UNIT 11  MULTIPLE CORRELATION

Structure

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   Objectives
11.2 Coefficient of Multiple Correlation
11.3 Properties of Multiple Correlation Coefficient
11.4 Summary
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11.1 INTRODUCTION

In Unit 9, you have studied the concept of regression and linear regression. Regression coefficient was also discussed with its properties. You learned how to determine the relationship between two variables in regression and how to predict value of one variable from the given value of the other variable. Plane of regression for trivariate, properties of residuals and variance of the residuals were discussed in Unit 10 of this block, which are basis for multiple and partial correlation coefficients. In Block 2, you have studied the coefficient of correlation that provides the degree of linear relationship between the two variables.

If we have more than two variables which are interrelated in someway and our interest is to know the relationship between one variable and set of others. This leads us to multiple correlation study.

In this unit, you will study the multiple correlation and multiple correlation coefficient with its properties. To understand the concept of multiple correlation you must be well versed with correlation coefficient. Before starting this unit, you go through the correlation coefficient given in Unit 6 of the Block 2. You should also clear the basics given in Unit 10 of this block to understand the mathematical formulation of multiple correlation coefficients.

Section 11.2 discusses the concept of multiple correlation and multiple correlation coefficient. It gives the derivation of the multiple correlation coefficient formula. Properties of multiple correlation coefficients are described in Section 11.3

Objectives

After reading this unit, you would be able to
- describe the concept of multiple correlation;
- define multiple correlation coefficient;
- derive the multiple correlation coefficient formula; and
- explain the properties of multiple correlation coefficient.
11.2 COEFFICIENT OF MULTIPLE CORRELATION

If information on two variables like height and weight, income and expenditure, demand and supply, etc. are available and we want to study the linear relationship between two variables, correlation coefficient serves our purpose which provides the strength or degree of linear relationship with direction whether it is positive or negative. But in biological, physical and social sciences, often data are available on more than two variables and value of one variable seems to be influenced by two or more variables. For example, crimes in a city may be influenced by illiteracy, increased population and unemployment in the city, etc. The production of a crop may depend upon amount of rainfall, quality of seeds, quantity of fertilizers used and method of irrigation, etc. Similarly, performance of students in university exam may depend upon his/her IQ, mother’s qualification, father’s qualification, parents income, number of hours of studies, etc. Whenever we are interested in studying the joint effect of two or more variables on a single variable, multiple correlation gives the solution of our problem.

In fact, multiple correlation is the study of combined influence of two or more variables on a single variable.

Suppose, $X_1, X_2$ and $X_3$ are three variables having observations on $N$ individuals or units. Then multiple correlation coefficient of $X_1$ on $X_2$ and $X_3$ is the simple correlation coefficient between $X_1$ and the joint effect of $X_2$ and $X_3$. It can also be defined as the correlation between $X_1$ and its estimate based on $X_2$ and $X_3$.

Multiple correlation coefficient is the simple correlation coefficient between a variable and its estimate.

Let us define a regression equation of $X_1$ on $X_2$ and $X_3$ as

$$X_1 = a + b_{12}X_2 + b_{13}X_3$$

Let us consider three variables $x_1, x_2$ and $x_3$ measured from their respective means. The regression equation of $x_1$ depends upon $x_2$ and $x_3$ is given by

$$x_1 = b_{12}x_2 + b_{13}x_3$$  \hspace{1cm} \ldots (1)$$

Where

$$X_1 - \overline{X}_1 = x_1, X_2 - \overline{X}_2 = x_2 \text{ and } X_3 - \overline{X}_3 = x_3$$

$$\therefore \sum x_1 = \sum x_2 = \sum x_3 = 0$$

Right hand side of equation (1) can be considered as expected or estimated value of $x_1$ based on $x_2$ and $x_3$ which may be expressed as

$$x_{1,23} = b_{12}x_2 + b_{13}x_3$$  \hspace{1cm} \ldots (2)$$

Residual $e_{1,23}$ (see definition of residual in Unit 5 of Block 2 of MST 002) is written as

$$e_{1,23} = x_1 - b_{12}x_2 - b_{13}x_3 = x_1 - x_{1,23}$$
The multiple correlation coefficient can be defined as the simple correlation coefficient between \( x_1 \) and its estimate \( e_{123} \). It is usually denoted by \( R_{123} \) and defined as

\[
R_{123} = \frac{\text{Cov}(x_1, x_{123})}{\sqrt{\text{V}(x_1)\text{V}(x_{123})}} \quad \text{... (4)}
\]

Now,

\[
\text{Cov}(x_1, x_{123}) = \frac{1}{N} \sum (x_i - \bar{x}_1)(x_{123} - \bar{x}_{123})
\]

(By the definition of covariance)

