UNIT 16 TESTS OF HYPOTHESIS – II

STRUCTURE

16.0 OBJECTIVES

After studying this unit, you should be able to:

- differentiate between exact tests i.e., small sample tests and approximate tests, i.e., large sample tests,
- be familiar with the properties and applications of t-distribution,
- find the interval estimation for mean using t-distribution,
- have an idea about the theory required for testing hypothesis using t-distribution,
- apply t-test for independent samples, and
- apply t-test for dependent samples.

16.1 INTRODUCTION

In the previous unit, we considered different aspects of the problems of inferences. We further noted the limitations of standard normal test or Z-test. As discussed in Unit 15, we can not apply normal distribution for estimating confidence intervals for population mean in case the population standard deviation is unknown and sample size does not exceed 30, i.e., small samples. We may further recall that as mentioned in Unit 15, we can not test hypothesis concerning population mean when the sample is small and population standard deviation is unspecified. In a situation like this, we use t-distribution which is also known as student's t-distribution. t-distribution was first applied by W.S. Gosset who used to work in 'Guinners Brewery' in Dublin. The workers of Guinners Brewery were not allowed to publish their research work. Hence Gosset was compelled to publish his research work under the penname 'student' and hence the distribution is known as student's t-distribution or simply student's distribution. Before we discuss t-distribution, let us differentiate between exact tests and approximate tests.

16.2 SMALL SAMPLES VERSUS LARGE SAMPLES

Normally a sample is considered as small if its size is 30 or less whereas, the sample with size exceeding 30 is considered as a large sample. All the tests under consideration may be classified into two categories namely exact tests and approximate tests. Exact tests are those tests that are based on the exact sampling distribution of the test statistic and no approximation is made about the form of parent population or the sampling distribution of the test statistic. Since exact tests are valid for any sample size and usually cost as well as labour increases with an increase in sample size; we prefer to take small samples for conducting exact tests. Hence, the exact tests are also known as **small sample tests**. It may be noted that while testing for population mean on the basis of a random sample from a normal distribution, we apply exact tests or small sample tests provided the population standard deviation is known. This was demonstrated in Unit 15.

However, there are situations when we have to compromise with an approximate test. It has been found that if a random sample is taken from a population characterized by a parameter θ and if T is a sufficient statistic for θ, then :

$$Z = \frac{T - \theta}{S.E\,(T)}$$

as well as

$$Z = \frac{T - \theta}{S.\hat{E}\,(T)}$$

is an approximate standard normal variate provided we have taken a sufficiently large sample. Thus for testing $H_0 : \theta = \theta_0$, we consider

$$Z_0 = \frac{T - \theta_0}{S.E\,(T)} \quad \text{or} \quad Z_0 = \frac{T - \theta_0}{S.\hat{E}\,(T)}$$

For a large sample size, Z would be approximately a standard normal deviate and as such we can prescribe Z-test. This test is known as an approximate test because it is not based on the exact sampling distribution of the test statistic T. Since this test is valid only if 'n' is sufficiently large, it is also known as a **large sample test**. In this connection, it may be pointed out that in the previous unit, two such large sample tests have already been discussed. In the first case while testing for population mean with an unknown population standard deviation, we consider :

$$Z = \frac{\sqrt{n}\,(\bar{x} - \mu)}{S'}$$

$$\text{where, } \left[S' = \sqrt{\frac{\Sigma (x_i - \bar{x})^2}{(n-1)}} \right]$$

which is a standard normal variate, approximately, for a large n. Similarly testing for population proportion, we used:

$$Z_0 = \frac{\sqrt{n}\,(p - p_0)}{\sqrt{P_0\,(1 - P_0)}}$$

P_0 being the specified population proportion, which again, for a large sample, is an approximate standard normal variable.

16.3 STUDENT'S t-DISTRIBUTION

Since we cannot use Z-test, for a small sample, for population mean when the population standard deviation is not known, we are on the look out for a new test statistic. It is necessary to know a few terms first.

Degree of freedom (df' or d.o.f'). If we are asked to pick up any five numbers, then there are no restrictions or constraints on the selection of the five numbers and as such we are at liberty to choose any five numbers. Statistically, this is analogous to stating that we have five degrees of freedom (5 d.f). But if we are asked to find any five numbers such that the total is 60, then basically we are to find four numbers and not five as the last number is automatically determined since the sum of the five numbers is provided. Hence, we have now 4 d.f. which is the difference between the number of observations and the number of constraints (in this case, 5–1 = 4). Similarly, if we are to pick up five numbers such that their sum is 100 and the sum of the squares of the numbers is 754, then we have 5–2 = 3 d.f.

Chi-square distribution: Let $x_1, x_2, x_3 \ldots x_n$ be 'n' independent standard normal variables. Then x_1^2 follows chi-square distribution with '1' d.f. This is denoted by

$$x_1^2 \sim \chi_1^2$$

Again $x_1^2 + x_2^2 = \sum_{i=1}^{2} x_i^2 \sim \chi_2^2$ and in general,

$$x_1^2 + x_2^2 + \ldots + x_n^2 = \sum_{i=1}^{n} x_i^2 \sim \chi_n^2 \qquad \ldots\ldots (16.1)$$

If we write $u = \sum_{i=1}^{n} x_i^2$, then the probability density function of u is given by :

$f(u) = \text{const. } e^{-u/2} \cdot u^{n/2-1}$

for $o < u < \alpha$

It can be shown that for u,

Mean (μ) = n; Standard Deviation (σ) = $\sqrt{2n}$

χ^2 distribution has a positive skewness and it is leptokurtic. Leptokurtic means when a curve is more peaked than the normal curve (mesokurtic). Look at the following Fig. 16.1 which depicts a χ^2 distribution.

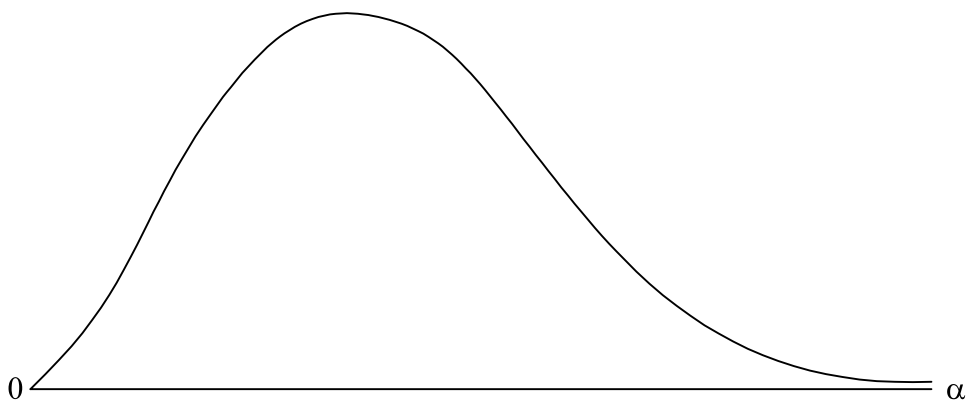


Figure 16.1: χ^2 distribution.

If $x_1, x_2, x_3 \ldots x_n$ are 'n' independent variables, each following normal distribution with mean (μ) and variance (σ^2), then $X_i = \frac{x_i - \mu}{\sigma}$ is a standard normal variable and as such

$$u = \sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{\sigma^2} \qquad \ldots(16.2)$$

follows χ^2 with m d.f.

The sample variance is given by :

$$S^2 = \frac{\Sigma(x_i - \bar{x})^2}{n}$$

or $\quad nS^2 = \Sigma(x_i - \bar{x})^2$

or, $\quad nS^2 = \Sigma(x_i - \mu)^2 + n(\bar{x} - \mu)^2 \qquad$ [From 15.9]

$$\frac{nS^2}{\sigma^2} = \frac{\Sigma(x_i - \mu)^2}{\sigma^2} - n\frac{(\bar{x} - \mu)^2}{\sigma^2}$$

As $\quad \frac{\Sigma(x_i - \mu)^2}{\sigma^2} \sim \chi_n^2$

and $\quad n\frac{(\bar{x} - \mu)^2}{\sigma^2} = \frac{(\bar{x} - \mu)^2}{\sigma^2 / n} = \left[\frac{(\bar{x} - \mu^2)}{\sigma^2 / \sqrt{n}}\right] \sim \chi_1^2$

$$\left[\text{since } \bar{x} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)\right],$$

Hence it follows that $\frac{ns^2}{\sigma^2} \sim \chi_{n-1}^2$

Student's t-distribution: Consider two independent variables 'y' and 'u' such that 'y' follows standard normal distribution and 'u' follows χ^2-distribution with md.f.

Then the ratio $t = \frac{y}{\sqrt{u/m}}$, follows t-distribution

with m d.f. The probability density function of t is given by :

$$f(t) = \text{const.}\left(1 + \frac{t^2}{m}\right)^{-\left(\frac{m+1}{2}\right)}$$

$$+ \text{ or } - \infty < t < \infty \qquad \ldots\ldots\ldots\ldots(16.3)$$

where, $t = \frac{(\bar{x} - \mu)}{s}\sqrt{n}$, const. = a constant required to make the area under the curve equal to unity; m = n–1, the degree of freedom of t.

It can be shown that for a t-distribution with m d.f.

$$\text{Mean } (\mu) = 0 \qquad \ldots\ldots\ldots\ldots(16.4)$$

Standard deviation $(\sigma)=\sqrt{\frac{m}{(m-2)}}, m>2$ …(16.5)

t-distribution is symmetrical about the point t = 0 and further the distribution is leptokurtic for m > 4. Thus compared to a standard normal distribution, t-distribution is more stiff at the peak. Figure 16.2 shows the probability curve of a t-distribution.

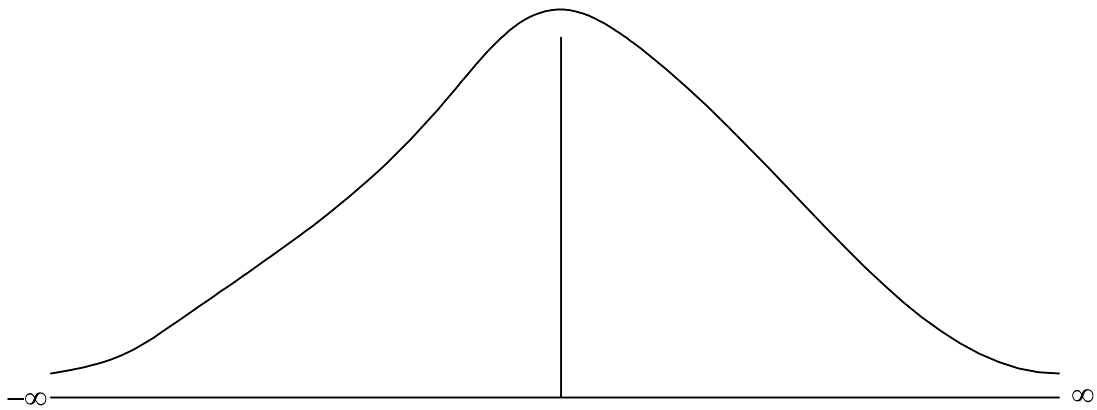


Figure 16.2: t-distribution

Since t-distribution is a function of m, we have a different t-distribution for every distinct value of 'm'. If 'm' is sufficiently large, we can approximate a t-distribution with md.f by a standard normal distribution.

