (a, 3) \} \]
\[ f_2 = \{ (a, 1), (b, 3), (c, 4) \} \]
\[ f_3 = \{ (a, 3), (b, 4) \} \]

Discuss: In \( f_1 \) all the element of \( A \) are related to some elements of \( B \). But \( a \) is related to two different elements 2 and 3 of \( B \). So, it is not a function. But, it is a relation.

Also, explain that if two ordered pairs have the same first entry, then it is not a function.

In \( f_2 \), every element of \( A \) is related to one and only one element of \( B \). Hence this relation is a function.

In \( f_3 \), \( c \in A \) and it is not related to any element of \( B \), so it is not a function.

Ask: What is the \( \text{dom } f_3 \) ?

Answer: \( \text{dom } f_3 = \{a, b\} \subset A \)

Explain: Since, \( \text{dom } f_3 \neq A \), it is not a function. If \( f \) is a function, then \( \text{dom } f = A \).
So, $f$ is a function from $A$ to $B$, if

(i) $\text{Dom } f = A$ and

(ii) No two ordered pairs have the same first entry.

If $x \in A$ and $x \in B$ and $f : A \rightarrow B$ such that $f : x \rightarrow y$, then we write it as $y = f(x)$ and say $y$ is image of $x$ or $f(x)$ is image of $x$.

If a relation is represented by an arrow diagram, then discuss with the students, whether it is a function or not?

**Explain:** A relation is represented by an arrow diagram. It is not a function, if (a) any element of $A$ is such that no arrow starts from it (fig. (ii)) or (b) any element of $A$ is such that two or more arrows start from it (fig. (ii)), otherwise it is a function (fig. (iii)).

(i) Not a function

(ii) Not a function

(iii) It is a function.

Fig. 13.3
**Explain:** (i) For any function $f$ from $A$ to $B$, domain of $f$ is $A$ and $B$ is called the co-domain of $f$.

(ii) Range of $f$ is a subset of $B$.

(iii) If the domain and co-domain of a function are the sets (or subsets) of real numbers, the function is called a real function.

**Ask:** Can you correlate the concept of a function with a daily life situation?

**Explain:** Explain that the relation between a temperature expressed in Fahrenheit and the temperature expressed in Celsius given by $C = \frac{5}{9} (F - 32)$, is a function. Graph of this function is a straight line. Corresponding to every value of $F$, we can find the value of $C$ and vice-versa. Area of a circle ‘$A$’ is a function of its radius ‘$r$’. This function is given by $A = \pi r^2$. Its graph is a part of a parabola. Similarly, many other real life situations can be explained using the concept of function.

**Methodology:** Discussion method is used to illustrate the meaning of a function, Mainly, the lecture method is used to define domain, co-domain and range.

**Check Your Progress:**

**Notes:**

a) Write your answer in the space given below each question.

b) Compare your answer with the one given at the end of the unit.

4) Which of the following relations is a function from $\{1, 2, 3\}$ to itself:

i) $\{ (1,2), (2,3), (3,1) \}$ ............................................

ii) $\{ (1,1), (1,2), (1,3) \}$ ............................................

iii) $\{ (1,1), (2,1), (3,1) \}$ ............................................

iv) $\{ (1,1), (2,1), (3,1) \}$ ............................................

**13.6 GRAPHS**

**Main Teaching Point**

Graphical Representation of Different Functions

**Teaching Learning Process:**

Rene Descartes, a French Mathematician, first showed the relationship of algebra and geometry by associating number pairs (ordered pairs) with points in plane. The ordered pair associated with a point on the plane is called the Cartesian coordinates of the point after the name of the mathematician.