Since, \( x_1, x_2, \text{ and } x_3 \) are measured from their respective means, so

\[
\sum x_1 = \sum x_2 = \sum x_3 = 0 \quad \Rightarrow \bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 0
\]

and consequently

\[
\bar{x}_{123} = b_{123} \bar{x}_2 + b_{132} \bar{x}_3 = 0 \quad \text{(From equation (2))}
\]

Thus,

\[
\text{Cov}(x_1, x_{123}) = \frac{1}{N} \sum x_i x_{123}
\]

\[
= \frac{1}{N} \sum x_i (x_i - e_{123}) \quad \text{(From equation (3))}
\]

\[
= \frac{1}{N} \sum x_i^2 - \frac{1}{N} \sum x_i e_{123} \quad \text{(By third property of residuals)}
\]

\[
= \frac{1}{N} \sum x_i^2 - \frac{1}{N} \sum e_{123}^2
\]

\[
= \sigma_1^2 - \sigma_{123}^2 \quad \text{(From equation (29) of Unit 10)}
\]

Now \( \text{V}(x_{123}) = \frac{1}{N} \sum (x_{123} - \bar{x}_{123})^2 \)

\[
= \frac{1}{N} \sum (x_{123})^2 \quad \text{(Since } \bar{x}_{123} = 0)\]

\[
= \frac{1}{N} \sum (x_i - e_{123})^2 \quad \text{(From equation (3))}
\]

\[
= \frac{1}{N} \sum (x_i^2 + e_{123}^2 - 2x_i e_{123})
\]
Regression and Multiple Correlation

\[ \frac{1}{N} \sum x_i^2 + \frac{1}{N} \sum e_{i,23}^2 - 2 \frac{1}{N} \sum x_i e_{i,23} \]

\[ = \frac{1}{N} \sum x_i^2 + \frac{1}{N} \sum e_{i,23}^2 - 2 \frac{1}{N} \sum e_{i,23} \]

(By third property of residuals)

\[ V(x_{1,23}) = \sigma_i^2 - \sigma_{i,23}^2 \]  
(From equation (29) of Unit 10)

Substituting the value of \( \text{Cov}(x_1, x_{1,23}) \) and \( V(x_{1,23}) \) in equation (4), we have

\[ R_{1,23} = \frac{\sigma_i^2 - \sigma_{i,23}^2}{\sqrt{\sigma_i^2 (\sigma_i^2 - \sigma_{i,23}^2)}} \]

\[ R_{2,13}^2 = \frac{(\sigma_i^2 - \sigma_{i,23}^2)^2}{\sigma_i^2 (\sigma_i^2 - \sigma_{i,23}^2)} \]

\[ R_{2,13}^2 = \frac{\sigma_i^2 - \sigma_{i,23}^2}{\sigma_i^2} = 1 - \frac{\sigma_{i,23}^2}{\sigma_i^2} \]

here, \( \sigma_{i,23}^2 \) is the variance of residual, which is

\[ \sigma_{i,23}^2 = \frac{\sigma_i^2}{1 - r_{23}^2} (1 - r_{23}^2 - r_{12}^2 - r_{13}^2 + 2 r_{12} r_{23} r_{13}) \]  
(From equation (30) of unit 10)

Then,

\[ R_{2,13}^2 = 1 - \frac{\sigma_i^2 (1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2 r_{12} r_{13} r_{23})}{\sigma_i^2 (1 - r_{23}^2)} \]

\[ R_{2,13}^2 = 1 - \frac{1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2 r_{12} r_{13} r_{23}}{1 - r_{23}^2} \]

\[ R_{2,13}^2 = \frac{1 - r_{23}^2 - 1 + r_{13}^2 + r_{13}^2 + r_{23}^2 - 2 r_{12} r_{13} r_{23}}{1 - r_{23}^2} \]

\[ R_{1,23}^2 = \frac{r_{12}^2 + r_{13}^2 - 2 r_{12} r_{13} r_{23}}{1 - r_{23}^2} \]

\[ R_{1,23} = \sqrt{\frac{r_{12}^2 + r_{13}^2 - 2 r_{12} r_{13} r_{23}}{1 - r_{23}^2}} \]

... (5)

which is required formula for multiple correlation coefficient.

where, \( r_{12} \) is the total correlation coefficient between variable \( X_1 \) and \( X_2 \),

\[ r_{23} \] is the total correlation coefficient between variable \( X_2 \) and \( X_3 \),
Multiple Correlation

$r_{13}$ is the total correlation coefficient between variable $X_1$ and $X_3$.

Now let us solve a problem on multiple correlation coefficients.