Since we have :

$$f(t)=\text{const}\left(1+\frac{t^2}{m}\right)^{\frac{-m+1}{2}}$$

for $-\infty<t<\infty$

$$\therefore \text{Log} f=k-\frac{m+1}{2}\text{Log}\left(1+\frac{t^2}{m}\right)$$ where, 'k' is a constant.

$$=k-\frac{m+1}{2}\left(\frac{t^2}{m}-\frac{t^4}{2m^2}+\ldots\ldots\text{to}\,\alpha\right)$$

[as Log (1+x) = $-\frac{x^2}{2}+\ldots\ldots\text{to}\,\alpha$

for –1 < x ≤ 1

and $\frac{t^2}{m}$ is rather small for a large m].

Hence $\text{Log} f=k-\frac{m+1}{m}.\frac{t^2}{2}+\frac{m+1}{4m^2}t^4 \ldots\ldots \text{to } \alpha$

since m is very large $\frac{m+1}{m}$ tends to 1 and other terms containing powers of 'm' higher than 2 would tend to zero. Thus we have:

$$\text{Log} f=k-\frac{t^2}{2}$$

or, $$f = e^{K - \frac{t^2}{2}} = e^k . e^{-t^2/2} = \text{const.}\, e^{-t^2/2}$$

for $-\infty < t < \infty$

which takes the form of a standard normal variable.

Looking from another angle, as the mean of t-distribution is zero and standard deviation = $\sqrt{\frac{m}{m-2}}$ which tends to unity for a large m,

we may replace a t-distribution with m d.f. by a standard normal distribution. Here lies the justification of applying z test for large samples.

If x_1, x_2, x_3 …x_n denote the n observations of a random sample drawn from a normal population with mean as μ and the standard deviation as σ, then x_1, x_2, x_3 …x_n can be described as 'n' independent random variables each following normal distribution with the same mean μ and a common standard deviation σ. If we consider the statistic:

$$\sqrt{n}\frac{(\bar{x} - \mu)}{S^{|}}$$

where, $\bar{x} = \frac{\Sigma x_i}{n}$, the sample mean; and $S^{|} = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n-1}}$ is the standard deviation with divisor as (n–1) instead of n, then we may write :

$$\sqrt{n}\frac{(\bar{x} - \mu)}{S^{|}} = \frac{\frac{\sqrt{n}(\bar{x} - \mu)}{\sigma}}{\frac{S^{|}}{\sigma}}$$, dividing both numerator and denominator by σ

$$= \frac{(\bar{x} - \mu/\sigma/\sqrt{n}}{\sqrt{\frac{\Sigma(x_i - \bar{x})^2}{(n-1)\sigma^2}}} = \sqrt{\frac{y}{u/(n-1)}}$$

As $\bar{x}$ follows normal distribution with mean μ and standard deviation $\sigma/\sqrt{n}$, therefore:

$y = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ is a standard normal variate

Also $u = \frac{\Sigma(x_i - x)^2}{\sigma^2}$ follows χ^2-distribution with (n–1) d.f

Hence, by definition $\sqrt{n}\frac{(\bar{x} - \mu)}{s'} = \frac{y}{\sqrt{u/(n-1)}}$

follows t-distribution with (n–1) d.f.

As such we can write :

$$t = \sqrt{n}\frac{(\bar{x} - \mu)}{s^{|}} \sim t_{n-1}$$

We apply t-distribution for finding confidence interval for mean as well as testing hypothesis regarding mean. These are discussed in Sections 16.4 and 16.5 respectively.

16.4 APPLICATION OF t-DISTRIBUTION TO DETERMINE CONFIDENCE INTERVAL FOR POPULATION MEAN

Let us assume that we have a random sample of size 'n' from a normal population with mean as μ and standard deviation as σ. We consider the case when both μ and σ are unknown. We are interested in finding confidence interval for population mean. In view of our discussion in Section 16.3, we know that :

$$t = \sqrt{n}\frac{(\bar{x}-\mu)}{s'}$$

follows t-distribution with (n–1) d.f. We may recall here that $\bar{x}$ denotes the sample mean and s', the sample standard deviation with divisor as (n–1) and not 'n'. We denote the upper α-point of t-distribution with (n–1) d.f as $t_{\alpha,}$ (n–1). Since t-distribution is symmetrical about t = 0, the lower α-point of t-distribution with (n–1) d.f would be denoted by $-t_{\alpha,}$ (n–1). As per our discussion in Unit 15, in order to get 100 (1–α)% confidence interval for μ, we note that :

$$P\left[-t_{\alpha/2},(n-1) \le \sqrt{n}\frac{(\bar{x}-\mu)}{s} \le t_{\alpha/2},(n-1)\right] = 1-\alpha$$

or $$P\left[-\bar{x}-\frac{s'}{\sqrt{n}}.t_{\alpha/2},(n-1) \le -\mu \le -\bar{x}+\frac{s'}{\sqrt{n}}.t_{\alpha/2},(n-1)\right] = 1-\alpha$$

or $$P\left[\bar{x}-\frac{s'.}{\sqrt{n}}t_{\alpha/2},(n-1) \le \mu \le \bar{x}+\frac{s'.t'_{\alpha/2},}{\sqrt{n}}(n-1)\right] = 1-\alpha$$

Thus 100 (1-α) % confidence interval to μ is :

$$\left[\bar{x}-\frac{s'.}{\sqrt{n}}t_{\alpha/2},(n-1), \bar{x}+\frac{s'.}{\sqrt{n}}t_{\alpha/2},(n-1)\right] \quad \text{...(16.6)}$$

100 (1-α) % Lower Confidence Limit to $\mu = \bar{x} - \frac{s'}{\sqrt{n}}.t_{\alpha/2},(n-1)$

and 100 (1-α) % Upper Confidence Limit to $\mu = \bar{x} + \frac{s'}{\sqrt{n}}.t_{\alpha/2},(n-1)$

Selecting α = 0.05, we may note that

95% Lower Confidence Limit to $\mu = \bar{x} - \frac{s'}{\sqrt{n}}.t_{0.025},(n-1)$

and 95% Upper Confidence Limit to $\mu = \bar{x} + \frac{s'}{\sqrt{n}}.t_{0.025},(n-1)$ ...(16.7)

In a similar manner, setting $\alpha = 0.01$, we get 99% Lower Confidence Limit to μ

$= \bar{x} - \frac{s'}{\sqrt{n}} . t_{0.005}, (n-1)$ and

99% Upper Confidence Limit to $\mu = \bar{x} + \frac{s'}{\sqrt{n}} . t_{0.005}, (n-1)$ …(16.8)

Values of t_{α}, m for m = 1 to 30 and for some selected values of a are provided in Appendix Table 5. Figures 16.3, 16.4 and 16.5 exhibit confidence intervals to μ applying t-distribution as follows :