Students are already familiar with geometrical representation of a linear equation in two variables $x$ and $y$. You can start with recapitulating the idea and explaining that the linear equation in two variables $x$ and $y$ can be written as $y = (\text{linear expression in } x)$, and so, it also represents a function.
**Ask:** Draw the graph of \( y = 2x + 1 \).

\[
\begin{array}{c|c|c|c}
  x & 0 & 1 & 2 \\
  y & 1 & 3 & 5 \\
\end{array}
\]

**Fig. 13.4**

**Explain:** Consider, the function \( f : \mathbb{R} \rightarrow \mathbb{R} \), such that \( f = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}, y = 2x + 1\} \). Clearly, \( f \) is a real function and the above graph geometrically represents this function. Every ordered pair belonging to \( f \) is represented by a point on this line and ordered pair associated with every point on this line belongs to the given function.

A real function \( f \) such that \( f(x) = ax + b \) is called a linear function because, it represents a straight line, geometrically.

**Ask:** What is the graph of the function \( y = [x] \), where \( [x] \) is the greatest integer less than or equal to \( x \).

According to the definition of \([x]\),

- If \( 1 \leq x < 2 \), \( y = 1 \), it is a line parallel to \( x \)-axis.
- If \( 2 \leq x < 3 \), \( y = 2 \), it is also a line parallel to \( x \)-axis and so on.

**Fig. 13.5**

**Explain:** The value of \( y \) is always an integer and the graph of \( y = [x] \) integer is a line parallel to \( x \)-axis.
Therefore, the graph of the function \( y = (x) \) consists of a number of line segments all parallel to x-axis, such that the points (1, 0), (2, 1), (3, 2) which are not included in the graph are represented in the graph by encircling these points.

**Ask:** Draw the graph of \( y = x^2 \), construct the following table:

<table>
<thead>
<tr>
<th>x</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>16</td>
<td>9</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
</tr>
</tbody>
</table>

Draw the graph as shown:

![Graph of \( y = x^2 \)](image)

**Explain:** The graph is called a parabola. It is symmetric about y-axis.

**Ask:** The students to draw the graph of \( y = -x^2 \) and other simple quadratic functions.

**Explain:** The function \( f = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}, y=ax^2+bx+c\} \) is called a quadratic function. Its graph is always a parabola with its axis of symmetry as \( x = \frac{-b}{2a} \). The parabola opens upwards if \( a > 0 \) and downwards if \( a < 0 \).

**Ask:** Draw the graph of the function.

\[
f = \{(x, y) : x \in \mathbb{R} - \{0\}, y \in \mathbb{R}, y = \frac{12}{x}\}
\]

To draw the graph of \( y = \frac{12}{x} \), construct the following table:

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>-1</th>
<th>-2</th>
<th>-3</th>
<th>-4</th>
<th>-6</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>12</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>-12</td>
<td>-6</td>
<td>-4</td>
<td>-3</td>
<td>-2</td>
</tr>
</tbody>
</table>

Plot these points and draw the graph as shown.

\[
y = \frac{12}{x}
\]
Explain: The function is not defined at x = 0. The graph is called a rectangular hyperbola. Here y varies inversely as x i.e. y approaches 0 as x becomes a large number and y becomes very large as x approaches 0.

Methodology: Skill can be developed only by drill method.

Check Your Progress:

Notes:

a) Write your answer in the space given below each question.

b) Compare your answer with the one given at the end of the unit.

5) Draw the graphs of the following functions:

(i) $y = 3 - 2x$  (ii) $y = \frac{x^2}{2}$  (iii) $y = \sqrt{x}, \; x \geq 0$

………………………………………………………………………

………………………………………………………………………

13.7 LET US SUM UP

In this unit, we have discussed many types of activities on sets, relations and functions. Many real life problems can be associated with the teaching of sets and functions. Regardless of the ability level of the students, appropriate activities can be found. Teachers should constantly gather material and ideas for teaching of this unit.

The concepts of Cartesian product of sets, relations and functions, their domain and range, and understanding the behavior of functions by graphically
representing them have been studied in this chapter. Remember that students should develop proper understanding of these concepts as they lay a foundation stone for the study of another important branch of Mathematics called ‘calculus’.