**Example 1:** From the following data, obtain $R_{123}$ and $R_{213}$.

<table>
<thead>
<tr>
<th></th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$(X_1)^2$</th>
<th>$(X_2)^2$</th>
<th>$(X_3)^2$</th>
<th>$X_1X_2$</th>
<th>$X_1X_3$</th>
<th>$X_2X_3$</th>
</tr>
</thead>
<tbody>
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<td>1</td>
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<td>9</td>
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<td>3136</td>
<td>81</td>
<td>3640</td>
<td>585</td>
<td>504</td>
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<td>2</td>
<td>72</td>
<td>58</td>
<td>11</td>
<td>5184</td>
<td>3364</td>
<td>121</td>
<td>4176</td>
<td>792</td>
<td>638</td>
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<td>8</td>
<td>2916</td>
<td>2304</td>
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<td>432</td>
<td>384</td>
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<td>793</td>
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<td>10</td>
<td>3025</td>
<td>2500</td>
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<td>2750</td>
<td>550</td>
<td>500</td>
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<td>2601</td>
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<td>408</td>
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<td>3025</td>
<td>121</td>
<td>4290</td>
<td>858</td>
<td>605</td>
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<td>8</td>
<td>58</td>
<td>48</td>
<td>10</td>
<td>3364</td>
<td>2304</td>
<td>100</td>
<td>2784</td>
<td>580</td>
<td>480</td>
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<td>57</td>
<td>52</td>
<td>11</td>
<td>3249</td>
<td>2704</td>
<td>121</td>
<td>2964</td>
<td>627</td>
<td>572</td>
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<td>10</td>
<td>51</td>
<td>42</td>
<td>7</td>
<td>2601</td>
<td>1764</td>
<td>49</td>
<td>2142</td>
<td>357</td>
<td>294</td>
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<td>Total</td>
<td>617</td>
<td>521</td>
<td>98</td>
<td>38753</td>
<td>27423</td>
<td>990</td>
<td>32495</td>
<td>6137</td>
<td>5178</td>
</tr>
</tbody>
</table>

**Solution:** To obtain multiple correlation coefficients $R_{123}$ and $R_{213}$, we use following formulae

$$R_{123}^2 = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1-r_{23}^2} \quad \text{and}$$

$$R_{213}^2 = \frac{r_{12}^2 + r_{23}^2 - 2r_{12}r_{23}r_{13}}{1-r_{13}^2}$$

We need $r_{12}$, $r_{13}$ and $r_{23}$ which are obtained from the following table:

Now we get the total correlation coefficient $r_{12}$, $r_{13}$ and $r_{23}$

$$r_{12} = \frac{\sum_1^N (x_{1i}x_{3i}) - (\sum_1^N x_{1i})(\sum_1^N x_{3i})}{\sqrt{[\sum_1^N (x_{1i}^2) - (\sum_1^N x_{1i})^2][\sum_1^N (x_{3i}^2) - (\sum_1^N x_{3i})^2]}}$$

$$r_{12} = \frac{(10\times32495) - (617\times521)}{\sqrt{[(10\times38753) - (617\times98)][(10\times27423) - (521\times5178)]}}$$
Regression and Multiple Correlation

\[ r_{12} = \frac{3493}{\sqrt{4641 \times 2789}} = \frac{3493}{4368.01} = 0.80 \]

\[ r_{13} = \frac{N(\sum X_1X_3 - (\sum X_1)(\sum X_3))}{\sqrt{N(\sum X_1^2 - (\sum X_1)^2)(\sum X_3^2 - (\sum X_3)^2)}} \]

\[ (10 \times 6137) - (617 \times 98) \]

\[ \frac{r_{13}}{\sqrt{[10 \times 38753] - (617 \times 617)[10 \times 990] - (98 \times 98)}} \]

\[ r_{13} = \frac{904}{\sqrt{6841 \times 296}} = \frac{904}{1423.00} = 0.64 \]

and

\[ r_{23} = \frac{N(\sum X_2X_3 - (\sum X_2)(\sum X_3))}{\sqrt{N(\sum X_2^2 - (\sum X_2)^2)(\sum X_3^2 - (\sum X_3)^2)}} \]

\[ (10 \times 5178) - (521 \times 98) \]

\[ \frac{r_{23}}{\sqrt{[10 \times 27423] - (521 \times 521)[10 \times 990] - (98 \times 98)}} \]

\[ r_{23} = \frac{722}{\sqrt{2789 \times 296}} = \frac{722}{908.59} = 0.79 \]

Now, we calculate \( R_{1,23} \)

We have, \( r_{12} = 0.80 \), \( r_{13} = 0.64 \) and \( r_{23} = 0.79 \), then

\[ R_{1,23}^2 = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2} \]

\[ = \frac{0.80^2 + 0.64^2 - 2 \times 0.80 \times 0.64 \times 0.79}{1 - 0.79^2} \]

\[ = \frac{0.64 + 0.41 - 0.81}{1 - 0.62} \]

\[ R_{1,23}^2 = \frac{0.24}{0.38} = 0.63 \]