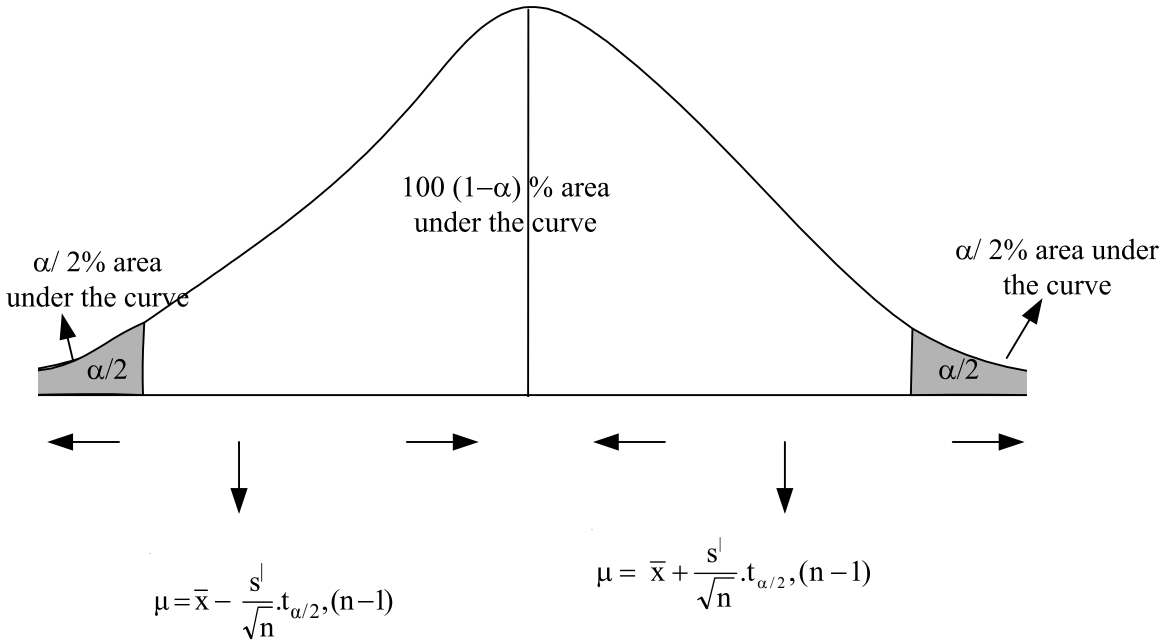


Fig. 16.3: 100 (1–α%) Confidence Interval to μ

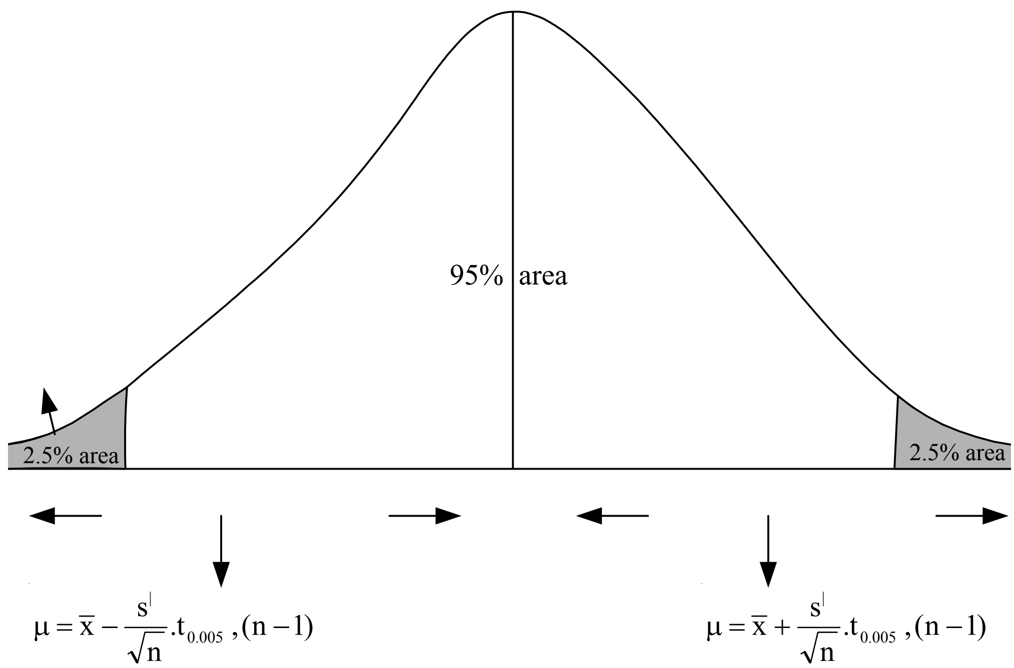
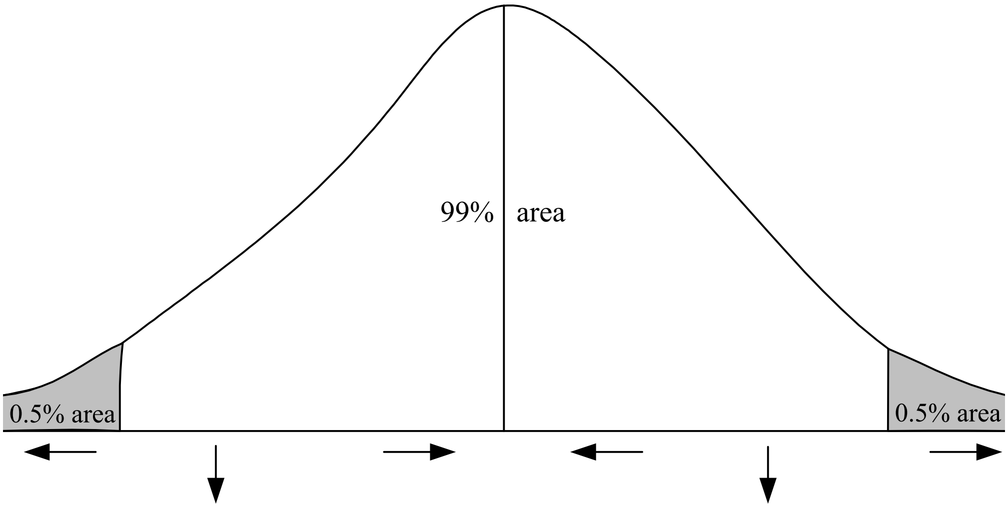


Fig. 16.4: 95% Confidence Interval to μ



$$\mu = \bar{x} - \frac{s'}{\sqrt{n}}.t_{0.005}\,,(n-1) \qquad \mu = \bar{x} + \frac{s'}{\sqrt{n}}.t_{0.005}\,,(n-1)$$

Fig. 16.5: 99% Confidence Interval to μ

Let us now take up some illustrations to understand this concept.

Illustration 1

Following are the lengths (in ft.) of 7 iron bars as obtained in a sample out of 100 such bars taken from SUR IRON FACTORY.

4.1, 3.98, 4.01, 3.95, 3.93, 4.12, 3.91

we have to find 95% confidence interval for the mean length of iron bars as produced by SUR IRON FACTORY.

Solution: Let x denote the length of iron bars. We assume that x is normally distributed with unknown mean μ and unknown standard deviation σ. Then 95% Lower Confidence Limit to μ

$$= \bar{x} - \frac{s'}{\sqrt{n}}.\sqrt{\frac{N-n}{N-1}}t_{0.005}\,,6$$

and 99% Upper Confidence Limit to $\mu = \bar{x} + \frac{s'}{\sqrt{n}}.\sqrt{\frac{N-n}{N-1}}.t_{0.025}\,,6$

where, $\bar{x} = \frac{\Sigma x_i}{n}$; $S' = \frac{\Sigma(x_i - \bar{x})^2}{n-1}$; n = sample size = 7; N = population size = 100

and $\sqrt{\frac{N-n}{N-1}}$ = finite population correction (fpc)

and $t_{0.025}$, 6 = Upper 2.5% point of t-distribution with 6 d.f.

= 2.447 [from Appendix Table-5 given at the end of this block, α = 0.025, m = 6]

Table 16.1: Computation of Sample Mean and S.D.

x_i	x_i^2
4.10	16.8100
3.98	15.8404
4.01	16.0801
3.95	15.6025
3.93	15.4449
4.12	16.9744
3.91	15.2881
28.00	**112.0404**

Thus, we have : $\bar{x} = \frac{28}{7} = 4$

$$\Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - n\bar{x}^2 = 112.0404 - 7 \times 4^2$$
$$= 0.0404$$

$$\therefore s' = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{0.0404}{6}} = 0.082057$$

$$\text{f.p.c.} = \sqrt{\frac{100-7}{100-1}} = 0.969223$$

Hence 95% Lower confidence Limit to $\mu = 4 - \frac{0.08\,2057}{\sqrt{7}} \times 0.969223 \times 2.365$

$$= 4 - 0.188091 = 3.811909$$

Similarly 95% Upper confidence limit to μ = 4 + 0.188091= 4.188091

So 95% Confidence Interval for mean length of iron bars = [3.81 ft, 4.19 ft].

Illustration 2

Find 90% confidence interval to μ given sample mean and sample S.D as 20.24 and 5.23 respectively, as computed on the basis of a sample of 11 observations from a population containing 1000 units.

Solution: As sample size (n) = 11 is small, we apply t-distribution. Further, we may ignore f.p.c as the population is rather large. Thus 90% confidence interval to μ is given by:

$$\left[\bar{x} - \frac{s'}{\sqrt{n}}.t_{0.05}, 10,\quad \bar{x} + \frac{s'}{\sqrt{n}}.t_{0.05}, 10\right]$$

As $S = \sqrt{\frac{\Sigma(x_i - \bar{x}^2)}{n}}$, the sample standard deviation (S.D).

$$\therefore nS^2 = \Sigma(x_i - \bar{x})^2$$

Hence $(s^{|})^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1}$

$$= \frac{nS^2}{n-1} \text{ [since } \Sigma(x_i - \bar{x}^2) = nS^2]$$

or, $s^{|} = \sqrt{\frac{n}{n-1}}.S$

$$= \sqrt{\frac{11}{10}} \times 5.23 = 5.4853$$

Consulting Appendix Table-5, given at the end of this block, we find $t_{0.05}$, 10 = 1.812

Thus 90% confidence interval to μ is given by:

$$\left[20.24 - \frac{5.4853}{3.1623} \times 1{,}812, \quad 20.24 + \frac{5.4853}{3.1623} \times 1.812\right] = [17.0969, \quad 23.3831]$$

Illustration 3

The study hours per week of 17 teachers, selected at random from different parts of West Bengal, were found to be:

6.6, 7.2, 6.8, 9.2, 6.9, 6.2, 6.7, 7.2, 9.7, 10.4, 7.4, 8.3, 7.0, 6.8, 7.6, 8.1, 7.8

Suppose, we are interested in computing 95% and 99% confidence intervals for the average hours of study per week per teacher in the state of West Bengal.

Solution: If μ denotes the average hours of study per week per teacher in West Bengal, then as discussed earlier,

95% confidence interval to $\mu = \left[\bar{x} - \frac{s^{|}}{\sqrt{n}}.t_{0.025}, \ (n-1), \quad \bar{x} + \frac{s^{|}}{\sqrt{n}}.t_{0.025}, \ (n-1)\right]$

and 99% confidence interval to $\mu = \left[\bar{x} - \frac{s^{|}}{\sqrt{n}}.t_{0.005},(n-1), \quad \bar{x} + \frac{s^{|}}{\sqrt{n}}.t_{0.005},(n-1)\right]$

Table 16.2: Computation of Sample Mean and Sample S.D.

Study Hours (x_i)	**x_i^2**
6.6	43.56
7.2	51.84
6.8	46.24
9.2	84.64
6.9	47.61
6.2	38.44
6.7	44.89
7.2	51.84
9.7	94.09
10.4	108.16
7.4	54.76
8.3	68.89
7.0	49.00
6.8	46.34
7.6	57.76
8.1	65.61
7.8	60.84
129.9	**1014.41**

We have n = 17

$$\bar{x} = \frac{\Sigma x_i}{n} = \frac{129.9}{17} = 7.64$$

$$s^{|} = \sqrt{\frac{\Sigma (x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{\Sigma x_i^2 - n.(\bar{x})^2}{n-1}}$$

$$= \sqrt{\frac{1014.41 - 17 \times (7.64)^2}{17-1}}$$

$$= \sqrt{\frac{1014.41 - 992.28}{16}} = 1.1761$$

From Appendix Table-5, given at the end of this block, $t_{0.025}$, 16 = 2.120; $t_{0.005}$, 16 = 2.921

Thus 95% confidence interval to μ

$$= \left[(7.64 - \frac{1.1761}{4} \times 2.12)^{hrs}, \quad (7.64 + \frac{1.1761}{4} \times 2.12)^{hrs} \right]$$

= [7.0167 hours, 8.2633 hours]

Similarly 99% confidence interval to μ

$$= \left[(7.64 - \frac{1.1761}{4} \times 2.921) \text{ hours}, \quad (7.64 + \frac{1.1761}{4} \times 2.921) \text{ hours} \right]$$

= [6.7812 hours, 8.4988 hours]

Illustration 4

In a sample of 26 items, 90% confidence limits to population mean are found to be 46.584 and 53.416 respectively. Find the sample mean and sample S.D.

Solution: As explained earlier, 90% confidence interval to μ, population mean is:

$$\left[\bar{x} - \frac{s^{|}}{\sqrt{n}}.t_{0.05},(n-1), \quad \bar{x} + \frac{s^{|}}{\sqrt{n}}.t_{0.05},(n-1) \right]$$

In this case, n = 26, From Appendix Table-5, given at the end of this block, $t_{0.05}$, 25 = 1.708

Hence we have $\bar{x} - \frac{s^{|}}{\sqrt{26}} \times 1.708 = 46.584$

or $\bar{x} - 0.33497 s^{|} = 46.584$ …(1)

and $\bar{x} + \frac{s^{|}}{\sqrt{26}} \times 1.708 = 53.416$

or, $\bar{x} + 0.33497\, s^{|} = 53.416$ …(2)

on adding equation (1) and (2) we get

$2\bar{x} = 100$ or $\bar{x} = 50$

replacing $\overline{x}$ by 50 in equation (1), we have

$$50 - 0.33497\, s^{|} = 46.584$$

or $$s^{|} = \frac{3.416}{0.33497} = 10.19793$$

Hence $S = \sqrt{\frac{n-1}{n}} s^{|}$ [from illustration 2]

$$= 0.98058 \times 10.19793 = 9.9999 \simeq 10$$

Thus the sample mean is 50 units and sample S.D is approximately 10 units.

Self Assessment Exercise A

1) State, with reasons, whether the following statements are true or false.

   a) t-distribution can be used for samples of any size.

   b) Exact tests can be applied for samples of any size whereas approximate tests can be applied for samples of large size only.

   c) The tests meant for large samples can not be used for small samples.

   d) For applying t-distribution, assumption of normality may not be necessary.

   e) When sample size exceeds thirty, we can replace t-test by z-test.

   f) For a population with unknown S.D, confidence interval for population mean would vary in accordance with sample size.

   g) For a population with known S.D, confidence interval remains unchanged for varying sample size.

   h) Large sample tests can be performed without the assumption of normality.

   i) Both the standard normal distribution and t-distribution have the same mean and same variance.

   j) If $x_1, x_2, x_3 \ldots x_n$ are n sample observations from N (μ, σ^2) then

   $$\frac{\Sigma(x_i - \overline{x})^2}{\sigma^2} \sim \chi^2_n$$

   k) Z test has the widest range of applicability among all the commonly used tests.