13.8 UNIT END ACTIVITIES

1) Which of the following collections are not sets and why?
   a) \( \{ x : x^2 - 5x + 6 = 0 \} \)
   b) \( \{ x : x \text{ is a good student of the class} \} \)
   c) \( \{ x : x \text{ is a capital city of a state in India} \} \)
   d) \( \{ x : x \text{ is a fair complexioned student of the class} \} \).

2) Write the following sets in Roster Form:
   a) \( \{ x : x^2 = 9 \} \)
   b) \( \{ x : x \text{ is a letter of the word ‘collection’ } \} \)
   c) \( \{ x : x \in \mathbb{N}, x \text{ is a prime number less than 20 } \} \)
   d) \( \{ x : x \in \mathbb{N}, x \text{ is a divisor of } 10 \} \).

3) Which of the following sets are equal?
   a) \( \{ I, G, N, O, U \} \text{ and } \{ N, G, I, O, U \} \)
   b) \( \{ 1, 2, 3, 5 \} \text{ and } \{ 3, 5, 7, 9 \} \)
   c) \( \{ 1, -1 \} \text{ and } \{ x, x^2 = 1 \} \)
   d) \( \{ a, b, c, d, \} \text{ and } \{ c, b, a, d \} \)

4) Write true or false:
   a) \( \{ 1, 2, 3 \} \subset \{ 1, 2, 3 \} \)
   b) Set of integers is a subset of the set of natural numbers.
   c) Empty set is an element of every set
   d) Empty set is a subset of every set.
   e) \( \{ a, b \} = \{ b, a \} \)
   f) \( (a, b) = (b, a) \)
   g) If \( A \subset B \text{ and } B \subset C \), then \( A \subset C \).

5) Draw arrow diagrams to illustrate the relation \( R \) from set \( A \) to set \( B \):
   a) \( A = \{ 1, 2, 3, 4 \}, \ B = \{ 4, 8, 12, 16, 20 \} \text{ and }
      R = \{ (1, 4), (2, 8), (3, 12), (4, 16) \} \).
   b) \( A = \{ 1, 2, 3, 4 \}, \ B = \{ 7, 8, 9 \} \text{ and }
      R = \{ (1, 9), (2, 8), (3, 7), (4, 7) \} \).
   c) \( A = \{ 1, 2, 3, 4 \}, \ B = \{ 5, 6, 7 \} \text{ and }
      R = \{ (1, 5), (2, 6), (3, 7) \} \).
   d) \( A = \{ 1, 2, 3, 4 \} \text{ and } B = \{ 1, 2, 3, 4, 5, 6 \} \text{ and }
      R = \{ (x, y) : x \in A, y \in B \text{ and } y = x + 3 \} \).
6) Draw the graphs of the following real functions:

a) \( f(x) = 3 + 2x \)

b) \( f(x) = 4 - x^2 \)

c) \( f(x) = \frac{x}{2} \)

d) \( f(x) = |x| - x \)

13.9 ANSWERS TO CHECK YOUR PROGRESS

1) (i) Not a set
(ii) A finite set

2) Power set = \{ {}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \}

3) (i) \( A' \) or \( \sim A \)
(ii) \( A \cap B \)
(iii) \( A \cap B' \) i.e. elements, which are A and not in B.

4) (i) and (iii) are functions.

5) (i) \( y = 3 - 2x \)

\[
\begin{array}{c|c|c|c}
 x  & 0 & 2 & 4 \\
 y  & 3 & -1 & -5 \\
\end{array}
\]

(ii) \( y = \frac{x^2}{2} \)

\[
\begin{array}{c|c|c|c|c|c|c|c}
 x  & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
 y  & \frac{9}{2} & 2 & \frac{1}{2} & 0 & \frac{1}{2} & 2 & \frac{9}{2} \\
\end{array}
\]
(iii) \( y = \sqrt{x} \) \((x \geq 0)\)

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>4</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

13.10 SUGGESTED READINGS