Then

\[ R_{1,23} = 0.79. \]

\[ R_{2,13}^2 = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{13}^2} \]

\[ = \frac{0.80^2 + 0.79^2 - 2 \times 0.80 \times 0.64 \times 0.79}{1 - 0.64^2} \]

\[ = \frac{0.64 + 0.62 - 0.81}{1 - 0.49} \]

\[ = \frac{0.45}{0.51} = 0.88 \]
Thus, \( R_{2,13} = 0.94 \)

**Example 2:** From the following data, obtain \( R_{1,23} \), \( R_{2,13} \) and \( R_{3,12} \)

<table>
<thead>
<tr>
<th>( X_1 )</th>
<th>2</th>
<th>5</th>
<th>7</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_2 )</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>( X_3 )</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
</tr>
</tbody>
</table>

**Solution:** To obtain multiple correlation coefficients \( R_{1,23} \), \( R_{2,13} \) and \( R_{3,12} \), we use the following formulae:

\[
R_{1,23}^2 = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2},
\]

\[
R_{2,13}^2 = \frac{r_{12}^2 + r_{23}^2 - 2r_{12}r_{23}r_{13}}{1 - r_{13}^2}
\]

\[
R_{3,12}^2 = \frac{r_{13}^2 + r_{23}^2 - 2r_{13}r_{23}r_{12}}{1 - r_{12}^2}
\]

We need \( r_{12} \), \( r_{13} \) and \( r_{23} \) which are obtained from the following table:

<table>
<thead>
<tr>
<th>S. No.</th>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( X_3 )</th>
<th>( (X_1)^2 )</th>
<th>( (X_2)^2 )</th>
<th>( (X_3)^2 )</th>
<th>( X_1X_2 )</th>
<th>( X_1X_3 )</th>
<th>( X_2X_3 )</th>
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<tr>
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<td>1</td>
<td>9</td>
<td>1</td>
<td>6</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
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<td>42</td>
<td>60</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>11</td>
<td>12</td>
<td>10</td>
<td>121</td>
<td>144</td>
<td>100</td>
<td>132</td>
<td>110</td>
<td>120</td>
</tr>
<tr>
<td>Total</td>
<td>25</td>
<td>31</td>
<td>20</td>
<td>199</td>
<td>289</td>
<td>146</td>
<td>238</td>
<td>169</td>
<td>201</td>
</tr>
</tbody>
</table>

Now we get the total correlation coefficient \( r_{12} \), \( r_{13} \) and \( r_{23} \)

\[
r_{12} = \frac{N(\sum X_1X_2) - (\sum X_1)(\sum X_2)}{\sqrt{[N(\sum X_1^2) - (\sum X_1)^2][N(\sum X_2^2) - (\sum X_2)^2]}}
\]

\[
r_{13} = \frac{N(\sum X_1X_3) - (\sum X_1)(\sum X_3)}{\sqrt{[N(\sum X_1^2) - (\sum X_1)^2][N(\sum X_3^2) - (\sum X_3)^2]}}
\]

\[
r_{23} = \frac{N(\sum X_2X_3) - (\sum X_2)(\sum X_3)}{\sqrt{[N(\sum X_2^2) - (\sum X_2)^2][N(\sum X_3^2) - (\sum X_3)^2]}}
\]

Thus,

\[
(4 \times 238) - (25 \times 31)
\]

\[
(4 \times 199) - (25 \times 25)
\]

\[
(4 \times 289) - (31 \times 31)
\]

\[
= 177 \quad = 177 \quad = 0.97
\]

\[
\sqrt{171 \times 195} = 182.61
\]

\[
\sqrt{146 \times 238 \times 169 \times 201}
\]

\[
= 0.97
\]

\[
\sqrt{182.61}
\]
Regression and Multiple Correlation

\[ r_{13} = \frac{(4 \times 169) - (25 \times 20)}{\sqrt{(4 \times 199) - (25 \times 25)}/(4 \times 146) - (20 \times 20)} \]

\[ r_{13} = \frac{176}{\sqrt{1171}} = 0.99 \]

and

\[ r_{23} = \frac{N(\sum X_2X_3) - (\sum X_2)(\sum X_3)}{\sqrt{[N(\sum X_2^2) - (\sum X_2)^2][N(\sum X_3^2) - (\sum X_3)^2]}} \]

\[ r_{23} = \frac{(4 \times 201) - (31 \times 20)}{\sqrt{(4 \times 289) - (31 \times 31)}/(4 \times 146) - (20 \times 20)} \]

\[ r_{23} = \frac{184}{\sqrt{195}} = 0.97 \]

Now, we calculate \( R_{1,23} \)