2) Differentiate between exact tests and approximate tests.

   ....

   ....

3) Discuss the points of similarity and dissimilarity between a standard normal distribution and a t-distribution.

   ....

   ....

   ....

4) A random sample of size 10 drawn from a normal population yields sample mean as 85 and sample S.D as 8.7. Compute 90% and 95% confidence intervals to population mean.

....

....

....

5) Find 99% confidence limits for 'μ' given that a sample of 19 units drawn from a population of 98 units provides sample mean as 15.627 and sample S.D as 2.348.

....

....

....

6) A sample of size 10 drawn from a normal population produces the following results.

$\Sigma x_i = 92$ and $\Sigma x_i^2 = 889$

Obtain 95% confidence limits to μ.

....

....

....

16.5 APPLICATION OF t-DISTRIBUTION FOR TESTING HYPOTHESIS REGARDING MEAN

Let us consider a situation where a small random sample is taken from a normal population with mean as μ and standard deviation as σ, both of which are unknown. Then as discussed in Unit-15, we are interested in testing :

$H_0 : \mu = \mu_0$

against $H : \mu \neq \mu_0$ i.e., the population mean is anything but μ_0.

or $H_1 : \mu > \mu_0$ i.e., the population mean is greater than μ_0.

or $H_2 : \mu < \mu_0$ i.e., the population mean is less than μ_0.

As we have noted in Section 16.1 the proper test to apply in this situation is undoubtedly t-test. If we denote the upper α-point and lower α-point of t-distribution with m.d.f. by t_α, m and $t_{1-a,m} = -t_{\alpha,m}$ (as t-distribution is symmetrical about 0) then for testing H_0, based on the distribution of t, it may be possible to find 4 values of t such that :

$P(t_0 \geq t_{\alpha/2}, m) = \alpha/2$ ……(16.9)

$P(t_0 \leq -t_{\alpha/2}, m) = \alpha/2$ ……(16.10)

$P(t_0 \geq t_\alpha, m) = \alpha$ ……(16.11)

and $P(t_0 \leq -t_\alpha, m) = \alpha$ ……(16.12)

where, t_0 is the value of t under $H_0 : \mu = \mu_0$

combining (16.9) and (16.10), we have

$P(t_0 \geq t_{\alpha/2}, m) + P(t_0 \leq -t_{\alpha/2}, m) = \alpha$ ……(16.13)

Thus, in order to test $H_0 : \mu = \mu_0$ against both sided or two-sided alternative $H : \mu \neq \mu_0$, selecting a low value of the probability of type-I error i.e., α, say $\alpha = 0.05$ or $\alpha = 0.01$, we find from (16.13) that the probability that t_0 is greater than $t_{\alpha/2}$, m or t_0 is less than $-t_{\alpha/2}$, m is likely to be very low.

Hence, on the basis of a small random sample drawn from the population, if it is found that t_0 is greater than $t_{\alpha/2}$, m or t_0 is less than $-t_{\alpha/2}$, m i.e., $|t_0| > t_{\alpha/2}$, m, then we may suggest that there is enough evidence to suggest that H_0 is untrue and H is true. Then we reject H_0 and accept H. The critical region for this both sided alternative is provided by :

$$\omega : t_0 \geq t_{\alpha/2}, m \text{ and } \omega : t_0 \leq -t_{\alpha/2}, m$$

$$\Rightarrow \omega : |t_0| \geq t_{\alpha/2}, m \qquad \ldots(16.14)$$

This is shown in the following Figure 16.6. Critical region lies on both the tails.

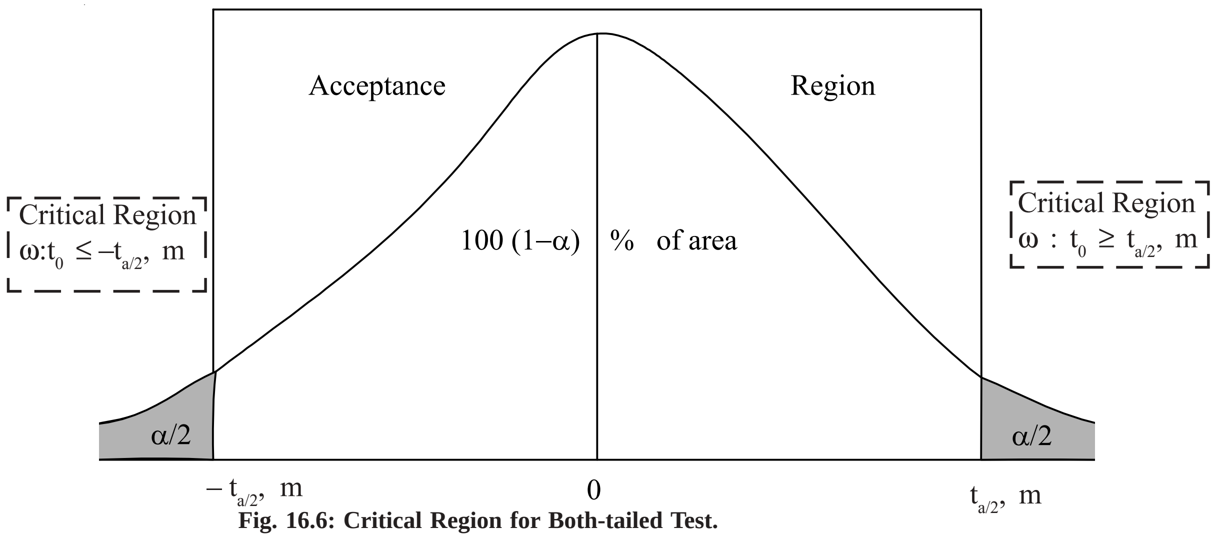


Fig. 16.6: Critical Region for Both-tailed Test.

Secondly, in order to test the null hypothesis against the right-sided alternative i.e., to test H_0 against $H_1 : \mu > \mu_0$, from (16.11) we note that, as before, if we choose a small value of α, then the probability that the observed value of t, would exceed the critical value t_α, m is very low. Thus one may have serious questions in this case, about the validity of H_0 if the value of t, as obtained on the basis of a small random sample, really exceeds t_α, m. We then reject H_0 and accept H_1. The critical region

$$\omega : t_0 \geq t_\alpha, m \qquad \ldots\ldots\ldots(16.15)$$

lies on the right-tail of the curve and the test as such is called right-tailed test. This is shown in Figure 16.7.

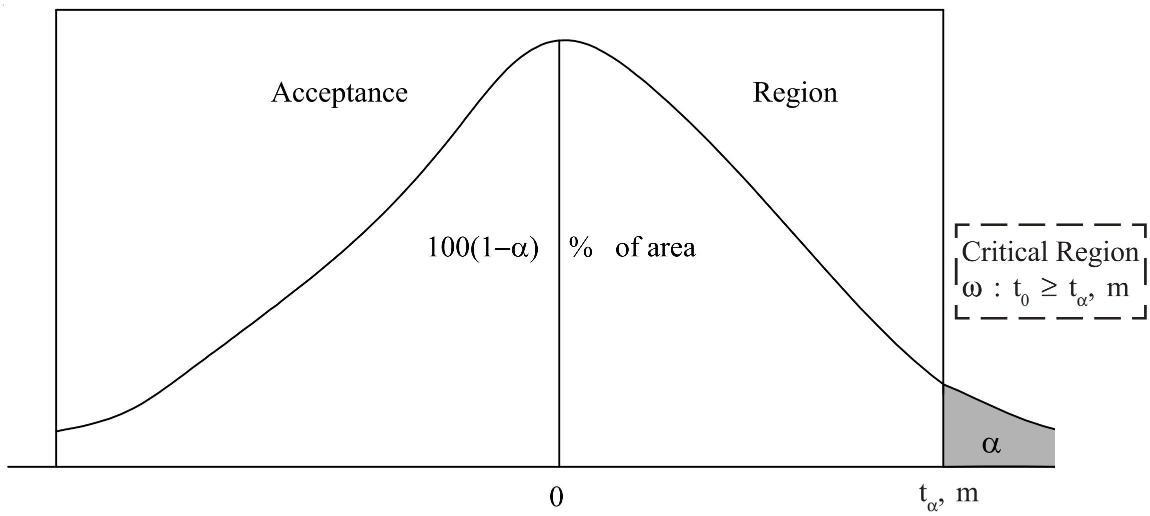


Fig. 16.7: Critical Region for Right-tailed Test.

Lastly, when we proceed to test H_0 against the left-sided alternative $H_2 : \mu < \mu_0$, we note that (16.12) suggests that if α is small, then the probability that t_0 would be less than the critical value $-t_\alpha$, m is very small. So if the value of t_0 as computed, on the basis of a small sample, is found to be less than $-t_\alpha$, m, we would doubt the validity of H_0 and accept H_2. The critical region

$$\omega : t_0 \leq - t_\alpha, m \quad \ldots(16.16)$$

would lie on the left-tail and the test would be left-tailed test. This is depicted in Fig. 16.8.

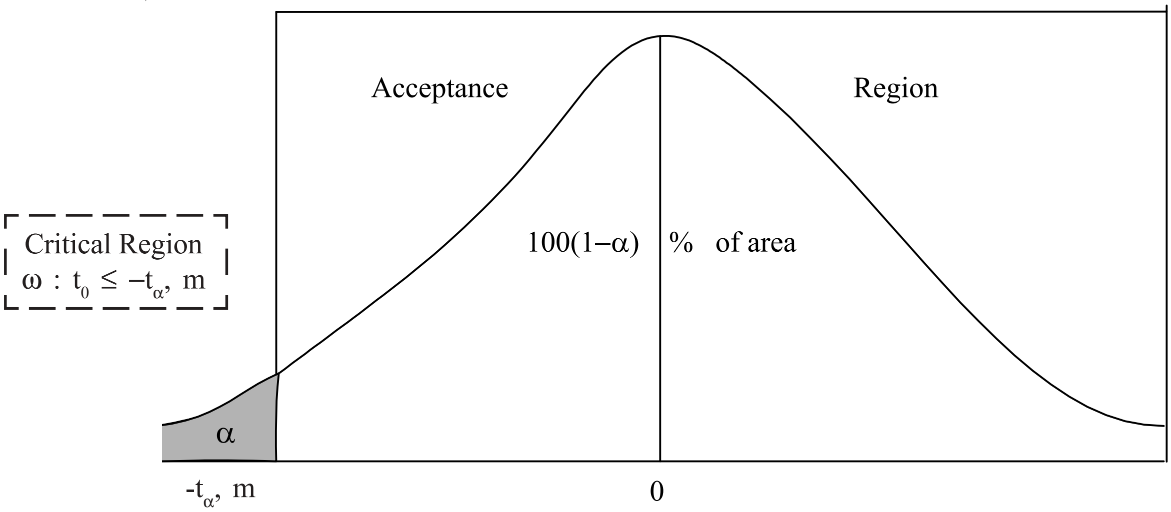


Fig. 16.8: Critical Region for Left-tailed Test.

16.6 t-TEST FOR INDEPENDENT SAMPLES

In order to apply t-test in a given situation, we need to verify the following points:

1) Whether the sample drawn is a random sample. A positive answer would confirm that the sample observations are independent.

2) Whether the sample is taken from a normal population. An affirmative answer is a pre-requisite for applying t-test.

3) Whether the population S.D is unknown. Here a negative answer would suggest Z-test, provided we get a positive answer to the first two questions. A 'yes' may mean t-test.

4) Whether the sample drawn is a small one. Again if the answer is 'no' i.e., n > 30, we would be satisfied with Z-test. However, if n ≤ 30 and the first three conditions are fulfilled, we should recommend t-test.

Putting this in a nutshell, t-test is suggested for population mean if a small random sample is drawn from a normal population with an unknown standard deviation. Under the above conditions, in order to test a null hypothesis, we use the test statistic:

$$t = \frac{\sqrt{n}\,(\bar{x} - \mu)}{s^{|}}$$

where, n = sample size; $\bar{x}$ = sample mean; and $s^{|}$ = sample S.D with divisor as (n-1). The test statistic follows t-distribution with (n–1) d.f

In order to test $H_0 : \mu = \mu_0$ against the both-sided alternative $H : \mu \neq \mu_0$ we compute :

$$t_0 = \frac{\sqrt{n}(\bar{x} - \mu_0)}{s^{|}}$$

if t_0 falls on the critical region defined by :

$$\omega : |t_0| \geq t_{\alpha/2}, (n-1)$$

t_α, m being the upper α-point of t-distribution with m d.f, then we reject H_0. In other words, H_0 is rejected and $H : \mu \neq \mu_0$ is accepted if the observed value of t, as computed from the sample, exceeds or is equal to the critical value $t_{\alpha/2}$, (n-1).

If we select α, the level of significance, as 0.05, then H_0 is rejected at 5% level of significance if :

$$|t_0| \geq t_{0.025}, (n-1)$$

on the other hand letting $\alpha = 0.01$, we reject H_0 at 1% level of significance if :

$$|t_0| \geq t_{0.005}, (n-1)$$

Figure 16.9 shows critical region at 5% level of significance while Figure 16.10 shows critical region at 1% level of significance.

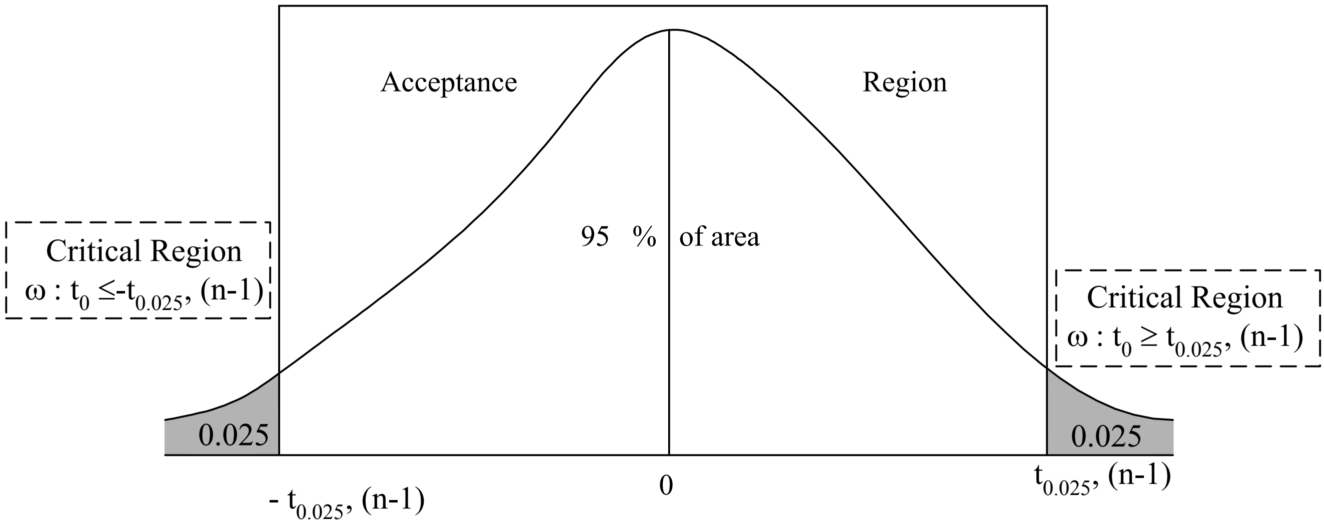


Fig.16.9: Critical Region for Both-tailed Test at 5% Level of Significance

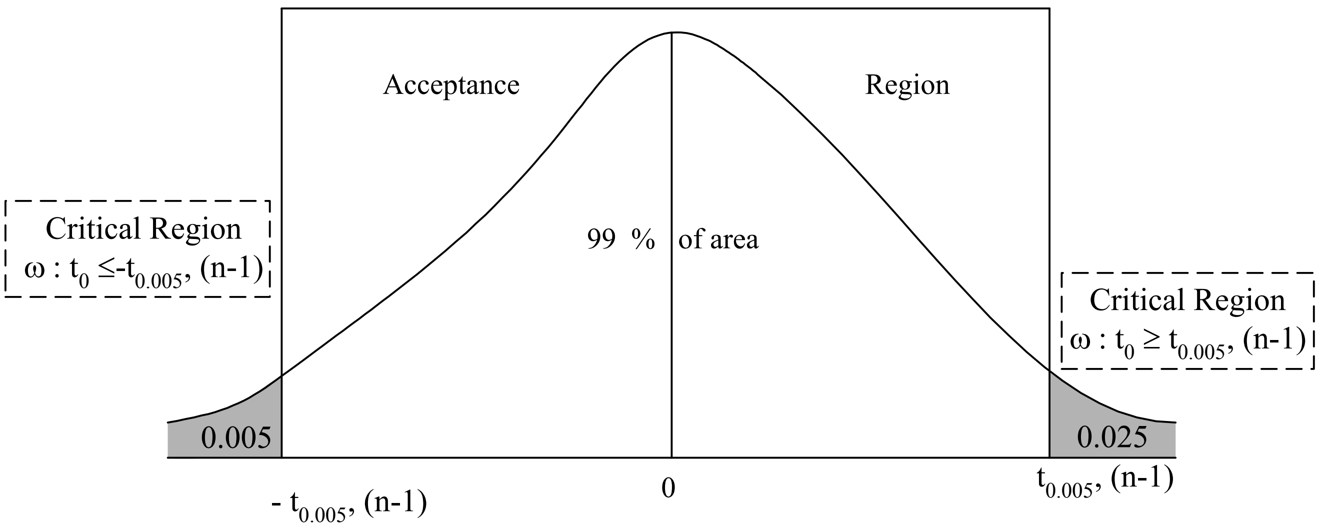


Fig. 16.10: Critical Region for Both-tailed Test at 1% Level of Significance

Similarly, for testing H_0 against the right-sided alternative $H_1 : \mu > \mu_0$, the critical region is given by :

$$\omega : t_0 \geq t_\alpha, (n-1)$$

The respective critical regions at 5% and 1% level of significance are given by :

$$\omega : t_0 \geq t_{0.05}, \ (n-1)$$

and $\quad \omega : t_0 \geq t_{0.01}, (n-1)$

The following Figures 16.11 and 16.12 show these two critical regions.

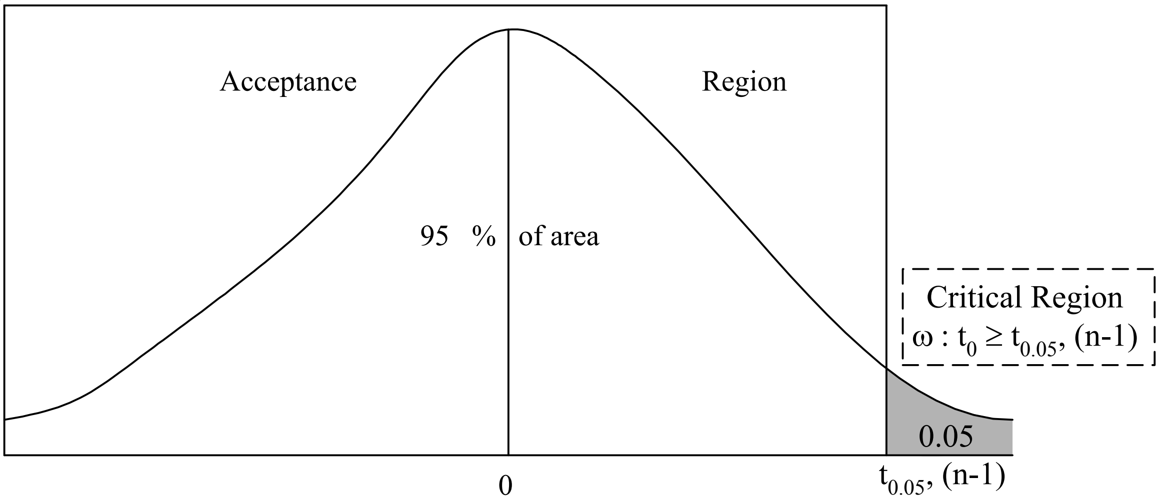


Fig. 16.11: Critical Region for Right-tailed Test at 5% Level of Significance

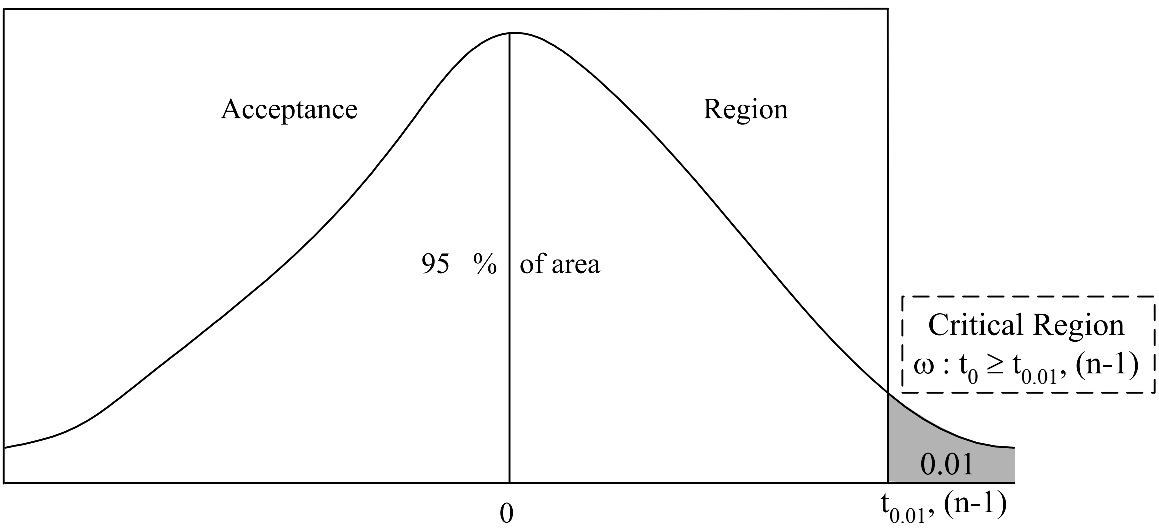


Fig. 16.12: Critical Region for Left-tailed Test at 1% Level of Significance

Lastly, when we test H_0 against the left-tailed test $H_2 : \mu < \mu_0$, the critical region would be:

$$\omega : t_0 \leq -t_{\alpha}, (n-1)$$

In particular, the critical region at 5% level of significance would be given by:

$$\omega : t_0 \leq -t_{0.05}, (n-1)$$

and the critical region at 1% level of significance would be:

$$\omega : t_0 \leq -t_{0.01}, (n-1)$$

These are depicted in the following Figure 16.13 and Figure 16.14 respectively.