We have, \( r_{12} = 0.97 \), \( r_{13} = 0.99 \) and \( r_{23} = 0.97 \), then

\[ R_{1,23}^2 = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2} \]

\[ = \frac{0.97^2 + 0.99^2 - 2 \times 0.97 \times 0.99 \times 0.97}{1 - 0.97^2} \]

\[ = \frac{0.058}{0.059} = 0.98 \]

Then

\[ R_{1,23} = 0.99 \]

\[ R_{2,13}^2 = \frac{r_{12}^2 + r_{23}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{13}^2} \]

\[ = \frac{0.97^2 + 0.97^2 - 2 \times 0.97 \times 0.99 \times 0.97}{1 - 0.99^2} \]

\[ = \frac{0.19}{0.20} = 0.95 \]

Thus,

\[ R_{2,13} = 0.97 \]

\[ R_{3,12}^2 = \frac{r_{13}^2 + r_{32}^2 - 2r_{13}r_{32}r_{23}}{1 - r_{13}^2} \]

\[ = \frac{0.99^2 + 0.97^2 - 2 \times 0.97 \times 0.99 \times 0.97}{1 - 0.97^2} \]
Thus, \( R_{3,12} = 0.99 \)

**Example 3:** The following data is given:

<table>
<thead>
<tr>
<th></th>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( X_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<tr>
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<td>62</td>
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</tr>
<tr>
<td>10</td>
<td>60</td>
<td>62</td>
<td>60</td>
</tr>
</tbody>
</table>

Obtain \( R_{1,23} \), \( R_{2,13} \) and \( R_{3,12} \).

**Solution:** To obtain multiple correlation coefficients \( R_{1,23} \), \( R_{2,13} \) and \( R_{3,12} \), we use following formulae:

\[
R^2_{1,23} = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2},
\]

\[
R^2_{2,13} = \frac{r_{12}^2 + r_{23}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{13}^2}.
\]

We need \( r_{12} \), \( r_{13} \) and \( r_{23} \) which are obtained from the following table:

<table>
<thead>
<tr>
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<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( X_3 )</th>
<th>( d_1 = X_1 - 60 )</th>
<th>( d_2 = X_2 - 50 )</th>
<th>( d_3 = X_3 - 70 )</th>
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<th>( (d_2)^2 )</th>
<th>( (d_3)^2 )</th>
<th>( d_1d_2 )</th>
<th>( d_1d_3 )</th>
<th>( d_2d_3 )</th>
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<td>85</td>
<td>110</td>
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</tbody>
</table>
Regression and Multiple Correlation

Here, we can also use shortcut method to calculate $r_{12}$, $r_{13}$ & $r_{23}$,

Let 

\[ d_1 = X_1 - 60 \]
\[ d_2 = X_2 - 50 \]
\[ d_3 = X_3 - 70 \]

Now we get the total correlation coefficient $r_{12}$, $r_{13}$ and $r_{23}$

\[
 r_{12} = \frac{N(\sum d_1 d_2) - (\sum d_1)(\sum d_2)}{\sqrt{N(\sum d_1^2) - (\sum d_1)^2} \sqrt{N(\sum d_2^2) - (\sum d_2)^2}}
\]

\[
 r_{12} = \frac{(10 \times 325) - (4) \times (15)}{\sqrt{(10 \times 454) - (4) \times (4)} \sqrt{(10 \times 499) - (15) \times (15)}}
\]

\[
 r_{12} = \frac{3190}{4642.94} = 0.69
\]

\[
 r_{13} = \frac{N(\sum d_1 d_3) - (\sum d_1)(\sum d_3)}{\sqrt{N(\sum d_1^2) - (\sum d_1)^2} \sqrt{N(\sum d_3^2) - (\sum d_3)^2}}
\]

\[
 r_{13} = \frac{(10 \times 85) - (4) \times (18)}{\sqrt{(10 \times 454) - (4) \times (4)} \sqrt{(10 \times 310) - (18) \times (18)}}
\]

\[
 r_{13} = \frac{778}{3543.81} = 0.22
\]

and

\[
 r_{23} = \frac{N(\sum d_2 d_3) - (\sum d_2)(\sum d_3)}{\sqrt{N(\sum d_2^2) - (\sum d_2)^2} \sqrt{N(\sum d_3^2) - (\sum d_3)^2}}
\]

\[
 r_{23} = \frac{(10 \times 110) - (15) \times (18)}{\sqrt{(10 \times 499) - (15 \times 15)} \sqrt{(10 \times 310) - (18 \times 18)}}
\]

\[
 r_{23} = \frac{830}{3636.98} = 0.23
\]