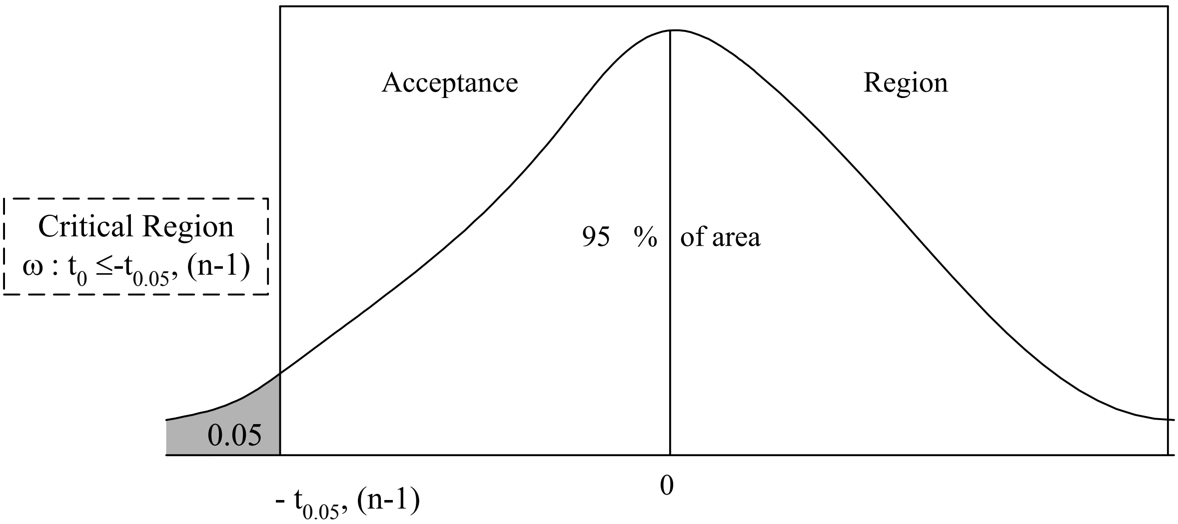


Fig. 16.13: Critical Region for Left-tailed Test at 5% Level of Significance

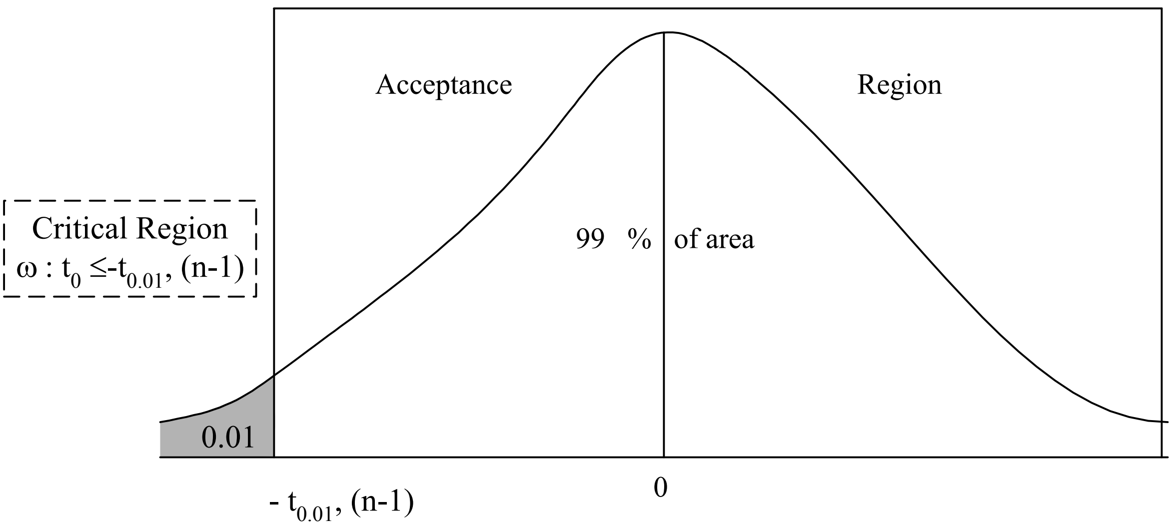


Fig. 16.14: Critical Region for Left-tailed Test at 1% Level of Significance

Let us take up some illustrations to understand the application of t-test for independent samples.

Illustration 5

An automatic machine was manufactured to pack 10 kilograms of oil. A random sample of 13 tins was taken to test the machine. Following were the weights in kilograms of the 13 tins.

9.7, 9.6, 10.4, 10.3, 9.8, 10.2, 10.4, 9.5, 10.6, 10.8, 9.1, 9.4, 10.7

Assuming normal distribution of the weights of the packed tins, examine whether the machine worked in accordance with the specifications.

Solution: Let x denote the weight of the packed tins of oil. Since,

1) x is assumed to be normally distributed
2) the population S.D. of the weight of the packed tin is unknown and
3) the sample size n = 13 is small, we apply t-test

Thus, we have to test $H_0 : \mu = 10$ against

$H_1 : \mu < 10$ (i.e., the machine packed less than 10 kg)

The test statistic is, as discussed in Section 16.5, t-test

Hence we compute :

$$t_0 = \frac{\sqrt{n}(\bar{x} - 10)}{s^{|}}, \text{ where, } \bar{x} = \frac{\Sigma x_i}{n}$$

$$s^{|} = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{\Sigma x_i^2 - n\bar{x}^2}{n-1}}$$

The critical region for this left-sided alternative is provided by :

$$\omega : t_0 \le -t_{\alpha,\ (n-1)}$$

choosing $\alpha = 0.05$, look at t-table (Appendix Table-5) suggests that $t_{0.05,}\ 12 = 1.782$

Thus the critical region is :

$$\omega : t_0 \le -1.782$$

From the given data, we have

$\Sigma x_i = 130.5$ and $\Sigma x_i^2 = 1313.65$

Thus, $\bar{x}$ = 130.5/13 = 10.038

$$S^{|} = \sqrt{\frac{1313.65 - 13(10.038)^2}{13-1}} = \sqrt{\frac{1313.65 - 1309.8987}{12}}$$

$$= 0.5591$$

Hence $$t_0 = \frac{\sqrt{13}\ (10.038 - 10)}{0.5591} = 0.245$$

which is greater than –1.782

As t_0 does not fall on the critical region w, we accept H_0. So, on the basis of the given data as obtained from the sample observations, we conclude that the machine worked in accordance with the given specifications.

Illustration 6

A company has been producing steel tubes of inner diameter of 4 cms. A sample of 15 tubes gives the average inner diameter as 3.96 cms with a S.D. of 0.032 cms. Is the sample mean significantly different from the population mean?

Solution: Let x denote the inner diameter of steel tubes as produced by the company. We are interested in testing

$H_0 : \mu = 4$ against
$H : \mu \neq 4$

Assuming that x follows normal distribution, we note that the sample size is 15 (<30) and the population S.D. is unknown. All these factors justify the application of t-distribution. Thus we compute our test statistic as:

$$t = \frac{\sqrt{n}(\bar{x} - 4)}{s^{|}}$$

As given, $\bar{x} = 3.96$; and $s = 0.032$

$$\therefore s^{|} = \sqrt{\frac{n}{n-1}}\, s = \sqrt{\frac{15}{14}} \times 0.032 = 0.033$$

$$t_0 = \frac{\sqrt{n}\,(\bar{x} - 4)}{s^{|}}$$

So, $t_0 = \dfrac{\sqrt{14}\,(3.96 - 4)}{0.033} = -4.536$

Hence $|t_0| = 4.536$

The critical region for the both-side test is :

$$\omega : |t_0| \geq t_{\alpha/2,(n-1)}$$

Selecting the level of significance as 1%, from the t-table (Appendix Table-5), we get $t_{0.01/2,}$ (15–1)

$= t_{0.005}$, 14 = 2.977

Thus, $\omega : |t_0| \geq 2.977$

Since the computed value of t i.e., $|t_0| = 4.536$, falls on w, we reject H_0. Hence the sample mean is significantly different from the population mean.

Illustration 7

The mean weekly sales of detergent powder in the department stores of the city of Delhi as produced by a company was 2,025 kg. The company carried out a big advertising campaign to increase the sales of their detergent powder. After the advertising campaign, the following figures were obtained from 20 departmental stores selected at random from all over the city (weight in kgs.).

2000	2023	2056	2048	2010	2025	2100
2563	2289	2005	2082	2056	2049	2020
2310	2206	2316	2186	2243	2013	

Based on above data, was the advertising successful?

Solution: Let us assume that x represents the weekly sales (in kg) of detergent powder as produced by the company. If μ denotes the average (i.e. mean) weekly sales in the city of Delhi, then we would like to test:

$H_0 : \mu = 2025$ i.e. there is no change due to the advertisement

$H_1 : \mu > 2025$ i.e. there is an increase in sales due to the advertisement.

As explained in the illustrations 5 and 6, we compute

$$t_0 = \frac{\sqrt{n}(\bar{x} - 2025)}{s^|}$$

and the critical region for the right-sided alternative is given by :

$$\omega : t_0 \geq (t_0, (n-1)$$

or $\omega : t_0 \geq 1.729$

[By selecting α = 0.05 and consulting Appendix Table-5, given at the end of this block, we find that for m = 20–1 = 19 and for α = 0.05, value of t is 1.729].

Table 16.3: Computation of mean and S.D.

x_i	$u_i = x_i - 2000$	u_i^2
2000	0	0
2023	23	529
2056	56	3136
2048	48	2304
2010	10	100
2025	25	625
2100	100	10000
2563	563	316969
2289	289	813521
2005	5	25
2082	82	6724
2056	56	3136
2049	49	2401
2020	20	400
2310	310	96100
2206	206	42436
2316	316	99856
2186	186	34596
2243	243	59049
2013	13	169
Total	**2600**	**762076**

From the above table, we have $\bar{x} = \left(2000 + \frac{2600}{20}\right) \text{kg} = 2130 \text{ kg}$

$$s^{|} = \sqrt{\frac{\Sigma u_i^2 - n\bar{u}^2}{n-1}}$$

$$= \sqrt{\frac{762076 - 20 \times (130)^2}{19}} = 149.3981 \text{ kg}$$

As $t_0 = \dfrac{\sqrt{n}\,(\bar{x} - 2025)}{s^{|}}$

$$\therefore t_0 = \frac{\sqrt{20}\,(2130 - 2025)}{149.3981} = 3.143$$

A glance at the critical region suggests that we reject H_0 and accept H_1. On the basis of the given sample we, therefore, conclude that the advertising campaign was successful in increasing the sales of the detergent powder produced by the company.

Illustration 8

A random sample of 26 items taken from a normal population has the mean as 145.8 and S.D. as 15.62. At 1% level of significance, test the hypothesis that the population mean is 150.

Solution: Here we would like to test $H_0 : \mu = 150$ i.e., the population mean is 150 against $H : \mu \neq 150$ i.e., the population mean is not 150. As the necessary conditions for applying t-test are fulfilled, we compute

$$t_0 = \frac{\sqrt{n}(\bar{x} - 150)}{s^{|}}$$

and the critical region at 1% level of significance is :

$$\omega : |t_0| \geq 2.787$$

In this case, as given

$$\bar{x} = 145.8; \quad s = 15.62 \;; \quad \text{and} \quad n = 26$$

$$\therefore s^{|} = \sqrt{\frac{n}{n-1}}\ S \quad = \sqrt{\frac{26}{25}} \times 15.62 = 15.9293$$

So, $t_0 = \dfrac{\sqrt{26}\,(145.8 - 150)}{15.9293} = -1.344$

thereby, $|t_0| = 1.344$

Looking at the critical region, we find acceptance of H_0. So on the basis of the given data, we infer that the population mean is 150.

16.7 t-TEST FOR DEPENDENT SAMPLE

One of the important applications of t-distribution is t-test for dependent samples or paired t-test. Let us consider a situation where a giant multinational company is claiming that their research wing has developed a new type of restorative that is going to increase the bodyweights of the babies suffering from malnutrition and this is a revolution in the world of medicine.

We may note that there are other factors such as age, height, food habits, living conditions etc., which could be attributed to a change in body weight. In case, we apply the drug to the same group of babies, these factors would be constant and if there is a significant increase in bodyweights, it would be due to the treatment i.e. the application of restorative except, may be, the chance factor. Thus, in order to verify the efficacy of the restorative, the best course of action would be to take a random sample of babies affected with rickets, measure their bodyweights before applying the restorative, and take their bodyweights for a second time, say a couple of months, after applying the restorative. The appropriate test to apply, in this case, is a paired t-test.

Similarly one may apply paired t-test to verify the necessity of a costly management training for its sales personnel by recording the sales of the selected trainees before and after the management training or the validity of special coaching for a group of educationally backward students by verifying their progress before and after the coaching programme or the increase in productivity due to the application of a particular kind of fertiliser by recording the productivity of a crop before and after applying this particular fertiliser and so on.

Let us now discuss the theoretical background for the application of paired t-test. In our earlier discussions, we were emphatic about the observations being independent of each other. Now we consider a pair of random variables which are dependent or correlated. Earlier, we considered normal distribution, to be more precise, univariate normal distribution. Similarly, we may think of bivariate normal distribution. Let x and y be two random variables following bivariate normal distribution with mean μ_1 and μ_2 respectively, standard deviations σ_1 and σ_2 respectively and a correlation co-efficient (ρ).

Thus 'x' and 'y' may be the bodyweight of the babies before and after the application of the restorative, sales before and after the training programme, marks of the weak students before and after the coaching, yield of a crop before and after applying the fertiliser and so on.

Let us consider 'n' pairs of observations on 'x' and 'y' and denote the 'n' pairs by (x_i, y_i) for i = 1, 2, 3, …, n.

Our null hypothesis is $H_0 : \mu_1 = \mu_2$ i.e., the restorative has no impact on the weight of the babies or the training has no importance or the coaching has not improved the standard of the students or the fertiliser has not increased the productivity significantly and so on. If we introduce a new random variable

$u = x–y$, then we may note that:

$E(u)$ or $\mu_u = E(x) - E(y)$

$= \mu_1 - \mu_2 = 0$, under H_0

Further, from the properties of normal distribution, it follows that 'u', being a linear function of two normal variables, also follows normal distribution with mean

mean $(\mu_u) = \mu_1 - \mu_2$ and variance $(\sigma_u^2) = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1, \sigma_2$

Thus testing $H_0 : \mu_1 – \mu_2$ is analogous to testing for population mean when the population standard deviation is unknown. In view of our discussion in Section 16.5, if the sample size is small, it is obvious that the appropriate test statistic would be:

$$t = \frac{\sqrt{n}(\bar{u} - \mu_u)}{s'_u} \quad \text{……(16.17)}$$

where n = sample size; $\bar{u} = \frac{\Sigma u_i}{n}$; $u = x - y$

$$s'_u = \frac{\sqrt{\Sigma(u_i - \bar{u})^2}}{n-1} = \sqrt{\frac{\Sigma u_i - n\bar{u}^2}{n-1}}$$

As before, under H_0, $t_0 = \frac{\sqrt{n}\bar{u}}{s'_u}$ follows t-distribution with (n–1) d.f.

Thus for testing $H_0 : \mu_u = 0$ against $H : \mu_u \neq 0$,

the critical region is provided by :

$\omega : t_0 \geq t_{\alpha/2\ (n-1)}$

For testing H_0 against $H_1 : \mu_1 > \mu_2$ i.e., $H_1 : \mu_u > 0$

We consider the critical region

$\omega : t \geq t_{\alpha,\ (n-1)}$

when the sample size exceeds 30, the assumption of normality for u may be avoided and the test statistic $\frac{\sqrt{n}\bar{u}}{s'_u}$ can be taken as a standard normal variable and accordingly we may recommend Z-test.

With the help of the above discussion, we take up some illustrations to understand the application of t-test for dependent samples.

Illustration 9

A drug is given to 8 patients and the increments in their blood-pressure were recorded to be :

4, 5, –1, 3, –2, 4, –7 and 0

Is it reasonable to believe that the drug has no effect on the change of blood-pressure?

Solution: Let x denote blood-pressure before applying the drug and y, the blood-pressure after applying the drug. Further let μ_1 denote the average blood-pressure in the population before applying the drug and μ_2, the average blood-pressure after applying the drug. Thus the problem is reduced to testing :

$H_0 : \mu_1 = \mu_2$ i.e., the drug has no effect on blood-pressure

Against $H_1 : \mu_1 > \mu_2$ i.e., the drug has reduced blood-pressure.

Denoting u = x – y = change in blood-pressure, we note that, under H_0,

$\frac{\sqrt{n}\,\bar{u}}{s'_u}$ follows t-distribution with (n–1) d.f under H_0.

Thus the critical region would be

$\omega : t_0 \geq t_{\alpha\ ,\ (n-1)}$

or $\omega : t_0 \geq 1.895$

By taking $\alpha = 0.05$, $t_{\alpha, (n-1)} = t_{0.05, 7} = 1.895$ from Appendix Table-5.

From the given data, we find that $n = 8$, $\Sigma u_i = 6$, $\Sigma u_i^2 = 120$

Hence $\bar{u} = \frac{\Sigma u_i}{n} = \frac{6}{8} = 0.75$

$$s'_u = \frac{\sqrt{\Sigma u_i^2 - n\bar{u}^2}}{n-1} = \frac{\sqrt{120 - 8 \times (0.75)^2}}{7} = 4.062$$

$$t_0 = \frac{\sqrt{n}\ \bar{u}}{s'_u}$$

$$\therefore t_0 = \frac{\sqrt{8} \times 0.75}{4.062} = 0.522$$

Looking at the critical region, we find that H_0 is accepted. Thus on the basis of the given data we conclude that the drug has been unsuccessful in reducing blood-pressure.

Illustration 10

A group of students was selected at random from the set of weak students in statistics. They were given intensive coaching for three months. The marks in statistics before and after the coaching are shown below.

Student's serial Number	**Marks in statistics**	
	Before coaching	**After coaching**
1	19	32
2	38	36
3	28	30
4	32	30
5	35	40
6	10	25
7	15	30
8	29	20
9	16	15

Could the coaching be considered as a success?

Test at 5% level of significance.

Solution: Let x and y denote the marks in statistics before and after the coaching respectively. If the corresponding mean marks in the population be μ_1 and μ_2 respectively, then we are to test :

$H_0 : \mu_1 = \mu_2$ i.e., the coaching has really improved the standard of the students, against the alternative hypothesis $H : \mu_1 < \mu_2$.

We compute :

$$t_0 = \frac{\sqrt{n}\bar{u}}{s'_u}$$ which follows t-distribution with (n–1) d.f under H_0.

where, n = no. of students selected = 9

u = x – y = difference in statistics marks

$$s'_u = \frac{\sqrt{\Sigma u_i^2 - n(\bar{u})^2}}{n-1}$$

since α = 0.05, n = 9,

consulting Appendix Table-5, we find that $t_{0.05}$, 8 = 1.86.

Thus the left-sided critical region is provided by w : $t_0 \le -1.86$.

Table 16.4: Computation of Sample Mean and Sample S.D.

Serial No. of student	Marks in Statistics		$u_i = (x_i - y_i)$	u_i^2
	(x_0) Before coaching	(y_0) After coaching		
1	19	32	–13	169
2	38	36	2	4
3	28	30	–2	4
4	32	30	2	4
5	35	40	–5	25
6	10	25	–15	225
7	15	30	–15	225
8	29	20	9	81
9	16	15	1	1
Total	–	–	–36	738

Thus $\bar{u} = \frac{\Sigma u_i}{n} = \frac{-36}{9} = -4$

$$s'_u = \sqrt{\frac{738 - 9 \times (-4)^2}{8}} = 8.6168$$

$$\therefore t_0 = \frac{\sqrt{8} \times -4}{8.6168} = -1.313$$

A glance at the critical region suggests that we accept H_0. On the basis of the given data, therefore, we infer that the coaching has failed to improve the standard of the students.

Illustration 11

Wilson company, known for producing fertilizers, recruited 15 candidates. After recording their sales, they were asked to attend a sales management course. Their sales, after attending the course, were recorded. The data are presented below.

Serial number of trainee	Sales ('000 Rs.)	
	Before the course	After the course
1	15	16
2	16	17
3	13	19
4	20	18
5	18	22.5
6	17	18.3
7	16	19.2
8	19	18
9	20	20
10	15.5	16
11	16.2	17
12	15.8	17
13	18.7	20
14	18.3	18
15	20	22

Was the training programme effective in promoting sales? Select $\alpha = 0.05$.

Solution: If we consider x and y as sales before and after attending the course respectively, then we are going to test :

$H_0 : \mu_1 = \mu_2$ against
$H_1 : \mu_1 < \mu_2$

μ_1 and μ_2 being the average sales in the population before the training and after the training. As before the critical region is :

$\omega : t_0 \leq -1.761$

as m = n–1 = 14 and $t_{0.05}$, 14 = 1.761

Table 16.5: Computation of Sample Mean and S.D.