Now, we calculate $R_{1,23}$

We have, $r_{12} = 0.69$, $r_{13} = 0.22$ and $r_{23} = 0.23$, then

\[
 R_{1,23}^2 = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2}
\]

\[
 = \frac{0.69^2 + 0.22^2 - 2 \times 0.69 \times 0.22 \times 0.23}{1 - 0.23^2}
\]

\[
 = \frac{0.4547}{0.9471} = 0.4801
\]

Then

\[
 R_{1,23} = 0.69
\]
\[ R^2_{2,13} = \frac{r^2_{12} + r^2_{23} - 2r_{12}r_{23}}{1 - r^2_{23}} = \frac{0.69^2 + 0.23^2 - 2 \times 0.69 \times 0.22 \times 0.23}{1 - 0.22^2} = \frac{0.4592}{0.9516} = 0.4825 \]

Thus,
\[ R_{2,13} = 0.69 \]

\[ R^2_{3,12} = \frac{r^2_{13} + r^2_{23} - 2r_{13}r_{23}}{1 - r^2_{13}} = \frac{0.22^2 + 0.23^2 - 2 \times 0.69 \times 0.22 \times 0.23}{1 - 0.69^2} = \frac{0.0315}{0.5239} = 0.0601 \]

Thus,
\[ R_{3,12} = 0.25 \]

Now let us solve some exercises.

**E1)** In bivariate distribution, \( r_{12} = 0.6, \ r_{23} = r_{31} = 0.54 \), then calculate \( R_{1,23} \).

**E2)** If \( r_{12} = 0.70, \ r_{13} = 0.74 \) and \( r_{23} = 0.54 \), calculate multiple correlation coefficient \( R_{2,13} \).

**E3)** Calculate multiple correlation coefficients \( R_{1,23} \) and \( R_{2,13} \) from the following information: \( r_{12} = 0.82, \ r_{23} = -0.57 \) and \( r_{13} = -0.42 \).

**E4)** From the following data,

\[
\begin{array}{cccccccc}
X_1 & 22 & 15 & 27 & 28 & 30 & 42 & 40 \\
X_2 & 12 & 15 & 17 & 15 & 42 & 15 & 28 \\
X_3 & 13 & 16 & 12 & 18 & 22 & 20 & 12
\end{array}
\]

Obtain \( R_{2,3}, \ R_{2,13} \) and \( R_{3,12} \)

**E5)** The following data is given:

\[
\begin{array}{cccccccccccc}
X_1 & 50 & 54 & 50 & 56 & 50 & 55 & 52 & 50 & 52 & 51 \\
X_2 & 42 & 46 & 45 & 44 & 40 & 45 & 43 & 42 & 41 & 42 \\
X_3 & 72 & 71 & 73 & 70 & 72 & 72 & 70 & 71 & 75 & 71
\end{array}
\]

By using the short-cut method obtain \( R_{1,23}, \ R_{2,13} \) and \( R_{3,12} \)
11.3 PROPERTIES OF MULTIPLE CORRELATION COEFFICIENT

The following are some of the properties of multiple correlation coefficients:

1. Multiple correlation coefficient is the degree of association between observed value of the dependent variable and its estimate obtained by multiple regression,

2. Multiple correlation coefficient lies between 0 and 1,

3. If multiple correlation coefficient is 1, then association is perfect and multiple regression equation may said to be perfect prediction formula,

4. If multiple correlation coefficient is 0, dependent variable is uncorrelated with other independent variables. From this, it can be concluded that multiple regression equation fails to predict the value of dependent variable when values of independent variables are known,

5. Multiple correlation coefficient is always greater or equal than any total correlation coefficient. If \( R_{123} \) is the multiple correlation coefficient than \( R_{123} \geq r_{12} \) or \( r_{13} \) or \( r_{23} \), and

6. Multiple correlation coefficient obtained by method of least squares would always be greater than the multiple correlation coefficient obtained by any other method.

11.4 SUMMARY

In this unit, we have discussed:

1. The multiple correlation, which is the study of joint effect of a group of two or more variables on a single variable which is not included in that group,

2. The estimate obtained by regression equation of that variable on other variables,

3. Limit of multiple correlation coefficient, which lies between 0 and +1,

4. The numerical problems of multiple correlation coefficient, and

5. The properties of multiple correlation coefficient.

11.5 SOLUTIONS / ANSWERS

E1) We have,

\[
\begin{align*}
& r_{12} = 0.6, \quad r_{23} = r_{31} = 0.54 \\
& R^2_{123} = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}} 
\end{align*}
\]
\[
\text{Multiple Correlation}
\]

\[
R_{1,23} = 0.65
\]

E2) We have

\[
R_{2,13}^2 = \frac{r_{12}^2 + r_{23}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{13}^2}
\]

\[
= \frac{0.49 + 0.29 - 0.56}{1 - 0.55}
\]

\[
= \frac{0.22}{0.45} = 0.49
\]

Thus

\[
R_{2,13} = 0.70.
\]

E3) We have

\[
r_{12} = 0.82, \quad r_{23} = -0.57, \quad r_{13} = -0.42.
\]

\[
R_{1,23}^2 = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{13}^2}
\]

\[
= \frac{0.67 + 0.18 - 0.39}{0.68}
\]

\[
= \frac{0.46}{0.68} = 0.68
\]

Then

\[
R_{1,23} = 0.82
\]

\[
R_{2,13}^2 = \frac{r_{12}^2 + r_{23}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{13}^2}
\]

\[
= \frac{0.67 + 0.32 - 0.39}{0.82}
\]

\[
= \frac{0.60}{0.82} = 0.73
\]

Thus,

\[
R_{2,13} = 0.85.
\]

E4) To obtain multiple correlation coefficients \( R_{1,23} \), \( R_{2,13} \) and \( R_{3,12} \),

we use following formulae:
Regression and Multiple Correlation

\[ R_{1,23}^2 = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2} \]
\[ R_{2,13}^2 = \frac{r_{12}^2 + r_{23}^2 - 2r_{12}r_{23}r_{13}}{1 - r_{13}^2} \]
\[ R_{3,12}^2 = \frac{r_{13}^2 + r_{23}^2 - 2r_{13}r_{23}r_{12}}{1 - r_{12}^2} \]

We need \( r_{12}, r_{13} \) and \( r_{23} \) which are obtained from the following table:

<table>
<thead>
<tr>
<th>S. No.</th>
<th>X₁</th>
<th>X₂</th>
<th>X₃</th>
<th>(X₁)²</th>
<th>(X₂)²</th>
<th>(X₃)²</th>
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<td>3656</td>
<td>1921</td>
<td>4378</td>
<td>3334</td>
<td>2430</td>
</tr>
</tbody>
</table>

Now, we get the total correlation coefficient \( r_{12}, r_{13} \) and \( r_{23} \):

\[ r_{12} = \frac{N(\sum X_1 X_2) - (\sum X_1)(\sum X_2)}{\sqrt{[N(\sum X_1^2) - (\sum X_1)^2][N(\sum X_2^2) - (\sum X_2)^2]}} \]
\[ r_{12} = \frac{(7 \times 4378) - (204) \times (144)}{\sqrt{[(7 \times 6486) - (204) \times (204)](7 \times 3656) - (144) \times (144)]} \]
\[ r_{12} = \frac{1270}{\sqrt{3786 \times 4856}} \]
\[ r_{12} = \frac{1270}{4287.75} = 0.30 \]

\[ r_{13} = \frac{N(\sum X_1 X_3) - (\sum X_1)(\sum X_3)}{\sqrt{[N(\sum X_1^2) - (\sum X_1)^2][N(\sum X_3^2) - (\sum X_3)^2]}} \]
\[ r_{13} = \frac{(7 \times 3334) - (204) \times (113)}{\sqrt{[(7 \times 6486) - (204) \times (204)](7 \times 1921) - (113 \times 113)]} \]
Multiple Correlation

\[ r_{13} = \frac{286}{\sqrt{3786 \times 678}} = \frac{286}{1602.16} = 0.18 \]

and

\[ r_{23} = \frac{N(\sum X_1 X_3) - (\sum X_2)(\sum X_3)}{\sqrt{N(\sum X_2^2) - (\sum X_2)^2} \sqrt{N(\sum X_3^2) - (\sum X_3)^2}} \]

\[ r_{23} = \frac{(7 \times 2430) - (144 \times 113)}{\sqrt{7 \times 3656) - (144 \times 144)} \sqrt{(7 \times 1921) - (113 \times 113)}} \]

\[ r_{23} = \frac{738}{\sqrt{4856} \sqrt{678}} \]

\[ r_{23} = \frac{738}{1814.49} = 0.41 \]

Now, we calculate \( R_{1,23} \)

We have, \( r_{12} = 0.30 \), \( r_{13} = 0.18 \) and \( r_{23} = 0.41 \), then

\[ R^2_{1,23} = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2} \]

\[ = \frac{0.30^2 + 0.18^2 - 2 \times 0.30 \times 0.18 \times 0.41}{1 - (0.41)^2} \]

\[ = \frac{0.0781}{0.8319} = 0.9380 \]

Then

\[ R_{1,23} = 0.30 \]

\[ R^2_{2,13} = \frac{r_{12}^2 + r_{23}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{13}^2} \]

\[ = \frac{0.30^2 + 0.41^2 - 2 \times 0.30 \times 0.18 \times 0.41}{1 - 0.18^2} \]

\[ = \frac{0.2138}{0.9676} = 0.221 \]