Serial No. of trainee	Sales ('000 Rs.)		$u_i = x_i - y_i$	u_i^2
	Before course (x_i)	After course (y_i)		
1	15	16	– 1	1
2	16	17	– 1	1
3	13	19	– 6	36
4	20	18	2	4
5	18	22.5	– 4.5	20.25
6	17	18.3	– 1.3	1.69
7	16	19.2	– 3.2	10.24
8	19	18	1	1
9	20	20	0	0
10	15.5	16	– 0.5	0.25
11	16.2	17	– 0.8	0.64
12	15.8	17	– 1.2	1.44
13	18.7	20	– 1.3	1.69
14	18.3	18	0.3	0.09
15	20	22	– 2	4
Total	**–**	**–**	**–19.l5**	**83.29**

$$s'_u = \sqrt{\frac{\Sigma \mu i - n\bar{u}^2}{n-1}}$$

From the above table 16.5, we have

$$\bar{u} = \frac{-19.5}{15} = -1.3$$

$$s'_u = \sqrt{\frac{61 - 6\,(0.833)^2}{5}} = 3.3715$$

Hence $t_0 = \frac{\sqrt{n}\bar{u}}{s'_u} = \frac{\sqrt{15} \times -1.3}{2.0343} = -2.428$

t_o being less than –1.761, we reject H_0. Thus on the basis of the given sample, we conclude that the training programme was effective in promoting sales.

Illustration 12

Six pairs of husbands and wives were selected at random and their IQs were recorded as follows:

Pair	:	1	2	3	4	5	6
IQ of Husband	:	105	112	98	92	116	110
IQ of Wife	:	102	108	100	96	112	110

Do the data suggest that there is no significant difference in average IQ between the husband and wife? Use 1% level of significance.

Solution: Let x denote the IQ of husband and y, that of wife. We would like to test

$H_0 : \mu_1 = \mu_2$ i.e., there is no difference in IQ.

Against $H_1 : \mu_1 \neq \mu_2$, i.e. there is significant difference in IQ.

The critical region for this two-sided test is given by :

$$\omega : \left|t_0\right| \geq t_{\frac{0.01}{2}}, (6-1)$$

i.e., $\omega : \left|t_0\right| \geq t_{0.05,5}$

i.e., $\omega : \left|t_0\right| \geq 4.032$

Table 16.6: Computation of Mean and S.D. of IQ.

Pair	**IQ**		**u_i $= x_i - y_i$**	**u_i^2**
	Husband (x_i)	**Wife (y_i)**		
1	105	102	3	9
2	112	108	4	16
3	98	100	–2	4
4	92	96	–4	16
5	116	112	4	16
6	110	110	0	0
Total	–	–	**5**	**61**

From the above Table, we get,

$$\bar{u} = \frac{5}{6} = 0.8333$$

$$s'_u = \sqrt{\frac{\Sigma u_i^2 - n(\bar{u})^2}{n-1}}$$

$$s'_u = \sqrt{\frac{61 - 6\,(0.833)^2}{5}} = 3.3715$$

$$t_0 = \frac{\sqrt{n}\ \bar{u}}{s'_u}$$

so, $$t_0 = \frac{\sqrt{5} \times 0.8333}{3.3715} = 0.553$$

Therefore, we accept H_0 and conclude that, on the basis of the given sample, there is no reason to believe that IQs of husbands and wives are different.

Self Assessment Exercise B

1) State with reasons, whether the following statements are true or false?

   a) t-test is an exact test whereas z-test is an approximate test.

   b) For small samples, one must always use t-test.

   c) In order to apply paired t-test, it is assumed that the data are taken from a bivariate normal population.

   d) t-test for independent sample and t-test for dependent sample are applied under different conditions.

   e) t-test is not applicable if the population S.D. is unknown.

2) Describe the different steps one should undertake in order to apply t-test.

   ....

   ....

   ....

   ....

3) Distinguish between large sample and small sample.

   ....

   ....

4) A manufacturer of ball-point pens claims that a certain kind of pen manufactured by him has a mean writing life of 500 pages. A purchasing agent takes a sample of 10 such pens and the writing life of the 10 selected pens (in pages) are found to be :

   502, 510, 498, 475, 482, 523, 476, 518, 523, 479

   Determine at 1% level of significance whether the purchaser should reject the claim.

   ....

   ....

....

....

....

....

....

....

....

....

....

....

5) A certain diet was introduced to increase the weight of pigs. A random sample of 12 pigs was taken and weighed before and after applying the new diet. The differences in weights were :

   7, 4, 6, 5, – 6, – 3, 1, 0, –5, –7, 6, 2

   can we conclude that the diet was successful in increasing the weight of the pigs?

....

....

....

....

16.8 LET US SUM UP

This unit is an extension of the previous unit where we discussed the problems of statistical inferences in detail. In this unit we have discussed first the distinction made between small samples and large samples. If the sample size does not exceed thirty, then it is known as a small sample and we apply exact tests also known as Small Sample tests. In particular, if the assumption of normality holds and the population S.D. is unknown, we apply t-test for testing mean. A variation of t-test is its applicability to dependent samples where the effect of a particular treatment can be tested after eliminating other sources of variation. If the sample size exceeds 30, we consider approximate test also termed as Large Sample Test. These large sample tests are applicable without a set of assumptions.

Next, we have considered the problem of finding 100 (1–α) % confidence interval for population mean applying t-distribution. Coming back to the problem of testing for population mean for testing $H_0 : \mu = \mu_0$ against both-sided alternative $H_1 : \mu \neq \mu_0$, the critical region at α % level of significance is given by :

$$\omega : \left|t_0\right| \geq t_{\alpha/2, (n-1)}$$

The critical region for the right-sided alternative $H_1 : \mu > \mu_0$ is given by :

$\omega : t_0 \geq t_{\alpha, (n-1)}$ and the critical region for the left-sided alternative

$H_2 : \mu < \mu_0$ is given by $\omega : t_0 \leq -t_{\alpha, (n-1)}$

We have concluded our discussion by describing paired t-test and the critical region for t-tests applied to dependent samples.

16.9 KEY WORDS AND SYMBOLS

Chi-square Distribution: If x_1, x_2 …, x_n are 'm' independent standard normal variables, then $u = \Sigma x_i^2$ follows χ^2-distribution with md.f and this is denoted by $u \sim \chi^2_m$.

Degree of Freedom (d.f.): no. of observations – no. of constraints.

Large Sample: when sample size (n) is more than 30.

Large Sample Tests or Approximate Tests: tests based on large samples.

Paired Samples: Another term used for dependent samples.

Small Sample: when sample size (n) is less than 30.

Small Sample Tests or Exact Tests: tests based on small samples only.

t-distribution: If x is a standard normal variable and u is a chi-square with m.d.f., and x and u are independent variables, then the ratio.

$$\frac{x}{\sqrt{u/m}} \text{ follows t-distribution}$$

with m.d.f. and is denoted by $t \sim t_m$

100 (1–α) % confidence interval to m

$$\left[\bar{x} - t_{1/-\alpha/2,(n-1)} \times \frac{s^{|}}{\sqrt{n}}, \bar{x} + t_{1-\alpha/2,(n-1)} \times \frac{s^{|}}{\sqrt{n}}\right]$$

For testing population mean from independent samples, we use the test statistic

$$t_0 = \frac{\sqrt{n}(\bar{x} - \mu_0)}{s^{|}}$$

and for testing for a particular effect, we use

$$t_0 = \frac{\sqrt{n}\bar{u}}{s^{|}_u}$$

where u_0 = specific value for mean; $s^{|}$ = simple S.D. with (n–1) divisor;
u = x–y = difference in paired sample; and $s^{|}_u$ = sample of S.D. of u with (n–1) divisor

16.10 ANSWERS TO SELF ASSESSMENT EXERCISES

A) 1 a) Yes, b) Yes, c)Yes, d) No. e) Yes
f) Yes, g) No, h) Yes, i) Yes, j) No
k) Yes.

4. 90% confidence interval = [69.1163; 100.8837]
95% confidence interval = [65.2952, 104.7048]

5. Lower confidence limit = 9.3613
Upper confidence limit = 21.8927

6. 7.5596 and 10.8404

B) 1. a) No. b) Yes, c) Yes d) No e) No

5. No, $t_0 = -0.226$

6. No, $t_0 = 0.518$

16.11 TERMINAL QUESTIONS/EXERCISES

1) Describe a situation where you can apply t-distribution.

2) How would you distinguish between a t-test for independent sample and a paired t-test?

3) Describe the role played by t-distribution to set up Confidence Interval to population mean.

4) Distinguish between large samples and small samples.

5) Describe the steps one should undertake in order to apply t-test.

6) A technician is making engine parts with axle diameter of 0.750 inch. A random sample of 14 parts shows a mean diameter of 0.763 inch and a S.D. of 0.0528 inch.

   i) Set the null hypothesis and the alternative hypothesis.

   ii) Choose level of significance

   iii) Describe the critical region

   iv) Compute test statistic

   v) Draw your conclusion

   Examine whether the work meets the specification or not at 95% as well as 99% confidence interval to population mean.

   Answer : $t_0 = 0.888$

   95% confidence interval = [0.7302", 0.7958"]

   99% confidence interval = [0.7189", 0.8071"]

7) St. Nicholas college has 500 students. The heights (in cm.) of 11 students chosen at random provides the following results:

   175, 173, 165, 170, 180, 163, 171, 174, 160, 169, 176

   Determine the limits of mean height of the students of St. Nicholas college at 1% level of significance.

   (Ans: 164.6038 cm. and 176.4870 cm.)

8) For a sample of 15 units drawn from a normal population of 150 units, the mean and S.D. are found to be 10.8 and 3.2 respectively. Find the confidence level for the following confidence intervals.

   (i) 9.415, 12.185

   (ii) 9.113, 12.487

   [Ans: (i) 90% (ii) 95%]

9) A random sample of 15 observations from a normal population yields mean as 52.3 and S.D. as 5.63. Can it be assumed that the population mean is 50 ?

10) The following data relates to the sales of a new type of toothpaste in 15 selected shops before and after a special sales promotion campaign.

Shop No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Sales ('000 Rs.) Before campaign	15	16	13	14	18	19	12	16	20	11	12	9	15	17	21
Sales ('000 Rs.) After campaign	17	17	12	15	20	19	14	15	24	12	10	12	18	17	34

Would you regard the campaign as a success ?

(Ans: Yes, t_o = –2.671)

11. A suggestion was made that husbands are more intelligence than wives. A social worker took a sample of 12 couples and applied I.Q. Tests to both husbands and wives. The results are shown below:

Sl.No.	**I.Q. of**	
	Husbands	**Wives**
1.	110	115
2.	115	113
3.	102	104
4.	98	90
5.	90	93
6.	105	103
7.	104	106
8.	116	118
9.	109	110
10.	111	110
11.	87	100
12.	100	98

Do the data support the suggestion ?

Answer: No. t_0 = –0.7452

Note: These questions/exercises will help you to understand the unit better. Try to write answers for them. But do not submit your answers to the university for assessment. These are for your practice only.

16.12 FURTHER READING

The following text books may be used for more indepth study on the topics dealt within this unit.

Levin and Rubin, 1996, *Statistics for Management*. Printice-Hall of India Pvt. Ltd., New Delhi.

Hooda, R.P., 2000, *Statistics for Business and Economics,* MacMillan India Ltd., Delhi.

Gupta, S.P., 1999, *Statistical Methods*, Sultan Chand & Sons, New Delhi.

Gupta, S.C., and Kapoor, V.K., *Fundamentals of Mathematical Statistics*, Sultan Chand & Sons, New Delhi.