Thus,

\[ R_{2,13} = 0.47 \]

\[ R^2_{3,12} = \frac{r_{13}^2 + r_{23}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{12}^2} \]

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Regression and Multiple Correlation

\[
R^2_{12} = \frac{0.18^2 + 0.41^2 - 2 \times 0.30 \times 0.18 \times 0.41}{1 - 0.30^2} = \frac{0.1562}{0.9100} = 0.1717
\]

Thus,

\[R_{3,12} = 0.41\]

E5) To obtain multiple correlation coefficients \(R_{1,23}, R_{2,13}\) and \(R_{3,12}\)
we use following formulae

\[
R^2_{1,23} = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2},
\]

\[
R^2_{3,12} = \frac{r_{13}^2 + r_{23}^2 - 2r_{13}r_{23}r_{3,12}}{1 - r_{23}^2}
\]

and

\[
R^2_{2,13} = \frac{r_{12}^2 + r_{23}^2 - 2r_{12}r_{23}r_{3,12}}{1 - r_{13}^2}
\]

We need \(r_{12}, r_{13}\) and \(r_{23}\) which are obtained from the following table:

<table>
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<th>S. No.</th>
<th>(X_1)</th>
<th>(X_2)</th>
<th>(X_3)</th>
<th>(d_1 = \frac{X_1 - 50}{X_1 \cdot 50})</th>
<th>(d_2 = \frac{X_1 - 60}{X_1 \cdot 60})</th>
<th>(d_3 = \frac{X_1 - 70}{X_1 \cdot 70})</th>
<th>((d_1)^2)</th>
<th>((d_2)^2)</th>
<th>((d_3)^2)</th>
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<td>83</td>
<td>25</td>
<td>44</td>
</tr>
</tbody>
</table>

Now, we get the total correlation coefficient \(r_{12}, r_{13}\) and \(r_{23}\)

\[
r_{12} = \frac{N\sum d_1d_2 - \left(\sum d_1\right)\left(\sum d_2\right)}{\sqrt{\left(\sum d_1^2\right) - \left(\sum d_1\right)^2}\sqrt{\left(\sum d_2^2\right) - \left(\sum d_2\right)^2}}
\]

\[
r_{12} = \frac{(10 \times 83) - (20 \times 30)}{\sqrt{(10 \times 86) - (20 \times 20)}\sqrt{(10 \times 124) - (30 \times 30)}}
\]
Multiple Correlation

\[ r_{12} = \frac{230}{\sqrt{460 \times 340}} = \frac{230}{395.47} = 0.58 \]

\[ r_{13} = \frac{N(\sum d_i d_{x1}) - (\sum d_i)(\sum d_{x1})}{\sqrt{[N(\sum d_i^2) - (\sum d_i)^2][N(\sum d_{x1}^2) - (\sum d_{x1})^2]}} \]

\[ r_{13} = \frac{(10 \times 25) - (20 \times 17)}{\sqrt{(10 \times 86) - (20 \times 20)} \times (17 \times 17)} \]

\[ r_{13} = \frac{-90}{\sqrt{460 \times 201}} = \frac{-90}{304.07} = -0.30 \]

and

\[ r_{23} = \frac{N(\sum X_2 X_3) - (\sum X_2)(\sum X_3)}{\sqrt{[N(\sum X_2^2) - (\sum X_2)^2][N(\sum X_3^2) - (\sum X_3)^2]}} \]

\[ r_{23} = \frac{(10 \times 44) - (30 \times 17)}{\sqrt{(10 \times 124) - (20 \times 20) \times (17 \times 17)}} \]

\[ r_{23} = \frac{-70}{\sqrt{340 \times 201}} = \frac{-70}{261.42} = -0.27 \]

Now, we calculate \( R_{123} \).

We have, \( r_{12} = 0.58 \), \( r_{13} = -0.30 \) and \( r_{23} = -0.27 \), then

\[
R^2_{123} = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2}
\]

\[
= \frac{0.58^2 + (-0.30)^2 - 2 \times 0.58 \times (-0.30) \times (-0.27)}{1 - (-0.27)^2}
\]

\[
= \frac{0.3324}{0.9271} = 0.36
\]

Then

\[ R_{13} = 0.60 \]

\[
R^2_{213} = \frac{r_{12}^2 + r_{23}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{13}^2}
\]
Regression and Multiple Correlation

Thus,

\[
\begin{align*}
R_{2,13} &= 0.59 \\
R_{3,12} &= \frac{r_{13}^2 + r_{23}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{12}^2} \\
&= \frac{(-0.30)^2 + (-0.27)^2 - 2 \times 0.58 \times (-0.30) \times (-0.27)}{1 - (0.58)^2} \\
&= \frac{0.0689}{0.6636} = 0.10 \\
R_{3,12} &= 0.32
\end{align*}
\]